Tutorial 2

Image formation
Reminder - Perspective projection
\[
\frac{Z_w}{f} = \frac{X_w}{X_i} \Rightarrow X_i = \frac{X_w \cdot f}{Z_w}
\]
Reminder - Parallel projection
The $X$ component (Distance from the camera) is ignored.

$$X_i = X_w$$

The $Z$ component (Distance from the camera) is ignored.
Exercise

Given a polygon of the following points:

\[(1, -2, 3) \; (1, 3, 3) \; (1.5, 0, 3.5) \; (2, 3, 4) \; (2, -2, 4)\]

*Camera A* is located at \((0, 0)\) directed at the \(Z\) axis. The image plane of camera A is \(z = -1\).

*Camera B* is located at \((-4, 0, 0)\) and is directed at the \(Y\) axis. The image plane of camera B is \(x = -4\).
In [4]: plot_q1((12, 8))

Out[4]: <matplotlib.axes._subplots.Axes3DSubplot at 0x7fc2f6d29cf8>
Section a

What are the coordinates of the polygon on Camera A image plane?
In [5]:

object_points = [(1, -2, 3), (1, 3, 3),
                 (1.5, 0, 3.5), (2, 3, 4), (2, -2, 4)]

image_a_pts = [(-1 * x / z, -1 * y / z, -1)
               for x, y, z in object_points]  # Answered in class

write_points(image_a_pts)

The points are:

(−0.33, 0.67, −1.00), (−0.33, −1.00, −1.00), (−0.43, 0.00, −1.00),
(−0.50, −0.75, −1.00), (−0.50, 0.50, −1.00)
In [7]: draw_poly(plot_q1((12,7)), image_a_pts, color='purple')

Out[7]: <matplotlib.axes._subplots.Axes3DSubplot at 0x7fc2f4a3d550>
section b

Where does the polygon image of camera A fall on Camera B image plane?
In [8]: draw_poly(plot_q1_b(plot_q1(((12,7)))), image_a_pts, color='purple')

Out[8]: <matplotlib.axes._subplots.Axes3DSubplot at 0x7fc2f4c085f8>
In [9]:
#camera b center (-2, 0, 0) - plane x=-4
image_b_pts = [(-4, -2 * y / (x+2), -2 * z / (x+2))
               for x, y, z in image_a_pts]  # Answered in class
write_points(image_b_pts)

The points are:

(-4.00, -0.80, 1.20), (-4.00, 1.20, 1.20), (-4.00, -0.00, 1.27),
(-4.00, 1.00, 1.33), (-4.00, -0.67, 1.33)
In [10]:
    axes = draw_poly(plot_q1_b(plot_q1((12, 7))), image_a_pts, color='purple')
    draw_poly(axes,
              image_b_pts,
              color='purple')

Out[10]: <matplotlib.axes._subplots.Axes3DSubplot at 0x7fc2f4b39668>
section c

Calculate section b using parallel projection. What is the shape that was formed?
In [11]:
    image_b_pts_par = [(-4, y, z)
                     for x, y, z in image_a_pts]  # Answered in class
    write_points(image_b_pts)

The points are:

(−4.00, −0.80, 1.20), (−4.00, 1.20, 1.20), (−4.00, −0.00, 1.27),
(−4.00, 1.00, 1.33), (−4.00, −0.67, 1.33)
In [12]:
    axes = draw_poly(plot_q1_b(plot_q1((12, 7))), image_a_pts, color='purple')
    draw_poly(axes,
               image_b_pts_par,
               color='purple')

Out[12]:
<matplotlib.axes._subplots.Axes3DSubplot at 0x7fc2f4ae8518>
Exercise

In a 2D world (only $X$ and $Z$ axis), we have a pinhole camera sitting at $X = 0$ directed towards the positive $Z$ axis and an unknown $Z$ value.

How many pictures do we need to find out the focus $f$ of the camera?

Show the calculations.

Note: The process of finding the intrinsic camera parameters is called Camera calibration and will be discussed in more detail later in the course.

Answer

We have two unknowns, the focal length of the camera $f$, and the the location of the camera $z_c$.

If we take pictures of two point of known locations $(x_1^w, z_1^w)$ and $(x_2^w, z_2^w)$ and identify their corresponding image locations $x_1^i$ and $x_2^i$, we can get two equations with two unknowns:
\[ f = \frac{x_1^i}{x_1^w} (z_1^w - z_c) \]
\[ f = \frac{x_2^i}{x_2^w} (z_2^w - z_c) \]
\[ z_c = z_1^w - \frac{x_1^w}{x_1^i} f \]

\[ z_c = z_2^w - \frac{x_2^w}{x_2^i} f \]
\begin{equation}
\frac{z_1^w}{x_1^i} f = \frac{z_2^w}{x_2^i} f
\end{equation}

\begin{equation}
f = \frac{(z_1 - z_2)}{\left(\frac{x_1^w}{x_1^i} - \frac{x_2^w}{x_2^i}\right)}
\end{equation}
Note that we didn't have to obtain $z_c$ in order to obtain $f$. 
Refresher on vectors
**Vector**

Length and direction. $\vec{v}$ can be expressed as $(l, \theta)$ or as $(x, y)$. 
Dot product | Scalar product

Given 2 vectors $\vec{v}_1 = (x_1, y_1)$ and $\vec{v}_2 = (x_2, y_2)$

$$\vec{v}_1 \cdot \vec{v}_2 = x_1 x_2 + y_1 y_2 = |\vec{v}_1||\vec{v}_2| \cos \alpha$$

Where $\alpha$ is the angle between $\vec{v}_1$ and $\vec{v}_2$
• $\vec{v}_{\perp \vec{u}} = \frac{\vec{v} \cdot \vec{u}}{|\vec{v}_2|}$ (Projection of $\vec{v}$ on $\vec{u}$)

• When $\alpha = 90^\circ$ $\vec{v}_1$
  
  $\cdot \vec{v}_2 = 0$

• When $\alpha = 0^\circ$ $\vec{v}_1$
  
  $\cdot \vec{v}_2 = |\vec{v}_1|$
  
  $\cdot |\vec{v}_2|$
Exam question

A camera is located at \((0, 0)\) with focal length \(f = 10\) and rotated \(30^\circ\) from the horizon. What will be the projected image point \(x_i\) of the world point \(p_w = (10, 20)\)?
Answer

\[ \tilde{z} = \left( \cos \frac{\pi}{6}, \sin \frac{\pi}{6} \right) \]

\[ \tilde{x} = \left( -\sin \frac{\pi}{6}, \cos \frac{\pi}{6} \right) \]

\[ \tilde{z}_w = p_w \cdot \tilde{z} \]

\[ \tilde{x}_w = p_w \cdot \tilde{x} \]

\[ x_i = \frac{f \cdot \tilde{x}_w}{\tilde{z}_w} \]
```python
# Code version - This will be answered in class

p_w = np.array([10, 20])
f = 10

z_tilde = np.array([np.cos(np.pi / 6), np.sin(np.pi / 6)])
x_tilde = np.array([-z_tilde[1], z_tilde[0]])

z_w_tilde = p_w @ z_tilde
x_w_tilde = p_w @ x_tilde

x_i = (x_w_tilde * f) / 10
```
Vector product | Cross product

Given two 3D vectors \( \vec{v}_1 = (x_1, y_1, z_1) \) and \( \vec{v}_2 = (x_2, y_2, z_2) \)

\[
\vec{v}_1 \times \vec{v}_2 = \text{det} \begin{pmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix} = \\
(y_1 z_2 - z_1 y_2, z_1 x_2 - x_1 z_2, x_1 y_2 - y_1 x_2)
\]
• Defined only for 3D vectors
• The result is perpendicular to both $\vec{v}_1$ and $\vec{v}_2$ with length $|\vec{v}_1| \cdot |\vec{v}_2| \sin \alpha$
• $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$
• When $\vec{u}$ and $\vec{v}$ are parallel, $\vec{u} \times \vec{v} = (0, 0, 0)$
Homogenius coordinates

Reminder

Homogenius coordinates is an extention for 2D points s.t 2D in the Homogenius space, are defined at \([wx, wy, w]\)

When \((\frac{wx}{w}, \frac{wy}{w}) \Leftrightarrow [wx, wy, w] \)
Motivation

For Homogenius points, line can be described as $(a, b, c)$ Where the points on the line are $ax + by + cz = 0$. 
Section a

Given two lines $l_1 = (a_1, b_1, c_1), \quad l_2 = (a_2, b_2, c_2)$, how can we find the intersection point?

What happens if the lines are parallel?
A point $p$ that lays on both lines, must comply to $p \cdot l_1 = 0$ and $p \cdot l_2 = 0$

Let $p = l_1 \times l_2$. By the rule $u \cdot (u \times v) = 0$ we know that the point satisfies the condition.
Section b

Given two points $p_1 = (x_1, y_1, z_1), p_2 = (x_2, y_2, z_1)$, how can we find the line crossing between them?
Same as before \( l = p_1 \times p_2 \)
Exercise

Given a camera in a 2d world sitting at \((0, 0)\). express the relation between image points to world points via homogenius coordinates.
\[
\begin{bmatrix}
\tilde{w}x_i \\
\tilde{w}
\end{bmatrix} =
\begin{bmatrix}
f & 0 & 0 \\
0 & 1 & 0
\end{bmatrix} \cdot
\begin{bmatrix}
w x \\
w y \\
w
\end{bmatrix}
\]