1a) The reduction was described in class.

1b) It cannot be used for $(s+1)$-coloring of unoriented trees. As a result of applying the reduction to an unoriented tree we get a graph which is not a tree. So an MIS algorithm for unoriented trees cannot be applied to this resulting graph.

2) The construction was given in class. In short, draw a graph from $G(n,\mu)$, $\mu = n^{\frac{1}{12}}$. With high probability, it has $N(n^{\frac{1}{12}})$ edges, and $o(n)$ cycles of length $\leq k$. Eliminate one edge from each cycle to get the desired
This construction implies that there exists a constant $c > 0$ so that a $(ck)$-spanner for general graphs requires $\Omega(n^{1+\frac{1}{k}})$ edges. The reason is that no edge can be removed from $G$ without increasing some distance by a factor of at least $\beta - 1$. So any $(k-1)$-spanner for $G$ requires $\Omega(n^{1+\frac{1}{(k-1)}})$ edges.

3) $(2+\varepsilon)n$-coloring can be computed in $O(n \cdot \log n)$ time.

We need to work out the dependence on $\varepsilon > 0$. During the construction of the $H$-decomposition, after removing each $H$-set we are left with at most $\frac{2}{2+\varepsilon}$ fraction of all vertices.
that were in the graph before this set was extracted.

Hence the number of H-sets in the decomposition is \( l \) s.t.

\[
\left( \frac{2}{2+\varepsilon} \right)^l \cdot n < 1 \quad \text{(the smallest } l \text{ that satisfies it)}.
\]

\[ l = O\left( \frac{\log n}{\varepsilon} \right). \]

Let \( \varepsilon = \frac{1}{2^{1/3}}. \) As a result we get a \( (20l + 1 + \varepsilon) \cdot (2 + \varepsilon) \cdot a^-2 \cdot O(a^{2/3}) \)-coloring in time \( O(a \cdot \frac{1}{\varepsilon} \cdot \log n) = O(a^{4/3} \cdot \log n). \)

4) Lin: The minim-bottleneck-spanning tree is the MST.

90:

Let \( T \) be the MST, and let \( T' \) be some another spanning tree for \( G. \) Let \( e \) be the heaviest edge in \( T. \) If \( e \) belongs to \( T' \) too, then the heaviest edge \( e' \) in \( T' \) satisfies...
\[ w(e') \geq w(e), \text{ as required.} \]

If \( e \) does not belong to \( T' \), then \( T' \cup \{e\} \) contains a cycle \( C \).

By the red rule, \( e \) is not the strictly heaviest edge in \( C \). So \( C \) contains another edge \( e' \) (from \( T' \)) with \( w(e') \geq w(e) \). Hence the heaviest edge \( e' \) in \( T' \) satisfies \( w(e') \geq w(e') = w(e) \), as required. \( \quad \text{QED} \)

Hence any of the algorithms that we studied for computing the MST does the job (and computes the min-bottleneck-spanning-tree).

By GHS alg \(^1\) it can be done in \( O(n\log n) \) time and \( O(EL + n\log n) \) communication.