Finding Largest Rectangles in Convex Polygons*

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Abstract

We consider the following geometric optimization problem: find a maximum-area rectangle and a maximum-perimeter rectangle contained in a given convex polygon with \( n \) vertices. We give exact algorithms that solve these problems in time \( O(n^3) \). We also give \( (1 - \varepsilon) \)-approximation algorithms that take time \( O(\varepsilon^{-3/2} + \varepsilon^{-1/2} \log n) \) for maximizing the area and \( O(\varepsilon^{-3} + \varepsilon^{-1} \log n) \) for maximizing the perimeter.

Keywords: geometric optimization; approximation algorithm; convex polygons; inscribed rectangles.

1 Introduction

Computing a largest rectangle contained in a polygon (with respect to some appropriate measure) is a well studied problem. Previous results include computing largest axis-aligned rectangles, either in convex polygons [2] or simple polygons (possibly with holes) [5], and computing largest fat rectangles in simple polygons [6].

Here we study the problem of finding a maximum-area rectangle and a maximum-perimeter rectangle contained in a given convex polygon with \( n \) vertices. We give exact \( O(n^3) \)-time algorithms and \( (1 - \varepsilon) \)-approximation algorithms that take time \( O(\varepsilon^{-3/2} + \varepsilon^{-1/2} \log n) \) for maximizing the area and \( O(\varepsilon^{-3} + \varepsilon^{-1} \log n) \) for maximizing the perimeter. (For maximizing the perimeter we allow the degenerate solution consisting of a single segment whose perimeter is twice its length.) To the best of our knowledge, apart from a straightforward \( O(n^4) \)-time algorithm, there is no other exact algorithm known so far.

Our approximation algorithm to maximize the area improves the previous results by Knauer et al. [7], where they give a deterministic \( (1 - \varepsilon) \)-approximation algorithm with running time \( O(\varepsilon^{-2} \log n) \) and a Monte Carlo \( (1 - \varepsilon) \)-approximation algorithm with running time \( O(\varepsilon^{-1} \log n) \). We are not aware of previous \( (1 - \varepsilon) \)-approximation algorithms to maximize the perimeter.

Notation. We use \( C \) for arbitrary convex bodies and \( P \) for convex polygons.

2 Preliminaries

Let \( U \) be the set of unit vectors in the plane. For each \( u \in U \) and each convex body \( C \), the directional width of \( C \) in direction \( u \), denoted by \( \text{dwidth}(u, C) \), is the length of the orthogonal projection of \( C \) onto any line parallel to \( u \). Thus

\[
\text{dwidth}(u, C) = \max_{p \in C} (p, u) - \min_{p \in C} (p, u),
\]

where \( \langle \cdot, \cdot \rangle \) denotes the scalar product.

For a convex body \( C \) and a parameter \( \varepsilon \in (0, 1) \), an \( \varepsilon \)-kernel for \( C \) is a convex body \( C_\varepsilon \subseteq C \) such that

\[
\forall u \in U : (1 - \varepsilon) \cdot \text{dwidth}(u, C) \leq \text{dwidth}(u, C_\varepsilon).
\]

The diameter of \( C \) is the distance between the two furthest points of \( C \). It is easy to see that it equals

\[
\max_{u \in U} \text{dwidth}(u, C).
\]

Ahn et al. [1] show how to compute an \( \varepsilon \)-kernel. Their algorithm uses the following type of primitive operations for \( C \):

- given a direction \( u \in U \), find an extremal point of \( C \) in the direction \( u \);
- given a line \( \ell \), find \( C \cap \ell \).

Let \( T_C \) be the time needed to perform each of those primitive operations. We will use \( T_C \) as a parameter in some of our running times. When \( C \) is a convex \( n \)-gon whose boundary is given as a sorted array of vertices or as a binary search tree, we have \( T_C = O(\log n) \) [4, 10]. Ahn et al. show the following result.

Lemma 1 (Ahn et al. [1]) Given a convex body \( C \) and a parameter \( \varepsilon \in (0, 1) \), we can compute in \( O(\varepsilon^{-1/2} T_C) \) time an \( \varepsilon \)-kernel of \( C \) with \( O(\varepsilon^{-1/2}) \) vertices.

Lemma 2 Let \( C_\varepsilon \) be an \( \varepsilon \)-kernel for \( C \). If \( \varphi \) is an invertible affine mapping, then \( \varphi(C_\varepsilon) \) is an \( \varepsilon \)-kernel for \( \varphi(C) \).

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Proof. The ratio of directional widths for convex bodies is invariant under invertible affine transformations. This means that
\[
\forall u \in U : \quad 1 - \varepsilon \leq \frac{\text{dwidth}(u, C_\varepsilon)}{\text{dwidth}(u, C)} = \frac{\text{dwidth}(u, \varphi(C))}{\text{dwidth}(u, \varphi(C))}
\]
and thus \(\varphi(C_\varepsilon)\) is an \(\varepsilon\)-kernel for \(\varphi(C)\). \(\square\)

Lemma 3 Assume that \(C\) contains the square \([-1, 1]^2\), that \(C\) has diameter \(d\), and that \(C_\varepsilon\) is an \(\varepsilon\)-kernel for \(C\). Then \(C_\varepsilon\) contains the axis-parallel square \(S = [-1 + d\varepsilon, 1 - d\varepsilon]^2\).

Proof. Assume, for the sake of contradiction, that \(S\) is not contained in \(C_\varepsilon\). This means that one vertex of \(S\) is not contained in \(C_\varepsilon\). Because of symmetry, we can assume that \(s = (1 - d\varepsilon, 1 - d\varepsilon)\) is not contained in \(C_\varepsilon\). Since \(C_\varepsilon\) is convex and \(s \notin C_\varepsilon\), there exists a closed halfplane \(h\) that contains \(C_\varepsilon\) but does not contain \(s\). Let \(\ell\) be the boundary of \(h\).

We next argue that \([-1, 1]^2\) has some vertex at distance at least \(d\varepsilon\) from \(h\) (and thus \(\ell\)): see Figure 1 for a couple of cases. If \(\ell\) has negative slope and \(h\) is its lower halfplane, then the distance from \((1, 1)\) to \(\ell\) is at least \(d\varepsilon\). If \(\ell\) has positive slope, then \((-1, 1)\) or \((1, -1)\) are at distance at least \(d\varepsilon\) from \(h\).

Since \([-1, 1]^2 \subseteq C\), for the direction \(u\) of \(\ell\) we have
\[
\text{dwidth}(u, C) - \text{dwidth}(u, C_\varepsilon) > d\varepsilon \geq \varepsilon \cdot \text{dwidth}(u, C),
\]
where we have used the assumption that \(\text{dwidth}(u, C) \leq d\). This means that
\[
(1 - \varepsilon) \cdot \text{dwidth}(u, C) > \text{dwidth}(u, C_\varepsilon),
\]
which contradicts that \(C_\varepsilon\) is an \(\varepsilon\)-kernel for \(C\). \(\square\)

3 Exact algorithms

Let \(e_1, \ldots, e_n\) be the edges of the convex polygon \(P\). For each edge \(e_i\) of \(P\), let \(h_i\) be the closed halfplane defined by the line supporting \(e_i\) that contains \(P\). Since \(P\) is convex, we have \(P = \bigcap_i h_i\).

We parameterize the set of parallelograms in the plane by points in \(\mathbb{R}^3\), as follows. We identify each 6-dimensional point \((x_1, x_2, u_1, u_2, v_1, v_2)\) with the triple \((x, u, v) \in (\mathbb{R}^2)^3\), where \(x = (x_1, x_2)\), \(u = (u_1, u_2)\), and \(v = (v_1, v_2)\). The triple \((x, u, v) \in \mathbb{R}^6\) corresponds to the parallelogram \(\Diamond(x, u, v)\) with vertices
\[
x, x + u, x + v, x + u + v.
\]

Thus, \(x\) describes a vertex of the parallelogram \(\Diamond(x, u, v)\), while \(u\) and \(v\) are vectors describing the edges of \(\Diamond(x, u, v)\). This correspondence is not bijective. Nevertheless, each parallelogram is \(\Diamond(x, u, v)\) for some \((x, u, v) \in \mathbb{R}^6\): the parallelogram given by the vertices \(p_1 p_2 p_3 p_4\) in clockwise (or counterclockwise) order is \(\Diamond(p_1, p_2 - p_1, p_4 - p_1)\).

We are interested in the parallelograms contained in \(P\). To this end we define
\[
\Pi(P) = \{(x, u, v) \in \mathbb{R}^6 \mid \Diamond(x, u, v) \subseteq P\}.
\]

Since \(P\) is convex, a parallelogram is contained in \(P\) if and only if each vertex of the parallelogram is in \(P\). Therefore \(\Pi(P)\) is
\[
\bigcap_i \{(x, u, v) \in \mathbb{R}^6 \mid x, x + u, x + v, x + u + v \in h_i\}.
\]

Since \(\Pi(P)\) is trivially bounded, it follows that \(\Pi(P)\) is a convex polytope in \(\mathbb{R}^6\) defined by \(4n\) linear constraints. The Upper Bound Theorem [9] implies that \(\Pi(P)\) has combinatorial complexity at most \(O(n^3)\). Furthermore, a triangulation of \(\Pi(P)\) can be computed in \(O(n^3)\) time. Chazelle’s algorithm [3] gives a triangulation of the boundary of \(\Pi(P)\), which can be easily extended to a triangulation of \(\Pi(P)\).

The set of rectangles is obtained by restricting our attention to triples \((x, u, v)\) with \(\langle u, v \rangle = 0\), where \(\langle \cdot, \cdot \rangle = 0\) denotes the scalar product of two vectors. This constraint is non-linear. Because of this, it is more convenient to treat each simplex of a triangulation of \(\Pi(P)\) separately. When \(\langle u, v \rangle = 0\), the area of \(\Diamond(x, u, v)\) is \(|u| \cdot |v|\).

Consider any simplex \(\triangle\) of the triangulation of \(\Pi(P)\). Finding the maximum area rectangle restricted to \(\triangle\) corresponds to the problem
\[
\text{opt}(\triangle) = \max_{\langle u, v \rangle = 0} \frac{|u|^2 \cdot |v|^2}{\text{area}(\triangle)}
\]
\[
\text{subject to } x, x + u, x + v, x + u + v \in h_i.
\]

This is a constant-size problem. It has 6 variables and a constant number of constraints; all constraints but one are linear. The optimization function has degree four. In any case, each such problem can be solved in constant time. When the problem is not feasible, we set \(\text{opt}(\triangle) = 0\).

Taking the best rectangle over all simplices of a triangulation of \(\Pi(P)\), we find a maximum area rectangle. Thus, we return \(\arg\max_{\triangle} \text{opt}(\triangle)\). We have shown the following.
Theorem 4 Let $P$ be a convex polygon with $n$ vertices. In time $O(n^2)$ we can find a maximum-area rectangle contained in $P$.

To maximize the perimeter, we apply the same approach. For each simplex $\triangle$ in a triangulation of $\Pi(P)$ we have to solve the following problem:

$$\text{opt}(\triangle) = \max |u| + |v| \quad \text{s.t.} \quad (x, u, v) \in \triangle, \quad \langle u, v \rangle = 0$$

Combining the solutions over all simplices of the triangulation we obtain the following.

Theorem 5 Let $P$ be a convex polygon with $n$ vertices. In time $O(n^2)$ we can find a maximum-perimeter rectangle contained in $P$.

4 Approximation Algorithm to Maximize the Area

The algorithm is very simple: we compute an $(\varepsilon/64)$-kernel $C_{\varepsilon/64}$ for the input convex body $C$, compute a maximum-area rectangle contained in $C_{\varepsilon/64}$ and return it. We next show that this algorithm indeed returns a $(1 - \varepsilon)$-approximation.

Let $R_{\text{opt}}$ be a maximum-area rectangle contained in $C$, and let $\varphi$ be an affine transformation such that $\varphi(R_{\text{opt}})$ is the square $[-1, 1]^2$.

Lemma 6 The diameter of $\varphi(C)$ is at most 32.

Proof. We will show that $\varphi(C)$ is contained in the disk centered at $(0,0)$ of radius 16, which implies the result.

Each convex body contains a rectangle with at least one half of its area; see [8]. Therefore $\frac{\text{area}(R_{\text{opt}})}{\text{area}(C)} \geq 1/2$.

Any invertible affine transformation does not change the ratio between areas of objects. Therefore

$$\frac{1}{2} \leq \frac{\text{area}(R_{\text{opt}})}{\text{area}(C)} = \frac{\text{area}(\varphi(R_{\text{opt}}))}{\text{area}(\varphi(C))} = \frac{4}{\text{area}(\varphi(C))}$$

and thus $\text{area}(\varphi(C)) \leq 8$.

Assume, for the sake of reaching a contradiction, that $\varphi(C)$ has a point $p = (p_x, p_y)$ at distance larger than 16 from $(0,0)$. Because of symmetry, we can assume that $p$ lies in the cone defined by $y \leq x$ and $y \geq -x$. Then the triangle defined by vertices $(-1, -1), (1, 1)$ and $p$ is contained in $\varphi(C)$ (by convexity of $\varphi(C)$) and has area

$$\frac{1}{2} \cdot (p_x + 1) \geq 16/\sqrt{2} > 8,$$

so we get a contradiction. It follows that $\varphi(C)$ is contained in a disk centered at $(0,0)$ of radius 16. □

Lemma 7 Let $C_\varepsilon$ be an $\varepsilon$-kernel for $C$. Then $C_\varepsilon$ contains a rectangle with area at least $(1 - 64\varepsilon) \cdot \text{area}(R_{\text{opt}})$.

Proof. Because of Lemma 2, $\varphi(C_\varepsilon)$ is an $\varepsilon$-kernel for $\varphi(C)$. Since $\varphi(C)$ contains $[-1, 1]^2$ and has diameter at most 32 due to Lemma 6, Lemma 3 implies that $\varphi(C_\varepsilon)$ contains the square $S = [-t, t]^2$, where $t = 1 - 32\varepsilon$.

Since $S$ is obtained by scaling $[-1, 1]^2 = \varphi(R_{\text{opt}})$ by $1 - 32\varepsilon$, its preimage $R = \varphi^{-1}(S)$ is obtained by scaling $R_{\text{opt}}$ by $1 - 32\varepsilon$ about its center. It follows that $R$ is a rectangle with area

$$\text{area}(R) = (1 - 32\varepsilon)^2 \cdot \text{area}(R_{\text{opt}}) \geq (1 - 64\varepsilon) \cdot \text{area}(R_{\text{opt}}),$$

and the lemma follows. □

Combining Lemma 7, the construction of an $(\varepsilon/64)$-kernel $C_{\varepsilon/64}$ of $C$ from Lemma 1, and the exact algorithm of Theorem 4 applied to $C_{\varepsilon/64}$, we obtain the following.

Theorem 8 Let $C$ be a convex body in the plane. For any given $\varepsilon \in (0, 1)$, we can find a $(1 - \varepsilon)$-approximation to the maximum-area rectangle contained in $C$ in time $O(\varepsilon^{-1/2}T_C + \varepsilon^{-3/2})$.

Corollary 9 Let $C$ be a convex polygon with $n$ vertices given as a sorted array or a balanced binary search tree. For any given $\varepsilon \in (0, 1)$, we can find a $(1 - \varepsilon)$-approximation to the maximum-area rectangle contained in $C$ in time $O(\varepsilon^{-1/2} \log n + \varepsilon^{-3/2})$.

Proof. In this case $T_C = O(\log n)$. □

5 Approximation Algorithm to Maximize the Perimeter

The approximation algorithm is the following: we compute an $(\varepsilon^2/8)$-kernel $C_{\varepsilon^2/8}$ for the input convex body $C$, compute a maximum-perimeter rectangle $R_{\varepsilon^2/8}$ contained in $C_{\varepsilon^2/8}$, and return it. We next show that this indeed computes a $(1 - \varepsilon)$-approximation.

Let $R_{\text{opt}}$ be a maximum-perimeter rectangle contained in $C$, and let $a \geq b$ be the side lengths of $R_{\text{opt}}$. We distinguish two cases depending on the aspect ratio $b/a \leq 1$ of $R_{\text{opt}}$. When $b/a \leq \varepsilon$, then the longest segment contained in $C$ is a good approximation to $R_{\text{opt}}$. When $b/a > \varepsilon$, then $R_{\text{opt}}$ is fat enough that we can use a method similar to the one employed for the area. We next proceed to the details.

Lemma 10 If $b/a \leq \varepsilon$, then $\text{peri}(R_{\varepsilon^2/8}) \geq (1 - \varepsilon) \cdot \text{peri}(R_{\text{opt}})$.
Proof. For the direction $u$ of the line supporting a long edge of $R_{\text{opt}}$, we have
\[
\text{peri}(R_{\epsilon^2/8}) \geq 2 \text{dwidth}(u, C_{\epsilon^2/8}) \geq 2(1 - \epsilon^2/8) \text{dwidth}(u, C) \geq 2(1 - \epsilon^2)a.
\]
Using $b \leq \epsilon a$, we thus have
\[
\frac{\text{peri}(R_{\epsilon^2/8})}{\text{peri}(R_{\text{opt}})} \geq \frac{2(1 - \epsilon^2)a}{2a + 2b} \geq \frac{2(1 - \epsilon^2)a}{2a + 2\epsilon a} = \frac{1 - \epsilon^2}{1 + \epsilon} = 1 - \epsilon,
\]
proving the lemma. \hfill \Box

We now consider the case when $b/a > \epsilon$. Let $\varphi$ be the affine transformation such that $\varphi(R_{\text{opt}})$ is the square $[-1,1]^2$.

Lemma 11 \textit{If $b/a > \epsilon$, then $\varphi(C)$ has diameter at most $8/\epsilon$.}

Proof. Let $D$ be the disk of radius $2a$ centered at the center of $R_{\text{opt}}$. We have $C \subset D$. Otherwise $C$ contains a segment of perimeter strictly larger than $4a \geq 2a + 2b = \text{peri}(R_{\text{opt}})$, contradicting the optimality of $R_{\text{opt}}$.

The transformation $\varphi$ has the following property: for a segment $s$ parallel to the sides of $R_{\text{opt}}$ of length $x \in [a,b]$ we have $\text{len}(\varphi(s)) = 2\text{len}(s)/x$. It follows that $\varphi(D)$ is an ellipse of aspect ratio $a/b \leq 1/\epsilon$ whose axes are parallel to the coordinate axes. The diameter of $D$ parallel to the sides of $R_{\text{opt}}$ of length $a$ becomes, under $\varphi$, the smaller axis of $\varphi(D)$ and has length 8. Therefore the larger axis of $\varphi(D)$ has length at most $8a/b \leq 8/\epsilon$. Since $\varphi(C) \subseteq \varphi(D)$, the result follows. \hfill \Box

Lemma 12 \textit{If $b/a > \epsilon$, then $\text{peri}(R_{\epsilon^2/8}) \geq (1 - \epsilon) \cdot \text{peri}(R_{\text{opt}})$.}

Proof. Because of Lemma 2, $\varphi(C_{\epsilon^2/8})$ is an $(\epsilon^2/8)$-kernel for $\varphi(C)$. Since $\varphi(C)$ contains $[-1,1]^2$ and has diameter at most $8/\epsilon$ due to Lemma 11, Lemma 3 implies that $\varphi(C_{\epsilon^2/8})$ contains the square $S = [-t,t]^2$, where
\[
t = 1 - \frac{8}{\epsilon} \cdot \frac{\epsilon^2}{8} = 1 - \epsilon.
\]
Since $S$ is obtained by scaling the square $[-1,1]^2$ by $1 - \epsilon$, the rectangle $R$ obtained by scaling $R_{\text{opt}}$ by $1 - \epsilon$ about its center is contained in $C_{\epsilon^2/8}$. The rectangle $R$ has sides $(1 - \epsilon)a$ and $(1 - \epsilon)b$, and so its perimeter is $2(1 - \epsilon)(a + b) = (1 - \epsilon)\text{peri}(R_{\text{opt}})$, and the result follows. \hfill \Box

Combining Lemmas 10 and 12, the construction of an $(\epsilon^2/8)$-kernel $C_{\epsilon^2/8}$ of $C$ from Lemma 1, and the exact algorithm of Theorem 5 applied to $C_{\epsilon^2/8}$, we obtain the following

Theorem 13 \textit{Let $C$ be a convex body in the plane. For any given $\epsilon \in (0,1)$, we can find a $(1 - \epsilon)$-approximation to the maximum-perimeter rectangle contained in $C$ in time $O(\epsilon^{-3}T_C + \epsilon^{-3})$.}

Corollary 14 \textit{Let $C$ be a convex polygon with $n$ vertices given as a sorted array or a balanced binary search tree. For any given $\epsilon \in (0,1)$, we can find a $(1 - \epsilon)$-approximation to the maximum-perimeter rectangle contained in $C$ in time $O(\epsilon^{-3} \log n + \epsilon^{-3})$.}

Proof. In this case $T_C = O(\log n)$. \hfill \Box

References