Assignment 1 in Geometric Algorithms

Submission instructions:
Deadline: December 11, to be submitted in class or in my mailbox 64 (building 37, floor 1).
Submissions must be hardcopy, individual and clearly typed either in Hebrew or in English. You are allowed to borrow solution ideas from fellow students or other external sources. However, the solutions must be self-contained and the external sources should be clearly mentioned in the submission.

1. Show that the Davenport-Schinzel function \( \lambda_s(\cdot) \) is 'convex' for all \( s \geq 1 \):
   - \( \lambda_s(n_1) + \lambda_s(n_2) \leq \lambda_s(n_1 + n_2) \) for all positive integers \( n_1, n_2 \)
   - \( a\lambda_s(n) \leq \lambda_s(an) \) for all positive integers \( a \) and \( n \)

2. Show that any \( DS(n, s) \) sequence, for integers \( n, s > 0 \), can be realized as a lower envelope of a family of \( n \) pseudo-algebraic functions of order \( s \). (Namely for every Davenport-Schinzel sequence \( \pi \) of order \( s \) over \([n]\) there is a family of \( n \) pseudo-algebraic functions of order \( s \) so that their lower envelope is described by \( \pi \).)

3. A circular sequence over \([n]\) is a circular Davenport-Schinzel sequence of order 2 if it contains no circularly consecutive appearances of the same symbol \( a \in [n] \), and it does not contain an alternating circular subsequence \( abab \). (Due to circularity, each such forbidden circular sub-sequence \( abab \) is also a sub-sequence \( baba \).)
   - Show that any circular Davenport Schinzel Sequence of order \( s = 2 \) over \([n]\) has length at most \( 2n - 2 \)
   - Show a circular Davenport Schinzel sequence of order \( s = 2 \), over \([n]\), whose length is exactly \( 2n - 2 \)

4. Recall that a geometric object is called \( x \)-monotone if its intersection with any vertical line is a connected interval (which can be unbounded).

   Let \( \Gamma \) be a family of \( n \) \( x \)-monotone pseudo-algebraic curves of order \( s \). (That is, each of these curves is a graph of some function over \( \mathbb{R} \).)
   - Show that every cell in the arrangement of \( \Gamma \) is \( x \)-monotone.
   - Consider a partition \( \Gamma = \Gamma_1 \sqcup \Gamma_2 \). Let \( \Sigma(\Gamma_1, \Gamma_2) \) be the collection of all the two dimensional cells in the arrangement of \( \Gamma \) that lie below each curve of \( \Gamma_1 \) and above each curve of \( \Gamma_2 \).
     Show that the cells of \( \Sigma(\Gamma_1, \Gamma_2) \) have a total of at most \( O(\lambda_s(n)) \) vertices on their boundaries.

5. **Bonus!** Let \( A \) and \( B \) be families of closed curves (with \( |A| = a \) and \( |B| = b \)) so that no pair of curves from the same family intersect. Consider a (2-dimensional) cell \( \Delta \) in the arrangement of \( A \cup B \). Describe a boundary component of \( \Delta \) by a circular sequence in which each of the boundary edges is labeled with the curve of \( A \cup B \) that contains it.

   Show that the circular sub-sequence of all the \( A \)-symbols, in which we 'merge' all the consecutive appearances of the same symbol to one, is a circular Davenport Schinzel sequence of order 2.