

Separating Lock-Freedom from Wait-Freedom

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ABSTRACT

A long-standing open question has been whether lock-freedom and wait-freedom are fundamentally different progress conditions, namely, can the former be provided in situations where the latter cannot? This paper answers the question in the affirmative, by proving that there are objects with lock-free implementations, but without wait-free implementations—using objects of any finite power.

We precisely define an object called *n-process long-lived approximate agreement* (*n-LLAA*), in which two sets of processes associated with two *sides*, 0 or 1, need to decide on a sequence of increasingly closer outputs. We prove that 2-LLAA has a lock-free implementation using reads and writes only, while *n-LLAA* has a lock-free implementation using reads, writes and $(n - 1)$ -process consensus objects. In contrast, we prove that there is no wait-free implementation of the *n-LLAA* object using reads, writes and specific $(n - 1)$ -process consensus objects, called $(n - 1)$ -window registers.

CCS CONCEPTS

• **Theory of computation** → **Shared memory algorithms; Concurrent algorithms;**

KEYWORDS

concurrency; shared memory; multi-core algorithms; wait-freedom; lock-freedom; nonblocking.

1 INTRODUCTION

Asynchronous shared-memory algorithms capture the behavior of concurrent systems, where failure-prone processes, each running at its own speed, communicate by applying primitives to shared base

objects. Such algorithms are often used to *implement* higher-level objects, supporting ongoing invocations. These implementations withstand adverse system behavior, due to scheduling anomalies and failures.

Two main progress conditions have been studied for asynchronous shared-memory implementations [11]. The first, *wait-freedom*, ensures individual progress to each process, i.e., its operations complete as long as it takes an infinite number of steps. The second, *lock-freedom*, requires only global progress, namely, if a process takes an infinite number of steps then some (possibly other) processes complete their operations.¹ Clearly, every wait-free implementation is also lock-free.

Many wait-free implementations, starting with the universal construction [11] and recently [13], are derived by first presenting a fairly simple lock-free implementation, which is then made wait-free through sophisticated techniques. It is therefore natural to ask whether every object with a lock-free implementation also has a wait-free implementation. For *one-shot* (or *bounded*) objects, which each process accesses at most once (or a bounded number of times), the answer is trivially positive: Every lock-free algorithm is also wait-free, since once a process completes its operation (or a bounded number thereof), it takes no more steps, and the global progress condition implies that other processes are guaranteed to complete their operations.

The question, however, is challenging for long-lived objects, as there are objects with a lock-free implementation but without a known wait-free implementation *using the same primitives* (e.g., queues from test&set primitives [2]). The fundamental nature of wait-freedom and lock-freedom, and their use in numerous papers, makes this an important question.

The question was seemingly answered by Herlihy [10], who argued that there is an object for which there is a lock-free implementation for two processes using only reads and writes, and there is no wait-free algorithm in the same setting. Besides several inaccuracies in the proofs (see below), the answer provided by [10] is limited: it does not show a separation for more than two processes and it does not show a separation when primitives stronger than reads and writes can be used.

This paper paints a more complete picture of the separation between lock-freedom and wait-freedom. We precisely define an object called *n-process long-lived approximate agreement* (in short, *n-LLAA*), supporting a single operation called `output()`, which has

¹ Implementations of this kind were initially called *nonblocking* [11], but we use the more contemporary terminology.

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no argument and returns a real number. Each of the n processes is assigned a *side*, either 0 or 1. Let S be a sequential execution with invocations of `output()`. The *current position* of side i in S is the output value of the last operation of a process in side i , or i if there are no operations of processes in side i . The sequential specification of n -LLAA includes every sequential execution in which the distance between the current positions of the two sides is at most $\frac{1}{2^r}$, where r is the total number of operations in the execution.

We present a lock-free linearizable implementation of 2-LLAA, a restriction of n -LLAA for two processes, using reads and writes only (Section 3). This implementation is extended to a lock-free linearizable implementation of n -LLAA, using reads, writes and $(n - 1)$ -process consensus objects (Section 4).²

A specific consensus object that can be used is a *window register*. These registers support read and write primitives: the write primitive is standard, but the read primitive, parameterized with an integer $k > 1$, returns a k -tuple composed of the last k values written, in the order in which they were written. A k -window register supports read primitives with parameter $\leq k$; note that a 1-window register is a standard register supporting read and write primitives. This object was initially defined in [14] to illustrate causal consistency. A similar object was defined concurrently in the context of space lower bounds for concurrent objects [8]. It is proved in [1] that a k -window register can solve consensus among exactly k processes, i.e., its *consensus number* is k .

Our main impossibility result shows that there is no wait-free implementation of the n -LLAA object, that uses only k -window registers, for $k < n$ (Section 5). Note that this includes read/write registers.

Related Work: The 2-LLAA object is a stronger version of the *iterated approximate agreement* object studied in [10], and its implementation is the same, except for a few small details. However, as explained in Section 3.2, 2-LLAA requires a more elaborate correctness proof, which determines the linearization points dynamically. Differently from 2-LLAA, the convergence requirement of the iterated approximate agreement object is that the r -th output operation of p_i is at distance at most $1/2^r$ from the the current position of the other process. Furthermore, the impossibility result for wait-free implementations in [10] relies on an unstated assumption (in the second paragraph of the proof of Lemma 14) that an infinite number of steps by one process must eventually fix the decision of the other process. Although this behavior, often referred to as *helping*, is a common way to turn lock-free implementations into wait-free ones, it is not a priori known to be a necessary condition for wait-freedom. The notion of helping was recently formalized [5, 6] and was shown to be a nontrivial property, allowing to prove nontrivial results not known to hold otherwise. Additionally, the final step of the impossibility proof in [10] (in the proof of Theorem 15) needs a stronger convergence requirement than the one required by the iterated approximate agreement object. Our impossibility proof addresses these challenges and extends to the general case of systems with $(n - 1)$ -consensus objects.

² The separation question is trivial if n -process consensus objects are available since they support a wait-free implementation for n -processes of every object with a sequential specification [11].

Aspnès and Herlihy [4] defined a class of objects for which there are wait-free implementations from read/write objects, as well as lock-free ones. Another class of objects, called *Common2*, includes the set of objects that have a wait-free implementation from objects with consensus number 2, e.g., `test&set` [2, 3].

Strong linearizability [9] was introduced as a way to circumvent some anomalies when used with randomized implementations. It was shown that there are deterministic lock-free and strongly linearizable implementations of snapshot, max-register or counter from multi-writer multi-reader registers, while there are no such wait-free implementations [7].

2 MODEL OF COMPUTATION

We consider a standard shared memory system with n asynchronous processes, p_0, \dots, p_{n-1} , which may crash at any time during an execution. Processes communicate with each other by applying atomic *primitives* to *base objects*. The base objects we consider are *window registers* of size $k \in \{1, \dots, n - 1\}$ (in short, k -window registers) to which *read* and *write* primitives are applied. The *write* primitive has a single parameter, the value to be written. The *read* primitive returns a k -tuple composed of the last k values written, in the order in which they were written. Missing values are replaced by a default value \perp . Each window register can be read and written to by all processes, i.e., they are *multi-reader multi-writer*. Note that a 1-window register is the standard register, supporting ordinary read and write operations.

A (*high-level*) *concurrent object*, or *data type*, is defined by a state machine consisting of a set of states, a set of operations, and a set of transitions between states. Such a specification is known as *sequential*. An *implementation* of a data type T is a distributed algorithm \mathcal{A} consisting of a local state machine \mathcal{A}_p , for each process p . For any process p , \mathcal{A}_p specifies which primitives p applies in order to return a response when it invokes an operation of T . Each of these primitive invocations is a step. For the rest of this section, fix an implementation \mathcal{A} of some data type T .

A *configuration* C of the system contains the states of all registers and processes. In an *initial* configuration, registers and processes are in their initial states. Given a configuration C and process p , $p(C)$ is the configuration after p takes its next step in C . Moreover, $p^0(C) = C$ and for every $n \in \mathbb{N}$, $p^{n+1}(C) = p(p^n(C))$.

An *execution* of \mathcal{A} is a possibly infinite sequence of steps (namely, invocations of primitives) and invocations and responses of high-level operations in \mathcal{A} , with the following properties: (1) Each process first invokes a high-level operation, and only when it has a corresponding response, it can invoke another high-level operation, i.e., executions are *well-formed*. (2) A process takes steps only between an invocation and a response. (3) For any invocation of process p , the steps of p between that invocation and the following response of p , if there is one, correspond to steps of p that are specified by \mathcal{A} .

An operation in an execution is *complete* if both its invocation and response appear in the execution. An operation is *pending* if only its invocation appears in the execution. An implementation is *wait-free* if every process completes each of its operations in a finite number of its steps. Formally, if a process executes infinitely many steps in an execution, all its operations are complete. An

implementation is *lock-free* if whenever processes execute steps, at least one of the operations terminates. Formally, in every infinite execution, infinitely many operations are complete. Thus, a wait-free implementation is lock-free but not necessarily vice versa.

An operation OP *precedes* another operation OP' if OP completes before OP' starts. OP and OP' are *concurrent* (or *overlapping*) if neither OP precedes OP' nor OP' precedes OP . OP *does not precede* OP' if either OP is concurrent with OP' or OP' precedes OP .

Linearizability [12] is the standard notion used to identify a correct implementation. Roughly speaking, an implementation is linearizable if each operation appears to take effect atomically at some time between its invocation and response, hence operations' real-time order is maintained. Formally, let \mathcal{A} be an implementation of a data type T . An execution α of \mathcal{A} is *linearizable* if there is a sequential execution S of T (i.e., a sequence of matching invocation-response pairs, starting with an invocation) such that: (1) S contains every complete operation of α and some pending operations. Hence, the output value in the matching responses of an invocation in S and α are the same. (2) For every pair of operations op and op' in α , if the response of op precedes the invocation of op' in α , then op appears before op' in S ; namely, S respects the real-time order in α . \mathcal{A} is *linearizable* if each of its executions is linearizable.

3 READ/WRITE LOCK-FREE 2-LLAA

This section presents a read/write lock-free 2-LLAA implementation, as well as a sketch of its correctness proofs. Some proofs are deferred to the appendix.

DEFINITION 1. Let S be a sequential execution with invocations of two processes p_0 and p_1 to an operation $\text{output}()$, each returning a real number. The current position of p_i in S is the output value of the last operation of p_i , or i if there are no operations of p_i . We say that S satisfies the convergence requirement if the distance between the current positions of the processes is at most $1/2^r$, where r is the total number of operations in the execution. The sequential specification of the 2-LLAA object contains every sequential execution of $\text{output}()$ operations by p_0 and p_1 such that each of its prefixes satisfies the convergence requirement.

The algorithm in Figure 1 is a lock-free implementation of 2-LLAA using read/write primitives. Each process p_i locally stores its current position and round (number of complete and pending operations of p_i , so far) in its local variable me , and stores the position and round of the other process, denoted p_j , in you . The initial position and round of p_i are i and 0. Processes communicate through the shared array M by writing and reading positions and rounds. Additionally, p_i uses two (closed) intervals $prev$ and $range$: $prev$ stores the interval committed in p_i 's previous operation, which indicates where any future position of p_i might be, and $range$ stores the current interval the position of p_j should belong to in order for p_i to decide.

When executing $\text{output}()$, p_i first increments its round counter and writes the new value in M (lines 01 and 02) and then iterates the while loop (lines 03 to 15). Each iteration tries to get closer to the position of p_j in M , until the condition in line 08 is satisfied. There are two (disjoint) cases that can make p_i 's current operation decided. In the first case, called *closeness rule*, the position of p_j read by p_i is in the interval $range$ computed in line 07, hence the distance

Shared variables:

$M[0, 1] \leftarrow [(\text{pos} = 0, \text{round} = 0), (\text{pos} = 1, \text{round} = 0)]$

Local variables:

$myID \leftarrow i$
 $yourID \leftarrow 1 - i$
 $me.\langle \text{pos}, \text{round} \rangle \leftarrow \langle myID, 0 \rangle$ %% My position and round
 $you.\langle \text{pos}, \text{round} \rangle \leftarrow \langle yourID, 0 \rangle$ %% Other's position and round
 $prev, range \leftarrow [myID \pm 1]$

Function $\text{output}()$:

```
(01)  $me.\text{round} \leftarrow me.\text{round} + 1$ 
(02)  $M[myID].\text{round} \leftarrow me.\text{round}$ 
(03) while true do
(04)    $M[myID].\text{pos} \leftarrow me.\text{pos}$ 
(05)    $you \leftarrow M[yourID]$ 
(06)    $rounds \leftarrow me.\text{round} + you.\text{round}$ 
(07)    $range \leftarrow [me.\text{pos} \pm 1/2^{rounds}]$ 
(08)   if  $you.\text{pos} \in range \vee you.\text{pos} \notin prev$  then
(09)      $prev \leftarrow range$ 
(10)     return  $me.\text{pos}$ 
(11)   elseif  $me.\text{pos} < you.\text{pos}$  then
(12)      $me.\text{pos} \leftarrow me.\text{pos} + 1/2^{rounds}$ 
(13)   else
(14)      $me.\text{pos} \leftarrow me.\text{pos} - 1/2^{rounds}$ 
(15)   endif
(16) endwhile
endFunction
```

Figure 1: A lock-free 2-LLAA algorithm for processes p_0 and p_1 . Code of process p_i .

between the positions is at most $1/2^r$, where r is the total number of operations p_i is aware of. This ensures that the distance between the positions of the processes obeys the convergence requirement of 2-LLAA. In the second case, called *invalid position rule*, the position of p_j is out of the previously-committed range $prev$ of p_i . Observe that this can happen because of asynchrony: p_i might have completed several operations getting closer to a position of p_j in M that is different from the position p_j is about to write. The rationale behind the invalid position rule is that as p_j was slow and its position is not in $prev$ of p_i , it would have to “catch up” and decide by the closeness rule in its current operation. As shown below (Lemma 1), whenever p_i decides by the invalid position rule, the current operation of p_j is pending and p_j decides by the closeness rule in every extension in which its operation is complete.

3.1 Correctness Proof

In the analysis of the algorithm, for simplicity and without loss of generality, the invocation of an operation is identified with its first shared-memory write operation outside the **while** loop of the output function (line 02), that is, it is executed atomically together with the local operations of line 01. The response of a decided operation is identified with its last shared-memory read step inside the loop (line 05). We also assume that the local computation inside the **while** loop (lines 05-15) is executed atomically together with the shared-memory read step in line 05.

For an execution α , we use the following notation and terminology. The k -th operation of a process p_i in α is denoted $P_i^k(\alpha)$, if there is such an operation. $P_i^0(\alpha)$ denotes a fictitious operation whose output is the initial position of p_i . An operation $P_i^k(\alpha)$ of p_i is *decided* if it takes its response (read) step in α . In this case, $v_i^k(\alpha)$ denotes the value of variable v at the end of $P_i^k(\alpha)$. We let $rad(prev_i^k(\alpha))$ denote the radius of the interval $prev_i^k(\alpha)$ with center $me_i^k.pos(\alpha)$. A decided operation $P_i^k(\alpha)$ decides by the *closeness rule* on $P_j^m(\alpha)$ (with $j = 1 - i$) if $you_i^k.pos(\alpha) \in range_i^k(\alpha)$ and $you_i^k.round(\alpha) = m$. $P_i^k(\alpha)$ decides by the *invalid position rule* on $P_j^m(\alpha)$ if $you_i^k.pos(\alpha) \notin prev_i^k(\alpha)$ and $you_i^k.round(\alpha) = m$. We omit α from the above definitions when it can be understood from the context.

To see that Algorithm 1 is lock-free, we note that two operations P_i^k and P_j^m cannot both take an infinite number of steps without completing, since eventually they agree on $rounds = k + m$ and their positions get sufficiently close to each other. The formal proof is omitted for lack of space

Algorithm Linearize (Figure 2) specifies how we linearize a quiescent execution (i.e. an execution without pending operations) α . Lemma 1 stated below (whose proof is in the appendix) forms the basis for showing that Algorithm Linearize produces correct linearizations. Linearize considers the operations of α in the order they decide. Let P_i^k and P_j^m be the operations considered at the beginning of the ℓ -th iteration of the while loop of Linearize. If P_j^m is decided, then P_j^m is already in seq_ℓ , and, as we shall see, P_i^k decides by the closeness rule. This implies that P_i^k and P_j^m satisfy the convergence requirement. If P_j^m is not decided (hence pending), then P_j^m can be safely placed before or after P_i^k because Lemma 1 implies that P_j^m will decide by the closeness rule. Finally, as the proof of Lemma 3 argues, if none of P_i^k and P_j^m are in seq_ℓ , then P_i^k and P_j^m start from a quiescent state of the algorithm (operations P^2 and Q^1 in Figure 3 are an example of this situation) and their order is decided considering the outputs of the previous operations.

LEMMA 1. *Let α be a finite execution whose last step is a read (in line 05) of an operation P_i^k of p_i that makes that operation decided (by either the closeness or the invalid position rules) on some undecided operation P_j^m of p_j , with $j = 1 - i$. Let $\alpha\beta$ be any extension of α in which P_j^m is decided. Then, P_j^m decides by the closeness rule in $\alpha\beta$ and $dist(me_i^k.pos, me_j^m.pos) \leq 1/2^{m+k}$.*

It follows directly from Algorithm Linearize that the latest point in which an operation is linearized is when it decides (by reading in line 05 of Figure 1). Moreover, once an operation is linearized, the operation (by the other process) it decides on is also linearized (if it was not linearized before).

LEMMA 2. *Let α be a finite execution without pending operations. The sequential execution produced by Linearize(α) respects the real-time order in α .*

Function Linearize(α):

```

(01)  $seq_0 \leftarrow \epsilon, \ell \leftarrow 0$ 
(02) while  $\alpha$  has more than  $\ell$  operations do
(03)    $\beta_{\ell+1} \leftarrow$  shortest prefix of  $\alpha$  with  $\ell + 1$  decided operations
(04)    $P_i^k \leftarrow$  operation that decides in the last (read) step of  $\beta_{\ell+1}$ 
(05)    $P_j^m \leftarrow$  operation that  $P_i^k$  decides on in  $\alpha$  (i.e.  $you_i^k.round = m$ )
(06)   if  $P_i^k$  is in  $seq_\ell$  then
(07)     if  $P_j^m$  is in  $seq_\ell$  then
(08)        $seq_{\ell+1} \leftarrow seq_\ell$ 
(09)     else
(10)        $seq_{\ell+1} \leftarrow seq_\ell \cdot P_j^m$ 
(11)     else
(12)       if  $P_j^m$  is in  $seq_\ell$  then
(13)          $seq_{\ell+1} \leftarrow seq_\ell \cdot P_i^k$ 
(14)       else if  $dist(me_i^k.pos, me_j^{m-1}.pos) \leq dist(me_j^m.pos, me_i^{k-1}.pos)$  then
(15)          $seq_{\ell+1} \leftarrow seq_\ell \cdot P_i^k \cdot P_j^m$ 
(16)       else
(17)          $seq_{\ell+1} \leftarrow seq_\ell \cdot P_j^m \cdot P_i^k$ 
(18)       endif
(19)     endif
(20)      $\ell \leftarrow \ell + 1$ 
(21)   endwhile
(22) Remove the (fictitious) operations  $P_0^0$  and  $P_1^0$  from  $seq_\ell$ 
(23) return  $seq_\ell$ 
endFunction

```

Figure 2: A sequential algorithm to linearize an execution α without pending operations.

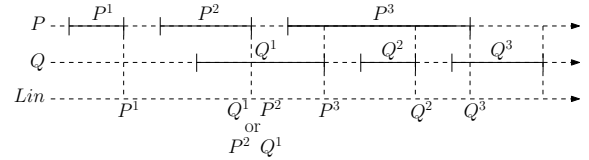


Figure 3: An example of the linearization produced by Algorithm Linearize.

PROOF. Consider two operations P_i^k and P_j^m in α , $k, m > 0$. Suppose that P_i^k precedes P_j^m in α . The latest point at which Algorithm Linearize linearizes P_i^k is when β_{l+1} in line 03 of Algorithm Linearize ends in the read step which makes P_i^k decide in α . None of the steps of P_j^m appear in β_{l+1} , hence P_i^k does not decide on P_j^m and consequently P_j^m is not linearized in lines 10, 15 or 17 of Algorithm Linearize (which are the only places operations are linearized). Furthermore, no prior operation by p_i could have decided on P_j^m in β_{l+1} , hence it was not linearized before. \square

Lemma 3, is the key technical lemma in the algorithm's correctness proof, establishing that Algorithm Linearize outputs correct linearizations.

LEMMA 3. *Let α be any finite execution without pending operations. Then, $Lin(\alpha)$ is a sequential execution of the 2-LLAA object.*

PROOF SKETCH. We consider a linearization $\text{Lin}(\alpha)$ of an execution α which, as we assume towards a contradiction, is not a sequential execution of 2-LLAA, and consider the first operation P_i^k that violates convergence. Let P_j^{m-1} be the latest preceding operation of p_j in $\text{Lin}(\alpha)$. Then, $\text{dist}(me_i^k.pos, me_j^{m-1}.pos) > 1/2^{m+k-1}$. Since $\text{Lin}(\alpha)$ respects the real-time order in α (Lemma 2), P_i^k does not precede P_j^{m-1} in α .

If P_j^{m-1} and P_i^k are concurrent in α , let β be the shortest prefix of α in which the first of them, say P_i^k , is decided. Hence P_j^{m-1} is pending and P_i^k decides on P_j^{m-1} . By Lemma 1,

$$\text{dist}(me_i^k.pos, me_j^{m-1}.pos) \leq 1/2^{m+k-1},$$

and the convergence requirement holds.

We therefore know that P_j^{m-1} precedes P_i^k in α . It can be shown that at the end of the previous (possibly fictitious) operation P_i^{k-1} of p_i , $\text{rad}(\text{prev}_i^{k-1}) \geq 1/2^{m+k-2}$. Moreover, when P_i^k starts, $\text{prev}_i = \text{prev}_i^{k-1}$, $me_i.pos = me_i^{k-1}.pos$ and $M[j].pos = me_j^{m-1}.pos$. Now, if no operation of p_j overlaps P_i^k , then if P_i^k decides by the invalid position rule, it does so in the very first iteration of the while loop, without any change to its position $me_i.pos$, and thus $me_i^k.pos = me_i^{k-1}.pos$. Hence, $me_j^{m-1}.pos \notin \text{prev}_i^{k-1}$ and $\text{dist}(me_i^{k-1}.pos, me_j^{m-1}.pos) > 1/2^{m+k-2}$, implying that either P_i^{k-1} or P_j^{m-1} (both preceding P_i^k in $\text{Lin}(\alpha)$) do not satisfy the convergence requirement, which is a contradiction. And if P_i^k decides by the closeness rule, the code implies that

$$\text{dist}(me_i^k.pos, me_j^{m-1}.pos) \leq 1/2^{m+k-1},$$

and the convergence requirement holds.

We are left with the case that operation P_j^m exists and overlaps

with P_i^k (and P_j^{m-1} precedes P_i^k). We first prove that

$$\text{dist}(me_i^k.pos, me_j^{m-1}.pos) \leq \text{dist}(me_j^m.pos, me_i^{k-1}.pos) \quad (1)$$

$$\text{dist}(me_i^k.pos, me_j^m.pos) \leq 1/2^{m+k} \quad (2)$$

To prove (1), note that P_j^m cannot precede P_i^{k-1} because it is concurrent with P_i^k . Also, it is easy to show that P_i^{k-1} precedes P_j^m in α . Thus, P_j^{m-1} precedes P_i^k and P_i^{k-1} precedes P_j^m , and hence both P_i^{k-1} and P_j^{m-1} are complete when the first among P_i^k and P_j^m starts in α . Without loss of generality, suppose that P_i^k decides before P_j^m , then, since they overlap, P_i^k decides on P_j^m . Since P_j^{m-1} precedes P_i^k and P_i^{k-1} precedes P_j^m , in the while iteration of Algorithm Linearize that linearizes P_i^k , the conditions of lines 06 and 12 are false, namely, P_i^k and P_j^m were not linearized yet, and they are linearized in lines 14-18. It can be shown that P_j^m is not linearized before P_i^k , and the condition of line 14 holds, implying (1).

Lemma 1 implies (2), since P_i^k and P_j^m are concurrent, similarly to the case when P_i^k and P_j^{m-1} are concurrent, handled before.

Finally, consider the outputs $me_i^{k-1}.pos$ and $me_j^{m-1}.pos$ of P_i^{k-1} and P_j^{m-1} . Suppose that $me_i^{k-1}.pos \leq me_j^{m-1}.pos$ (the other case is similar). Let $\mathcal{I} = [me_i^{k-1}.pos, me_j^{m-1}.pos]$. It can be shown that

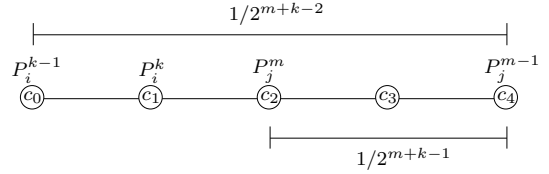


Figure 4: An example of the last case in the proof of Lemma 3: P_i^k decides before P_j^m and outputs c_1 , whose distance to the output of P_j^{m-1} is strictly larger than $1/2^{m+k-1}$. By (2), the output of P_j^m is in the interval $[c_0, c_2]$. This is a contradiction since the distance between any point in $[c_0, c_2]$ and the output of P_i^{k-1} is at most $1/2^{m+k-1}$, and thus $\text{dist}(me_i^k.pos, me_j^{m-1}.pos) > \text{dist}(me_i^k.pos, me_j^{m-1}.pos)$.

$me_i^k.pos, me_j^m.pos \in \mathcal{I}$. By assumption, P_i^{k-1} and P_j^{m-1} do satisfy the convergence requirement, namely,

$$\text{dist}(me_i^{k-1}.pos, me_j^{m-1}.pos) \leq 1/2^{m+k-2},$$

and hence $|\mathcal{I}| \leq 1/2^{m+k-2}$. The interesting case is when $|\mathcal{I}| = 1/2^{m+k-2}$. Lemma 12 (in the appendix) implies that the first to decide among P_i^k and P_j^m must decide a position (in \mathcal{I}) that is an integer multiple of $1/2^{m+k}$. There are only five integer multiples of $1/2^{m+k}$ in \mathcal{I} .³ A case by case analysis shows that if $\text{dist}(me_i^k.pos, me_j^{m-1}.pos) > 1/2^{m+k-1}$, then inequality (2) implies that $\text{dist}(me_i^k.pos, me_j^{m-1}.pos) > \text{dist}(me_j^m.pos, me_i^{k-1}.pos)$, which is a contradiction, given inequality (1) (see an example in Figure 4). \square

Consider any finite execution α of the algorithm from Figure 1. Let $\alpha\beta$ be an extension of α in which all operations in α are complete (and no new operation is started). This extension exists since the algorithm is lock-free. By Lemma 2, the sequential execution $\text{seq}(\alpha\beta)$ produced by $\text{Lin}(\alpha\beta)$ respects the real-time order of $\alpha\beta$; by Lemma 3, it is a sequential execution of the 2-LLAA object. Thus, $\text{seq}(\alpha\beta)$ is a linearization of $\alpha\beta$, and of α as well, implying the main theorem of this section:

THEOREM 4. *Algorithm 1 is a lock-free linearizable implementation of the 2-LLAA object, using reads and writes.*

3.2 A Simpler Linearization Proof?

This section discussed whether a simpler linearization argument exists. The linearization proof in the previous subsection is nontrivial, partly because Algorithm Linearize may place operations that are pending in α before operations that are complete. Furthermore, as we described previously, to linearize a non-quiescent execution α , we first complete all pending operations in α and then apply Algorithm Linearize to the resulting quiescent execution. Consequently, it is possible that the linearization S' we obtain for an extension α' of α is not an extension of the linearization S we obtain for

³ It is useful to consider the initial configuration, where $m = k = 1$ and $me_i^{k-1}.pos = 0$, $me_j^{m-1}.pos = 1$.

α , implying that our implementation is not necessarily strongly linearizable [9].⁴

In contrast, the linearizability order used in the correctness proof of the lock-free algorithm [10, Figure 5, Section 4] is simple and shows that the algorithm is strongly linearizable. Below we argue that this simple approach does not work for our stronger 2-LLAA object.

The linearization order used in [10] (right above Lemma 11) is the following. For every operation OP in an execution, let $rounds(OP)$ be the first value the operation sets to the local variable $rounds$ (that variable is called i in [10]). Then, if $rounds(OP) < rounds(OP')$, OP is linearized before OP' , and if $rounds(OP) = rounds(OP')$, OP and OP' can be linearized in any order.

Consider the following execution of the algorithm in Figure 1: (1) p_0 starts an invocation to $output()$, completes one iteration of the loop (note that it does not decide as $1/2^{rounds} = 1/2^1$) and stops before writing its new preference at the first step of the second iteration of the loop. Note that p_0 is about to write $1/2$ and the position of p_0 in M is still 0. (2) p_1 executes, until completion, two consecutive invocations to $output()$ (p_0 takes no steps while p_1 completes the two invocations). Observe that p_1 obtains $1/4$ and $1/8$ from the two invocations, respectively. (3) p_0 runs solo until it completes its pending $output()$ invocation. Note that p_0 obtains $1/4$ from the invocation.

It is easy to see that, for the execution α defined above, $rounds(p_0^1) = 1$, $rounds(p_1^1) = 2$ and $rounds(p_1^2) = 3$, and thus, using the linearization order defined in [10], α would be linearized as follows: $\langle p_0^1 : 1/4 \rangle \langle p_1^1 : 1/4 \rangle \langle p_1^2 : 1/8 \rangle$. However, this linearization is incorrect because $\text{lin}(\alpha)$ is not a sequential execution of 2-LLAA: the prefix $\langle p_0^1 : 1/4 \rangle$ of the sequential execution is invalid, because the distance between the current position of p_0 and the current (initial) position of p_1 , which are respectively $1/4$ and 1, is larger than $1/2^1$. Consequently, $\text{lin}(\alpha)$ is not a valid sequential execution of 2-LLAA either.

Finally, it can be verified that Algorithm Linearize produces the linearization $\langle p_1^1 : 1/4 \rangle \langle p_0^1 : 1/4 \rangle \langle p_1^2 : 1/8 \rangle$ from α , which is a sequential execution of 2-LLAA.

4 LOCK-FREE n -LLAA, USING $(n - 1)$ -CONSENSUS OBJECTS

We formally define the n -LLAA object and explain how it can be implemented in a lock-free manner using $(n - 1)$ -process consensus objects; full correctness proofs are omitted for lack of space.

DEFINITION 2. *Each of the n processes belongs to a side, either 0 or 1. The n -LLAA object supports a single operation called $output()$. Let S be a sequential execution with invocations of $output()$, each returning a real number. The current position of side i in S is the output value of the last operation of a process in side i , or i if there are no operations of processes in side i . We say that S satisfies the convergence requirement if the distance between the current positions of the two sides is at most $\frac{1}{2^r}$, where r is the total number of operations in the execution. The sequential specification of the n -LLAA object contains every sequential execution of $output()$ operations by the*

⁴ An algorithm is *strongly linearizable* if there is a function f from executions of the algorithm to sequential executions such that (1) for every execution α , $f(\alpha)$ is a linearization of α and (2) for every execution $\alpha\beta$, $f(\alpha)$ is a prefix of $f(\alpha\beta)$.

n processes such that each of its prefixes satisfies the convergence requirement.

The lock-free implementation of n -LLAA is based on the lock-free 2-LLAA algorithm. Consider first the case in which there is at least one process in each side. Then, using $(n - 1)$ -process consensus objects, the $\leq n - 1$ processes in side i simulate operations of process p_i in Algorithm 1; each time an operation terminates, the processes decide who takes this output value, also using consensus objects. When there are n processes in the same side, the processes can safely decide on the initial value of the other side.

Figure 5 shows how processes in the same side i simulate process p_i of Algorithm 1. Lines 05–31 simulate a single operation $output()$ of p_i ; each time this loop starts, a new operation of p_i is simulated. Lines 08 and 11 correspond to the write operations of Algorithm 1 and Lines 12–16 simulate the read operation of Algorithm 1, with the help of the consensus objects in $READS$. Each process first reads the pair $(pos, round)$ of the processes in the other side and considers the most current one (from its point of view), with the help of the timestamps in the local variables me . This is its proposal to the consensus which decides the next read value of the simulated process p_i . Once the output value of the current simulated operation of p_i has been computed (the condition in Line 19 is true), the processes in the same side i compete for that output value in Line 22, with the help of the consensus objects in OPS . The winning process outputs that value for its current operation, while the others start a new simulated operation of p_i . The goal of Lines 01–04 is to handle the case in which all processes belong to the same side: when a process discovers that all processes are in its side, it knows that the other side is not moving (it has not moved since the beginning of the execution) hence it can safely decide the initial position of the other side.

The way processes advertise their side in Line 01, and then check the others' side in Line 03, ensures that in every execution, at most $n - 1$ processes invoke the same consensus object (in Lines 16 and 22). The following property directly follows from the agreement property of consensus.

LEMMA 5. *Consider an execution of Algorithm 5. Then, for every pair of processes p_ℓ and p_j in the same side, if both p_ℓ and p_j assign the k th value to a local variable v , which is either me , you , $prev$ or $range$, then $v_j^k = v_\ell^k$.*

Using this lemma, we can now precisely define the notion of a *simulated operation*. Let α be a (finite or infinite) execution of Algorithm 5. For every side i and integer $k \geq 1$, the k -th simulated operation of side i , denoted sOP_i^k , is the subsequence of α containing all steps corresponding to steps in Lines 08, 11, 12, 16 or 22 of any process such that when it executes the step, its $me.round$ local variable is equal to k . The following lemma relates the executions of Algorithms 1 and 5, where the two processes in Algorithm 1 are denoted q_0 and q_1 .

LEMMA 6. *For every finite or infinite execution α of Algorithm 5, there is an execution α^* of Algorithm 1 such that:*

(1) *There is a function f from the set of operations of α onto the set of operations in α^* ; moreover, for each operation OP in side i , $f(OP)$ is an operation of q_i .*

Shared variables:

$M[0, 1][0, \dots, n-1] \leftarrow$ each entry of $M[j]$, $j \in \{0, 1\}$,
 initialized to $(pos = j, round = 0, tmst = 0)$
 $SIDE[0, \dots, n-1] \leftarrow$ each entry initialized to \perp
 $READS[0, 1][\infty] \leftarrow$ bi-dimensional array
 with infinitely many $(n-1)$ -process consensus objects
 $OPS[0, 1][\infty] \leftarrow$ bi-dimensional array
 with infinitely many $(n-1)$ -process consensus objects

Local variables:

$mySide \leftarrow$ initialized to 0 or 1
 $otherSide \leftarrow 1 - mySide$
 $me.\langle pos, round, tmst \rangle \leftarrow \langle mySide, 0, 0 \rangle$
 %% My side's position, round and timestamp
 $you.\langle pos, round, tmst \rangle \leftarrow \langle otherSide, 0, 0 \rangle$
 %% Other's position, round and timestamp
 $prev, range \leftarrow [mySide \pm 1]$
 $count_reads, count_ops \leftarrow 0$

Function output():

```

(01)  $SIDE[ID] \leftarrow mySide$ 
(02)  $s \leftarrow snapshot(SIDE)$ 
(03) if  $|\{ID : s[ID] = mySide\}| = n$  then
(04)   return  $otherSide$ 
(05) while true do
(06)    $me.round \leftarrow me.round + 1$ 
(07)    $me.tmst \leftarrow me.tmst + 1$ 
(08)    $M[mySide][ID] \leftarrow me$ 
(09)    $decided \leftarrow false$ 
(10)   while  $\neg decided$  do
(11)      $M[mySide][ID] \leftarrow me$ 
(12)      $t \leftarrow snapshot(M[otherSide])$ 
(13)      $max \leftarrow \max(\{tmst' : t[ID].tmst = tmst'\})$ 
(14)      $prop \leftarrow$  any  $t[ID]$  such that  $t[ID].tmst = max$ 
(15)      $count\_reads \leftarrow count\_reads + 1$ 
(16)      $you \leftarrow READS[mySide][count\_reads].decide(prop)$ 
(17)      $rounds \leftarrow me.round + you.round$ 
(18)      $range \leftarrow [me.pos \pm 1/2^{rounds}]$ 
(19)     if  $you.pos \in range \vee you.pos \notin prev$  then
(20)        $prev \leftarrow range$ 
(21)        $count\_ops \leftarrow count\_ops + 1$ 
(22)        $owner \leftarrow OPS[mySide][count\_ops].decide(ID)$ 
(23)       if  $owner = ID$  then
(24)         return  $me.pos$ 
(25)        $decided \leftarrow true$ 
(26)     elseif  $me.pos < you.pos$  then
(27)        $me.pos \leftarrow me.pos + 1/2^{rounds}$ 
(28)        $me.tmst \leftarrow me.tmst + 1$ 
(29)     else
(30)        $me.pos \leftarrow me.pos - 1/2^{rounds}$ 
(31)        $me.tmst \leftarrow me.tmst + 1$ 
(32)     endif
(33)   endwhile
(34) endwhile
endFunction

```

Figure 5: A lock-free n -process LLAA object algorithm.

(2) An operation OP is pending if and only if $f(OP)$ is pending; moreover, OP has infinitely many steps if and only if $f(OP)$ has infinitely many steps too.

(3) The outputs of a completed operation OP and its associated completed operation $f(OP)$ are the same.

(4) If OP precedes OP' in α then $f(OP)$ precedes $f(OP')$ in α^* .

PROOF SKETCH. By Lemma 5, every pair of processes performing the steps of the same simulated operation sOP , write the same sequence of values (possibly at distinct times, due to asynchrony). Thus, for each simulated operation sOP in α , we can map an operation $g(sOP)$ in an execution of α^* of Algorithm 1 by removing redundant steps from α (essentially, we keep the first ℓ th simulated step of sOP , for every ℓ). For any operation OP of process p_i in α , let $h(OP)$ be the simulated operation in α , the last step of OP by p_i belongs to. Then, if OP is completed, $h(OP)$ is the simulated operation p_i is simulating when it wins in the consensus at Line 22. Finally, we let $f(OP) = g(h(OP))$ \square

THEOREM 7. Algorithm 5 is a lock-free linearizable implementation of the n -LLAA object, using reads, writes and $(n-1)$ -process consensus objects.

PROOF. Lemma 6 implies that if there is an infinite execution α of Algorithm 5 in which only finitely many operations are completed, then there is an infinite execution α^* of Algorithm 1 in which only finitely many operations are completed, in contradiction to the lock-freedom of Algorithm 1. This implies that Algorithm 5 is lock-free as well.

Next, consider any finite execution α and let $\alpha\beta$ be an extension in which all operations in α are completed. Let λ be an execution of Algorithm 1 associated with $\alpha\beta$, as in Lemma 6. The output of a completed operation OP is equal to the output of its associated completed operation $f(OP)$, and if OP precedes OP' in $\alpha\beta$ then, by Lemma 6, $f(OP)$ precedes $f(OP')$ in λ . Thus, by replacing each OP with $f(OP)$, we can obtain a linearization $\text{lin}(\alpha\beta)$ of $\alpha\beta$ from any linearization $\text{lin}(\lambda)$ of λ , since the convergence requirement in both 2-LLAA and n -LLAA is a function of the number of all previous operations. Thus, Algorithm 5 is linearizable. \square

5 IMPOSSIBILITY OF WAIT-FREE n -LLAA USING $(n-1)$ -WINDOW REGISTERS

An execution β is an *extension* of a finite execution α if α is a prefix of β . A configuration C is *reachable* if there is a finite execution that ends with all processes and variables in the same states as in C . A configuration C' is *reachable* from a configuration C if there is a finite execution α that ends with C' and a prefix of the execution α ends with C . Two configurations C and C' are *indistinguishable* to process p , denoted $C \stackrel{p}{\sim} C'$, if the state of every shared variable and the state of p are the same in C and C' .

Fix an implementation \mathcal{A} of the n -LLAA object. Below, we consider only executions of \mathcal{A} in which process p_0 is on side 0, performing a single `output()` operation, and all other processes are on side 1.

Let α be a finite execution of \mathcal{A} ; α is *v -potent* if there is an extension α' of α in which the operation of p_0 is complete and returns v ; α is *v -univalent* (or just *univalent* when v is irrelevant)

if α is v -potent and for all $w \neq v$, α is not w -potent; α is *bivalent* if it is both v -potent and w -potent, for two values $v \neq w$.

DEFINITION 3. An execution α is *ambivalent* for process $q \neq p_0$ (or simply *ambivalent* when q is immaterial), if α is v -univalent and for some $w \neq v$, there is a w -univalent execution α' that is indistinguishable from α to process q . If α ends with a configuration C , we say that C is ambivalent for process q .

DEFINITION 4. An execution α , ending with configuration C , has a *critical q -extension* (or simply *critical extension* when q is immaterial) for a process $q \neq p_0$, if there is a process $q' \notin \{p_0, q\}$ such that, for all $\ell \geq 0$, $p_0(q^\ell(C))$ is v -univalent and $p_0(q'(q^\ell(C)))$ is w -univalent with $v \neq w$.

DEFINITION 5. An execution α , ending with configuration C , is (v, q, v') -critical (or simply *critical* if v, q and v' are immaterial), if $p_0(C)$ is v -univalent, there is a process $q \neq p_0$ such that $p_0(q(C))$ is v' -univalent, for $v' \neq v$, and for all finite extensions $\alpha\beta$ of α ending in configuration C' , $p_0(C')$ is univalent.

LEMMA 8. If α is a (v, p_1, v') -critical execution of \mathcal{A} ending in configuration C , then either some extension of α is ambivalent or some extension of α has a critical extension.

PROOF. The proof considers the next steps that p_0 and p_1 are about to take in C . If both p_0 and p_1 access different window registers, or p_1 's step is a read, then $p_1(p_0(C))$ and $p_0(p_1(C))$ are indistinguishable to p_0 . Hence, a solo execution of p_0 after both configurations returns the same value, contradicting the fact that α is (v, p_1, v') -critical, that is, $p_1(p_0(C))$ is v -univalent and $p_0(p_1(C))$ is v' -univalent.

If p_0 's next step is a read of a window register, then $p_1(p_0(C))$ and $p_0(p_1(C))$ are indistinguishable to all processes other than p_0 , $p_1(p_0(C))$ is v -valent and $p_0(p_1(C))$ is v' -valent, so $p_1(p_0(C))$ and $p_0(p_1(C))$ are both ambivalent for all processes other than p_0 .

The remaining case is when both steps are writes to the same window register R . If $n = 2$, then R is an ordinary register, so $p_0(p_1(C))$ and $p_0(C)$ are indistinguishable to p_0 . Since α , ending with C , is (v, p_1, v') -critical, $p_0(C)$ is v -valent and $p_0(p_1(C))$ is v' -valent, which is a contradiction.

For $n > 2$, we can prove the following claim:

CLAIM 9. Let $\alpha\beta$ be a finite extension of α ending in configuration C' , such that p_0 and p_1 do not take steps in β and no process writes to R in β . Then $p_0(C')$ is v -univalent and $p_0(p_1(C'))$ is v' -univalent.

PROOF. The proof is by induction on the length of β , with a trivial base case when β is empty. Suppose the lemma holds for when the length of β is ℓ , and let us consider an extension $\alpha\beta\gamma$ of α , where γ is a single step of some process $q \notin \{p_0, p_1\}$. Let C' be the configuration at the end of $\alpha\beta$, and C'' be the configuration at the end of $\alpha\beta\gamma$. By the induction hypothesis, $p_0(C')$ is v -univalent and $p_0(q(C'))$ is v' -univalent. Note that no process writes to R in $\beta\gamma$.

If q applies a primitive to a window register different than R , then $p_0(C'') = p_0(q(C')) = q(p_0(C'))$ is v -univalent and $p_0(p_1(C'')) = p_0(p_1(q(C'))) = q(p_0(p_1(C')))$ is v' -univalent.

Otherwise, since no process writes to R in $\beta\gamma$, q reads R . Then $p_0(C'') = p_0(q(C'))$ and $p_0(C')$ are indistinguishable to p_0 . As

$p_0(C')$ is v -univalent, $p_0(C'')$ is v -potent, since the solo executions by p_0 starting from C'' and from C' are the same, hence both must return v . Moreover, since α is critical, $p_0(C'')$ is univalent. Therefore, $p_0(C'')$ is v -univalent. Similarly, $p_0(p_1(C'')) = p_0(p_1(q(C')))$ and $p_0(p_1(C'))$ are indistinguishable to p_0 and $p_0(p_1(C'))$ is v' -univalent by the induction hypothesis. Hence, $p_0(p_1(C''))$ is v' -univalent. \square

Assume first that we can let each process p_2, \dots, p_{n-1} , in turn, take steps until it is about to write to R , reaching a configuration D in which the next step of every process is a write to R . By Claim 9, $p_0(D)$ is v -valent and $p_0(p_1(D))$ is v' -valent. However, since R is an $(n-1)$ -window register, $p_{n-1}(\dots(p_2(p_0(D))))$ and $p_{n-1}(\dots(p_2(p_0(p_1(D))))$ are indistinguishable to all processes except p_1 , contradicting the fact that $p_{n-1}(\dots(p_2(p_0(D))))$ is v -valent, whereas $p_{n-1}(\dots(p_2(p_0(p_1(D))))$ is v' -valent.

Otherwise, for some process $q \neq p_1$, there is an infinite extension $\alpha\beta\gamma$ of α such that q runs solo in γ and no process writes to R in $\beta\gamma$. Let C' be the configuration at the end of $\alpha\beta$. By Claim 9, for all $\ell \geq 0$, $p_0(q^\ell(C'))$ is v -univalent and $p_0(p_1(q^\ell(C')))$ is v' -univalent, i.e., γ is a critical q -extension of $\alpha\beta$. \square

We next use a standard valency argument to prove that if \mathcal{A} is wait-free, then it has a critical execution.

LEMMA 10. If \mathcal{A} is a wait-free linearizable implementation of the n -LLAA object, then it has a critical execution.

PROOF. Since \mathcal{A} is linearizable, in a solo execution by p_0 starting from C_0 and completing one output operation, the current position of side 0 must be at least $\frac{1}{2}$. On the other hand, in the execution from C_0 in which p_1 first executes solo two complete output operations and then p_0 runs solo and executes its output operation to completion, the current position of side 0 can be at most $\frac{3}{8}$. It follows that the empty execution (from C_0) is bivalent.

Starting with the empty execution as α , we repeatedly extend α to a bivalent execution $\alpha\alpha''$, such that p_0 executes a step in α'' (α'' may include steps by other processes). Since \mathcal{A} is wait-free, eventually we reach a bivalent execution $\alpha\alpha'$ ending with configuration C' such that for any extension $\alpha\alpha'\beta$ of $\alpha\alpha'$ ending with configuration C'' , $p_0(C'')$ is univalent. In particular, this holds for the empty β .

It follows that $p_0(C')$ is univalent, so it is v -valent for some v . Since $\alpha\alpha'$ is bivalent, it has an extension $\alpha\alpha'\beta_1$, ending with a configuration C_1 , such that $p_0(C_1)$ is v' -valent, for some $v' \neq v$. Let β_2 be the longest prefix of β_1 such that $\alpha\alpha'\beta_2$ ends with a configuration C_2 for which $p_0(C_2)$ is v -valent. Let q be the process that takes the next step in $\alpha\alpha'\beta_1$ after the execution of $\alpha\alpha'\beta_2$. From our construction, $p_0(C_2)$ is v -valent, $p_0(q(C_2))$ is v' -valent for $v' \neq v$, and for any finite extension $\alpha\alpha'\beta_3$ of $\alpha\alpha'\beta_2$, ending with a configuration C_3 , $p_0(C_3)$ is univalent. It follows from Definition 5 that $\alpha\alpha'\beta_2$ is (v, q, v') -critical. \square

THEOREM 11. There is no wait-free linearizable implementation of the n -LLAA object, which uses only $(n-1)$ -window registers. (Note that this includes read and write registers.)

PROOF. Assume, by way of contradiction, that \mathcal{A} is a wait-free linearizable implementation of the n -LLAA object. By Lemma 10, \mathcal{A} has a critical execution α . By Lemma 8, we have two cases:

Case 1: Some extension α' of α , ending with a configuration C' , is ambivalent for a process $q \neq p_0$. By Definition 3, α' is v -valent for some v , and there is a $w \neq v$ and an execution α_w , ending with a configuration C_w , such that α_w is w -univalent and configurations C' and C_w are indistinguishable to q .

Pick an integer k such that $\frac{1}{2^k} < |v-w|$. Let $\alpha'\beta_v^q$ be an extension of α' such that, in β_v^q , q runs solo and executes to completion $k+3$ output operations. Let x be the value returned by the last operation of q . Let $\alpha'\beta_v^q\beta_w^{p_0}$ be an extension of $\alpha'\beta_v^q$ such that, in $\beta_w^{p_0}$, p_0 runs solo until it completes its output operation. Since α' is v -univalent, p_0 's operation returns v .

Consider a linearization of $\alpha'\beta_v^q\beta_w^{p_0}$. It contains at least $k+3$ operations of processes in side 1, including all the ones by q , and one operation by p_0 . Suppose first that the last operation of q is linearized after the operation of p_0 . By Definition 2, $|x-v| \leq \frac{1}{2^{k+4}} < \frac{1}{2^{k+1}}$.

Suppose now that the last operation of q is linearized before the operation of p_0 . Let k' denote the number of operations in side 1 linearized between the last operation of q and the operation of p_0 and let y be the value returned by the operation linearized right before the operation of p_0 . By Definition 2,

$$\begin{aligned} |x-v| &\leq |x-0| + |0-y| + |y-v| \\ &\leq 2^{-(k+3)} + 2^{-(k+3+k')} + 2^{-(k+3+k'+1)} \\ &< 2^{-(k+1)}. \end{aligned}$$

Let $\alpha_w\beta_w^q$ be an extension of α_w such that, in β_w^q , q runs solo after α_w and executes to completion $k+3$ output operations. Since α' and α_w are indistinguishable to q , $\beta_w^q = \beta_v^q$ and so q 's $(k+3)$ 'th output operation in β_w^q also returns x . Now consider the extension $\alpha_w\beta_w^q\beta_w^{p_0}$ such that, in $\beta_w^{p_0}$, p_0 runs solo until it completes its output operation. Since α_w is w -univalent, p_0 's output operation returns w . Consider a linearization of $\alpha_w\beta_w^q\beta_w^{p_0}$. Using a similar reasoning as above, we can show that $|w-x| < 2^{-(k+1)}$. Therefore,

$$|w-v| \leq |w-x| + |x-v| < \frac{2}{2^{k+1}} = \frac{1}{2^k},$$

which is a contradiction.

Case 2: Some extension α' of α , ending with configuration C' , has a critical q -extension. That is, there are two processes $q \neq q'$, different from p_0 , and two values $v \neq w$, such that for all $\ell \geq 0$, $p_0(q^\ell(C'))$ is v -univalent and $p_0(q'(q^\ell(C')))$ is w -univalent.

The rest of the reasoning is similar to the previous case. Pick an integer k such that $\frac{1}{2^k} < |v-w|$. Let $\alpha'\beta$ be an extension of α' ending in a configuration C'' , such that in β , q runs solo and executes $k+3$ output operations to completion. Let x be the value returned by q 's last operation.

Let $\alpha'\beta\gamma_v$ and $\alpha'\beta\gamma_w$ be two extensions of $\alpha'\beta$ such that, in γ_v , p_0 runs solo until it completes its output operation, returning v , since $p_0(C'')$ is v -univalent; and in γ_w , q' executes a single step and then p_0 runs solo until it completes its output operation, returning w , since $p_0(q'((C'')))$ is w -univalent. Considering a linearization of $\alpha'\beta\gamma_v$ and a linearization of $\alpha'\beta\gamma_w$, it can be shown that

$$|v-w| \leq |v-x| + |w-x| < \frac{2}{2^{k+1}} = \frac{1}{2^k},$$

which is a contradiction. \square

6 SUMMARY AND FUTURE WORK

This paper shows a computational separation between lock-freedom and wait-freedom, two dominant progress conditions of concurrent algorithms: while there is no wait-free implementation of the n -LLAA object from k -window registers, with $k < n$ (whose consensus number is exactly k), there is a lock-free implementation of the n -LLAA object using read/write primitives and $(n-1)$ -process consensus objects. To the best of our knowledge, this is the first time such a separation is shown.

An interesting open question is whether there exists an n -LLAA implementation from x -process consensus objects for $x < n-1$, e.g., reads and writes, which would strengthen the separation. Noting that the n -LLAA object is non-deterministic, a second avenue for future research is to investigate whether or not the separation that we have shown holds also for *deterministic* long-lived objects. Finally, the n -LLAA specification assumes that the object “knows” the ID of the process invoking an operation on it. It would be interesting to see if the separation holds also for objects that do not assume this.

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A ADDITIONAL PROOFS FOR SECTION 3

We present additional details for the correctness analysis of Algorithm 1; some technical proofs are omitted for lack of space.

For an execution α , we use the following notation and terminology. If α is finite, v_i^α denotes the value of local variable v of a process p_i at the end of α , while M^α denotes the value of the shared array M at the end of α . I_i^α denotes the interval $[M^\alpha[i].pos \pm 1/2^{me_i^\alpha.round+you_i^\alpha.round}]$. If α is finite and ends with a read step of p_i that makes its current operation decided, then $rad(prev_i^\alpha)$ is the radius of the interval $prev_i^\alpha$ with center $me_i^\alpha.pos$.

The following lemma can be easily proved by induction on the length of the execution α .

LEMMA 12. In every finite execution α , for $i \in \{0, 1\}$, $M^\alpha[i].pos = y/2me_i^\alpha.round + you_i^\alpha.round$, for some integer $y \geq 0$.

LEMMA 13. Let $\lambda = \alpha e_1 e_2 \dots e_\ell$ be a finite execution. Then, $I_i^\lambda \subseteq prev_i^\alpha$.

PROOF. The proof of the lemma is based on the following two claims.

CLAIM 14. For every finite execution α , $I_i^\alpha \subseteq prev_i^\alpha$.

PROOF SKETCH. By induction on the length of the prefixes of α , one can prove that $me_i^\alpha.pos \in prev_i^\alpha$ and the distance between $me_i^\alpha.pos$ and any extreme of $prev_i^\alpha$ is at least $1/2me_i^\alpha.round + you_i^\alpha.round$. This invariant proves the lemma when $M^\alpha[i].pos = me_i^\alpha.pos$, because then $I_i^\alpha \subseteq prev_i^\alpha$ by definition. Otherwise, it must be that the last step of p_i in α is a read step in line 05 that modifies its position $me_i.pos$ after executing lines 11–15, in which case we consider the execution α' obtained from α by removing the last step of p_i and proceed as in the first case. \square

CLAIM 15. For every finite execution αe , $prev_i^{\alpha e} \subseteq prev_i^\alpha$.

PROOF. We prove the claim by induction on the length of the prefixes of αe . Note that the first step f in αe must be a write in line 02, which does not change the value of any local variable, and hence $prev_i^f = prev_i^\epsilon$, where ϵ is the empty execution. This is the base of the induction.

Suppose that $prev_i^{\beta f} \subseteq prev_i^\beta$, for a prefix βf of α . We show that $prev_i^{\beta f g} \subseteq prev_i^{\beta f}$, where g is the next step in αe . If g is not a step of p_i or is a write step of p_i (in line 02 or line 04), then such a step does not change the local variables of p_i , and $prev_i^{\beta f g} = prev_i^{\beta f}$.

Otherwise, g is a read step of p_i (Line 05) and the local variables of p_i might change when p_i executes lines 07–15. The claim clearly holds if p_i does not decide, since it executes line 07 and then lines 11–15; therefore, $prev_i$ is not modified, namely, $prev_i^{\beta f g} = prev_i^{\beta f}$.

If p_i decides, that is, the condition of line 08 is satisfied, then note that p_i does not modify its position after the read step, hence $me_i^{\beta f g}.pos = me_i^{\beta f}.pos = M^{\beta f}[i].pos$. By Claim 14, $I_i^{\beta f} \subseteq prev_i^{\beta f}$ holds. Also, following the execution of line 07, $range_i^{\beta f g} = I_i^{\beta f g} \subseteq I_i^{\beta f}$ holds (note that the containment may be proper, because $you_i^{\beta f g}.round \geq you_i^{\beta f}.round$) and $prev_i^{\beta f g} = range_i^{\beta f g}$ (line 09), and thus $prev_i^{\beta f g} \subseteq prev_i^{\beta f}$. \square

By Claim 14, $I_i^\lambda \subseteq prev_i^\lambda$. By repeatedly applying Claim 15, $prev_i^{\alpha e_1 e_2 \dots e_\ell} \subseteq prev_i^{\alpha e_1 e_2 \dots e_{\ell-1}} \subseteq \dots \subseteq prev_i^\alpha$. Therefore, $I_i^{\alpha e_1 e_2 \dots e_\ell} \subseteq prev_i^\alpha$ and the lemma follows. \square

LEMMA 16. Let α be a finite execution. Then, for each $i \in \{0, 1\}$ and $j = 1 - i$, either $M^\alpha[i].pos \in prev_j^\alpha$ or $M^\alpha[j].pos \in prev_i^\alpha$.

PROOF SKETCH. By strong induction on the length of the executions, we can show that either $me_i^\alpha.pos \in prev_j^\alpha$ or $me_j^\alpha.pos \in prev_i^\alpha$. This proves the lemma because if neither $M^\alpha[i].pos \in prev_j^\alpha$ nor $M^\alpha[j].pos \in prev_i^\alpha$, then we consider the execution α' obtained from α by removing, if necessary, the last read step of each process so that $M^{\alpha'}[i].pos = me_i^{\alpha'}.pos$ and $M^{\alpha'}[j].pos = me_j^{\alpha'}.pos$, hence reaching a contradiction. \square

Lemma 1 (repeated) Let α be a finite execution whose last step is a read (in line 05) of an operation P_i^k of p_i that makes that operation decided (by any of the two rules) on some undecided operation P_j^m of p_j , with $j = 1 - i$. Let $\alpha\beta$ be any extension of α in which P_j^m is decided. Then, P_j^m decides by the closeness rule in $\alpha\beta$ and $dist(me_i^k.pos, me_j^m.pos) \leq 1/2^{m+k}$.

PROOF. First, we show that if $M^\alpha[i].pos \in prev_j^\alpha$, then P_j^m must decide by the closeness rule in $\alpha\beta$. Note that $me_i^\alpha.pos = M^\alpha[i].pos$. Assume otherwise towards a contradiction, then P_j^m decides by the invalid position rule, hence at some point in the execution p_j evaluates $you_j.pos \notin prev_j$ as true in line 08. Thus, there must exist a prefix λ of $\alpha\beta$ such that p_i 's last read in λ (in line 05) results in it executing lines 11–15 and writing to $me_i^\lambda.pos$ a value such that $me_i^\lambda.pos \notin prev_j^\lambda$ (with α a prefix of λ and $prev_j^\alpha = prev_j^\lambda$). Observe that p_i changes $me_i^\lambda.pos$ in order to get closer to p_j 's position, hence $you_i^\lambda.pos = M^\lambda[j].pos \notin prev_j^\lambda$, implying that $I_j^\lambda \not\subseteq prev_j^\lambda$, contradicting Lemma 13. Thus, P_j^m cannot decide by the invalid position rule.

We prove that $M^\alpha[i].pos \in prev_j^\alpha$, by considering two possibilities. (1) P_i^k decides by the closeness rule. Note that $me_i^\alpha.pos = M^\alpha[i].pos$ and $you_i^\alpha.pos = M^\alpha[j].pos$. The fact that P_i^k decides by the closeness rule implies that at the end of α , when p_i executes line 08, it sees that $M^\alpha[j].pos = you_i^\alpha.pos \in range_i^\alpha$. Since the round counter of each process increases monotonically, we have that $rad(range_i^\alpha) \leq rad(I_j^\alpha)$, hence $M^\alpha[i].pos = me_i^\alpha.pos \in I_j^\alpha$. By Lemma 13, $I_j^\alpha \subseteq prev_j^\alpha$, from which follows that $M^\alpha[i].pos \in prev_j^\alpha$. (2) P_i^k decides by the invalid position rule. Hence, at the end of α , when p_i executes Line 08, it sees that $you_i^\alpha.pos = M^\alpha[j].pos \notin prev_j^\alpha$. Lemma 16 directly implies that $M^\alpha[i].pos \in prev_j^\alpha$. In both cases, $M^\alpha[i].pos \in prev_j^\alpha$ and hence P_j^m decides by the closeness rule.

Now, since the last step in α is a read by p_i in line 05 that makes P_i^k decided, p_i executes lines 07–10 at the end of α , and so we have that $me_i^\alpha.pos$ is the center of $prev_i^\alpha$, $rad(prev_i^\alpha) = 1/2^{m+k}$, $me_i^k.pos = me_i^\alpha.pos$ and $prev_i^k = prev_i^\alpha$. We prove that $me_j^m.pos \in prev_i^\alpha$, which implies that $dist(me_i^k.pos, me_j^m.pos) \leq 1/2^{m+k}$.

Let λ be the shortest prefix of $\alpha\beta$ in which P_j^m is decided (hence α is a prefix of λ). The last step of λ is a read step by p_j in line 05 that makes P_j^m decided. Let $x = you_j^\lambda.round$. Thus, P_j^m decides (by the closeness rule) on the operation P_i^x , for $x \geq k$. At the end of λ , after p_j executes lines 07–10, we have that $me_j^m.pos = me_j^\lambda.pos$.

By Lemma 13, $I_i^\lambda \subseteq prev_i^\alpha$. Since P_j^m decides by the closeness rule, at the end of λ the condition of line 08 is satisfied when p_j checks it, and thus $dist(me_j^\lambda.pos, you_j^\lambda.pos) \leq 1/2^{m+x}$. Notice that $you_j^\lambda.pos = M^\lambda[i].pos$, and hence $me_j^m.pos \in I_i^\lambda \subseteq prev_i^\alpha$, which concludes the proof. \square