Bounds on the performance of back-to-front airplane boarding policies

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Abstract

We provide bounds on the performance of back-to-front airplane boarding policies. In particular, we show that no back-to-front policy can be more than 20% better than the policy which boards passengers randomly.

1 Introduction

The process of airplane boarding is experienced daily by millions of passengers worldwide. Reductions in gate delays would yield significant economic benefits from more efficient use of aircraft and airport infrastructure and would also improve passenger experience. See Van Landeghem and Beuselinck, [7], Marelli *et al.*, [4] and Van den Briel *et al.*, [5, 6] for an extensive discussion.

Airplane boarding has been studied using detailed computer simulations by Van Landeghem and Beuselinck, [7], Marelli *et al.*, [4] , Van den Briel *et al.* [5, 6] and Ferrari and Nagel, [3]. Bachmat *et al.*, [1, 2], have introduced an analytical model which was shown to be in nearly complete agreement with the results of the aforementioned simulation studies.

Airlines have adopted a variety of boarding strategies in the hope of shortening the boarding process for airplanes. Many airlines practice back-to-front boarding policies, namely, the airline boards passengers from the back of the airplane first. These strategies are parametrized by the choice of which groups of rows are allowed to join the boarding queue at any given time. Several policies of this type have been studied both via simulations and analytically, [2-3,5-7], and the results showed that these policies provide no improvement, and may even be detrimental. In this letter we try to explain this phenomenon by proving bounds on the effectivness of back-to-front boarding policies in the setting of the analytical model of [1, 2]. The analytical methods have identified a congestion parameter k which plays a crucial role in assessing the effectivness of boarding policies. The parameter k depends on the design of the airplane, namely, on the distance between successive rows (leg room) and the number of passengers per row. We show that for an airplane whose design leads to a congestion factor $k \geq 1$ back-to-front policies can reduce boarding time in comparison to random boarding by at most a factor of

$$\frac{\sqrt{k-1}}{\sqrt{k}+\frac{1-\ln 2}{\sqrt{k}}} \ . \tag{1}$$

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As was argued in [2], in reality the congestion factor k is around 4. For this value of the congestion factor, our result shows that no back-to-front policy can improve upon random boarding by more than 20%. Moreover, since the expression (1) tends to 1 as k grows to infinity, our lower bound is, in fact, asymptotically optimal.

2 Modeling the airplane boarding process

In this section we explain how to estimate analytically the boarding time of a given back-to-front policy, using the mathematical model of [1]. We represent a back-to-front policy by a monotone decreasing sequence of numbers $\bar{r} = (r_0, r_1, \ldots, r_m), 1 = r_0 > r_1 > \ldots > r_{m-1} > r_m = 0$. The sequence $\bar{r} = (r_1, \ldots, r_{m-1})$ is referred to as a *partition of size* m. Assume that the airplane has n rows and n' passengers. We will assume that the airplane is full, and so $n = \Theta(n')$. The set of passengers who are seated between rows $r_{i-1} \cdot n$ and $r_i \cdot n$ is called the ith *group* of passengers. The back-to-front policy corresponding to the partition allows the passengers from the first group of rows to join the queue first, followed by passengers from the second group and so on.

We represent passengers by points (q, r) in the unit square $[0, 1]^2$. The row coordinate r represents the row of the passenger divided by n. The queue coordinate q represents the position of the passenger in the boarding queue divided by n'.

The boarding policy determines a joint density function $p(q, r)$, which describes the probability that a passenger sitting in row r will have queue position q. We note that in a back-to-front policy with parameters $\bar{r}$, passengers in the ith group occupy positions $(1 - r_i)n'$ to $(1 - r_{i-1})n'$ in the queue, therefore, the coordinates of passengers (q, r) in the ith group satisfy

$$r_{i-1} \geq r \geq r_i \tag{2}$$

and

$$1 - r_{i-1} \leq q \leq 1 - r_i. \tag{3}$$

We denote the square given by these inequalities by S_i. The set of squares S_i, $i = 1, \ldots, m$, contains the anti-diagonal segment given by $q + r = 1, 0 \leq q, r \leq 1$. For each $i = 1, 2, \ldots, m$, let B_i be the bottom edge of the square S_i. See Figure 1.

Since a passenger in a row r is equally likely to have any of the allowable queue positions, the probability density function p is defined by $p(q, r) = 1/(r_{i-1} - r_i)$ if $(q, r) \in S_i$, $i = 1, \ldots, m$, and $p(q, r) = 0$ otherwise (outside the squares S_i).

In addition to the density function p the model also uses a *congestion* parameter k. The congestion parameter is a certain function of the number of passengers per row, the average aisle length occupied by a single passenger, and the aisle distance between a pair of successive rows (the "leg-room"). Substituting realistic values of these parameters, one obtains a value of k roughly equal to four ($k = 4$) [2]. Given the probability density function $p = p_{\bar{r}}$ which is determined by the boarding policy at hand, and the congestion parameter k of the airplane, the model defines the boarding time of the policy as follows.

Set $\alpha(q, r) = \int_r^1 p(q, z) dz$. The boarding time $T(\bar{r}, k)$ is now given by the solution to the following variational problem. Consider the set Φ of all piecewise differentiable functions $\varphi(q)$ defined on an interval $[q', q'']$, $0 \leq q' < q'' \leq 1$, with values in the unit interval $[0, 1]$, and which satisfy

$$\varphi'(q) + k \cdot \alpha(q, \varphi(q)) \geq 0. \tag{4}$$

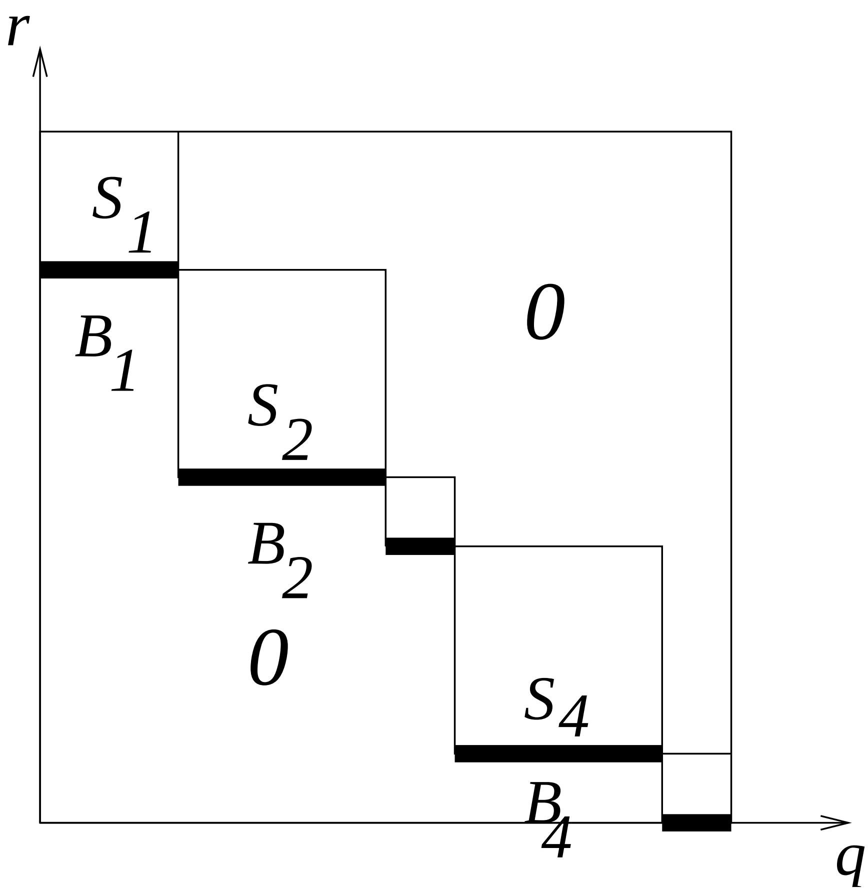

Figure 1: A graphic illustration of a partition into five boarding groups of different sizes. Each square corresponds to one group. The bottom edges $B_1, B_2, \ldots, B_5$ of the squares $S_1, S_2, \ldots, S_5$, respectively, are depicted by a solid thick line. This is a partition of size 5.

Let

$$T(\bar{r},k) = T(p_{\bar{r}},k) = \max_{\varphi\in\Phi}\ L(\varphi)\ , \tag{5}$$

where

$$L(\varphi) = \int_{q'}^{q''} \sqrt{p(q,\varphi(q))(\varphi'(q) + k\cdot\alpha(q,\varphi(q))))}dq. \tag{6}$$

This estimate T for the boarding time was validated in [2] against detailed trace driven simulations, particularly those of van Landeghem and Beuselink, [7]. We note that this variational problem has a natural interpretation in terms of spacetime (Lorentzian) geometry. $L(\varphi)$ is the length (proper-time) of the graph of φ with respect to the Lorentzian metric $ds^2 = dq(dq+k\cdot\alpha dr)$. The class of functions over which the maximum is taken consists of the time-like curves with respect to the metric, hence T is the proper time of the maximal curve in the model. See [1] for further details.

The value $T = T(p_{\bar{r}},k)$ is given by the maximum of the functional $L(\varphi)$ over a large class of functions $\varphi(q)$. Our strategy for obtaining lower bounds on T is to present a particular curve $\varphi\in\Phi$ with a large value $L(\varphi)$.

Let $b(q)$ denote the piecewise linear function defined by the union of the bottom edges of S_i, namely, $b(q) = r_i$ for $1-r_{i-1} \le q < 1-r_i$, $i = 1,2,\ldots,m$ (see Figure 1).

Lemma 2.1 *For all points (q,r), $0\le q\le 1$, $0\le r\le b(q)$, it holds that $\alpha(q,b(q)) = 1$.*

Proof: By definition, $\alpha(q,r) = \int_r^1 p(q,z)dz$. Let $i = i(q)$ be the index such that $1-r_{i-1}\le q\le 1-r_i$. Then by definition of the density function $p(q,r)$,

$$\int_r^1 p(q,z)dz \ = \ \int_{r_i}^{r_{i-1}} 1/(r_{i-1}-r_i)dz \ = \ 1\ .$$

∎

Definition 2.2 *Given a partition $\bar{r} = (1 = r_0, r_1, \ldots, r_{m-1}, r_m = 0)$, and an index $j\in\{1,2,\ldots,m\}$, we define a piecewise linear continuous function $\varphi_{(\bar{r},j)}(q) = \varphi_j(q)$ as follows. The variable q is in the range $[0,1-r_j]$. The graph of the function φ_j is composed of $h = h(j,\bar{r})$ linear segments, $\psi_1,\ldots,\psi_h$ where h is an integer, $1\le h\le 2\cdot j$.*

The segments are of two types. A segment ψ of the first type *is a horizontal segment (that is, a segment with slope 0), and it is necessarily a subsegment of some bottom edge B_i for an index i between 1 and j. Moreover, the segment ψ contains the left endpoint $(1-r_{i-1},r_i)$ of the segment B_i. Finally, the right-most segment ψ_h is of the first type and consists of the entire bottom edge B_j of the square S_j.*

A segment ψ of the second type *is a segment with slope $(-k)$ that ends in a point $(1-r_{i-1},r_i)$ for some index i, $1\le i\le j$. Moreover, for all values of q for which $\psi(q)$ is defined, the inequality $\psi(q)\le b(q)$ holds.*

Fix an index j, $1\le j\le m$. The curve $\varphi_j(q)$ is the unique piecewise linear continuous curve in which segments of the first and second types alternate. The sequence $(\psi_1,\psi_2,\ldots,\psi_h)$ is called the segment decomposition *of the curve φ_j. See Figures 2 and 3 for an illustration.*

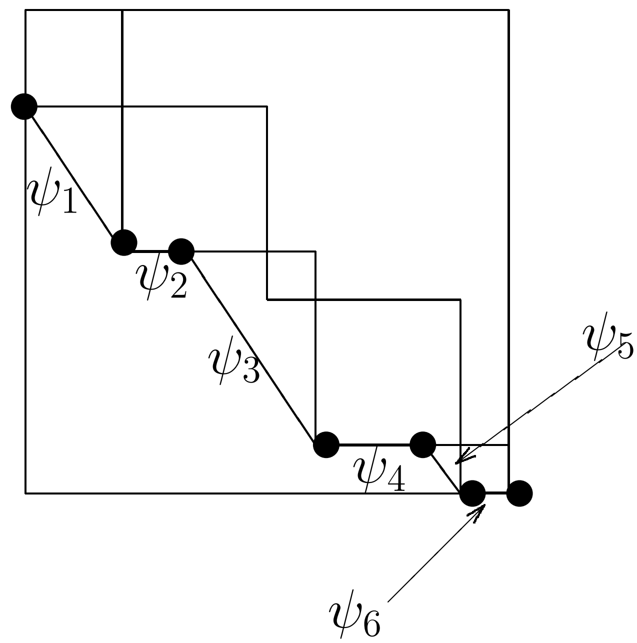

Figure 2: The curve φ_m with $m = 5$ and $h = 6$. The segments of the second type are depicted by diagonal lines, and the segments of the first type are depicted by horizontal lines. These segments alternate.

The next observation follows from Definition 2.2 by a basic geometric argument. See Figure 4 for an illustration.

Observation 2.3 *Consider an index j', $j < j' \leq m$. If $(\tilde{q}, \tilde{r})$ is a point of $\varphi_{j'}$ which belongs to the bottom edge B_j of the square S_j then the curves φ_j and $\varphi_{j'}$ coincide in the range $[0, \tilde{q}]$.*

For a fixed partition $\bar{r}$, let $\Omega = \{\varphi_1, \varphi_2, \ldots, \varphi_m\}$ be the family of m curves as above. For curves $\varphi \in \Omega$ there is a combinatorial description of the functional $L(\varphi)$. Specifically, consider a curve $\varphi \in \Omega$, and let $\psi_1, \psi_2, \ldots, \psi_h$, be its decomposition into linear segments. By definition, $L(\varphi) = \sum_\ell L(\psi_\ell)$. Since $p(q, r) = 0$ for all points (q, r) with $0 \leq q \leq 1$ and $r < b(q)$, it follows that for any segment ψ of the second type, we have $L(\psi) = 0$. Consider now a segment ψ of the first type defined on an interval $[q'(\psi), q''(\psi)]$, with $q' = q'(\psi) = 1 - r_{i-1}$. The segment ψ is contained in the bottom edge B_i of the square S_i, for some index i between 1 and m, and so $q'' \leq 1 - r_i$. By (6) we obtain

$$L(\psi) = \sqrt{k}(q'' - q')\sqrt{\frac{1}{r_{i-1} - r_i}} \, . \tag{7}$$

Hence $L(\varphi_j)$ is the sum of the contributions of the horizontal segments.

We recall that T is defined as the maximum of the functional $L(\varphi)$ over all piecewise differentiable functions which satisfy condition (4). Obviously, segments of the first type satisfy the condition (4). By Lemma 2.1, segments of the second type also satisfy the condition (4). Consequently, the curves $\varphi_{(\bar{r},j)}$ satisfy condition (4). We will show that *for every partition* $\bar{r}$, the cost $L(\varphi_m)$ of the curve φ_m is at least $\sqrt{k-1}$, and conclude that $T \geq \sqrt{k-1}$.

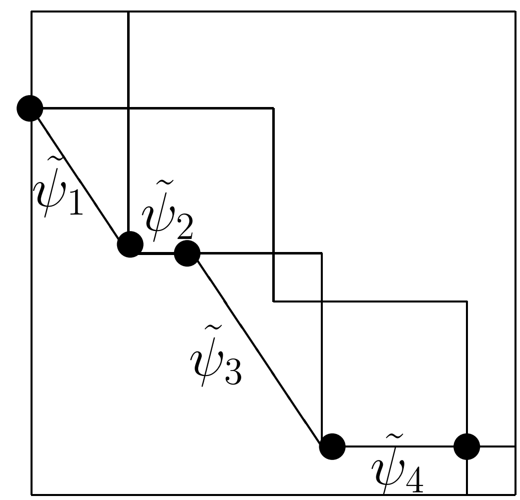

Figure 3: The curve φ_4 for the same partition $\bar{r}$. Note that $h = h(\bar{r}, 4) = 4$. The linear segments of φ_4 are denoted by $\tilde{\psi}_1$, $\tilde{\psi}_2$, $\tilde{\psi}_3$, and $\tilde{\psi}_4$. Comparing this curve with the curve φ_5 (see Figure 2) we see that $\tilde{\psi}_1 = \psi_1$, $\tilde{\psi}_2 = \psi_2$, $\tilde{\psi}_3 = \psi_3$, but $\tilde{\psi}_4 \neq \psi_4$. Specifically, $\tilde{\psi}_4$ is the entire bottom edge B_4 of the square S_4, while ψ_4 is a (proper) subsegment of B_4.

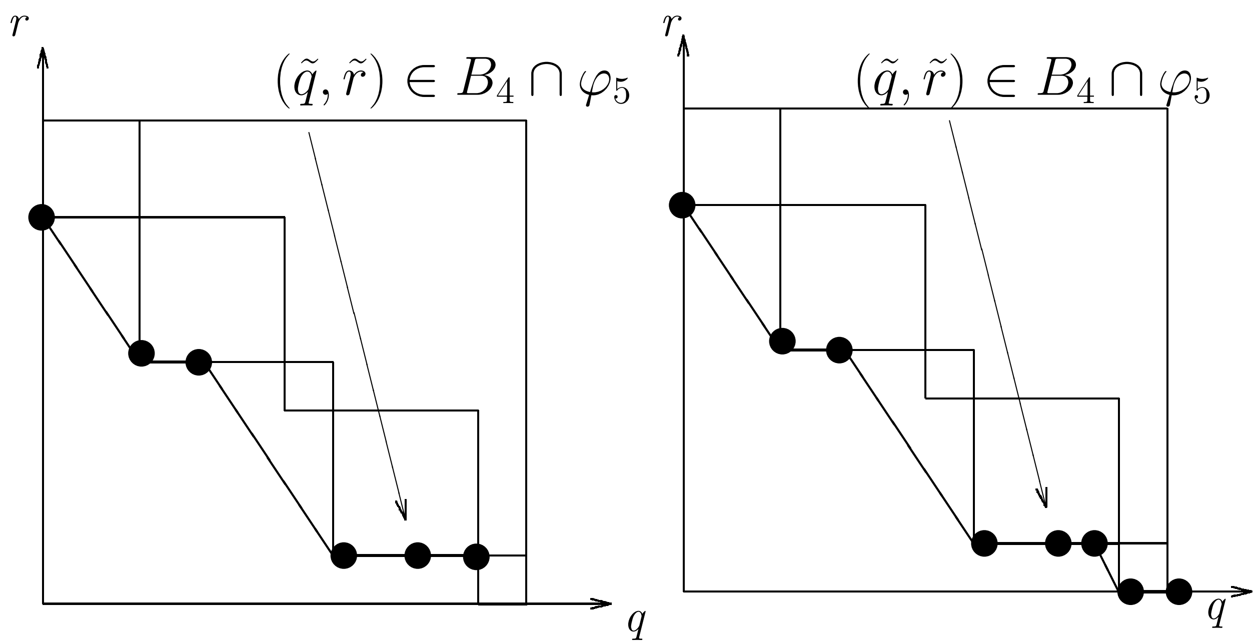

Figure 4: The curve φ_4 (respectively, φ_5)) is depicted on the left-hand (resp., right-hand) figure. In terms of the notation in the text, $j = 4$, $j' = 5$. The condition $\tilde{q}, \tilde{r}) \in B_4 \cap \varphi_5$ implies that the curves φ_4 and φ_5 coincide for $q \in [0, \tilde{q}]$.

To illustrate our approach we consider the case $m = 1$. In this case there is the unique partition $\bar{r} = (r_0 = 1, r_1 = 0)$. For this partition, S_1 is the entire unit square, and the density function $p(q, r)$ is given by $p(q, r) = 1$ for all points (q, r) in the square S_1. This partition corresponds to the policy of allowing passengers to board the airplane in random order, in other words, the airline does not employ a boarding policy. We will compare all other policies with this one. It has been shown in [1] that for this partition $T = \sqrt{k} + \frac{1-\ln(2)}{\sqrt{k}}$. In this case the family $\Phi = \{\varphi_1\}$ of curves contains just one single curve $\varphi_1(q) = 0$ for all q, $0 \le q \le 1$, and by equation (7), $L(\varphi_1) = \sqrt{k}$.

Theorem 2.4 *If $k > 1$ then for any partition $\bar{r} = (r_0 = 1, r_1, \ldots, r_{m-1}, r_m = 0)$, we have $L(\varphi_{(\bar{r},m)}) \ge \sqrt{k-1}$.*

Proof: The proof is by induction on m. The induction base $m = 1$ was established above. Let

$$F_m \;=\; \min\{L(\varphi_{(\bar{r}',m')}) \mid m' \le m,\ \bar{r}' \text{ is a partition of size } m'\}\ .$$

Let $\bar{r} = (r_0 = 1, r_1, .., r_m, r_{m+1} = 0)$ be a partition of size $m + 1$, and consider $\varphi_{(\bar{r},m+1)}$.

Let $(\psi_1, \psi_2, \ldots, \psi_h)$ be the segment decomposition of the curve $\varphi_{(\bar{r},m+1)}$. We split the argument into two cases, depending on the value of r_m. First suppose that $r_m \ge \frac{k-1}{k}$. By definition, the last linear segment ψ_h of φ_{m+1} has the form $\psi_h(q) = 0$, for $1 - r_m \le q \le 1$, and so

$$L(\varphi_{m+1}) \ge L(\psi_h) \;=\; \sqrt{kr_m} \ge \sqrt{k-1}\ . \tag{8}$$

We therefore assume that

$$r_m \;<\; \frac{k-1}{k}. \tag{9}$$

Consider the line ℓ given by the equation $r(q) = -kq + k(1 - r_m)$ which passes through the point $E = (q_E, r_E) = (1-r_m, 0)$ and has slope $-k$. Let $j = j(\bar{r}) \le m$ be the largest index such that the line ℓ intersects the bottom edge B_j of the square S_j. Let $D = (q_D, r_D) = ((1 - r_m) - r_j/k, r_j)$ be the intersection point of the line ℓ with B_j. By definition of the curve φ_{m+1}, the next to last segment ψ_{h-1} coincides with the segment of the line ℓ connecting the points D and E. (See Figure 5.) Let $C = (q_C, r_C) = (1 - \frac{k}{k-1} r_m, \frac{k}{k-1} r_m)$ be the intersection point of ℓ with the anti-diagonal. Note that since the squares S_i cover the anti-diagonal, the q coordinate of the point D is no smaller than that of C, i.e.,

$$q_C = 1 - \frac{k}{k-1} r_m \le q_D \le 1 - r_m\ .$$

Moreover, the r coordinate of D, r_D, is no larger than the r coordinate of C, r_C, i.e.,

$$r_m \le r_j = r_D \le r_C = \frac{k}{k-1} r_m. \tag{10}$$

Let $\tilde{\varphi}_{m+1} = \tilde{\varphi}_{m+1}(\bar{r})$ be the part of φ_{m+1} consisting of $\psi_1, \ldots \psi_{h-2}$, i.e., the curve φ_{m+1} restricted to the range $[0, q_D]$. By (9) and (10), this range is not empty. Since ψ_{h-1} is a segment of the second type, $L(\psi_{h-1}) = 0$. Consequently, $L(\varphi_{m+1}) = L(\tilde{\varphi}_{m+1}) + L(\psi_{h-1}) + L(\psi_h) = L(\tilde{\varphi}_{m+1}) + L(\psi_h)$. By (8), $L(\psi_h) = \sqrt{kr_m}$.

Next, we estimate $L(\tilde{\varphi}_{m+1})$. The index $j = j(\bar{r})$ determines the curve φ_j (see Definition 2.2). Since the point D lies on the bottom edge B_j, by Observation 2.3, the curve $\tilde{\varphi}_{m+1}$ is also the

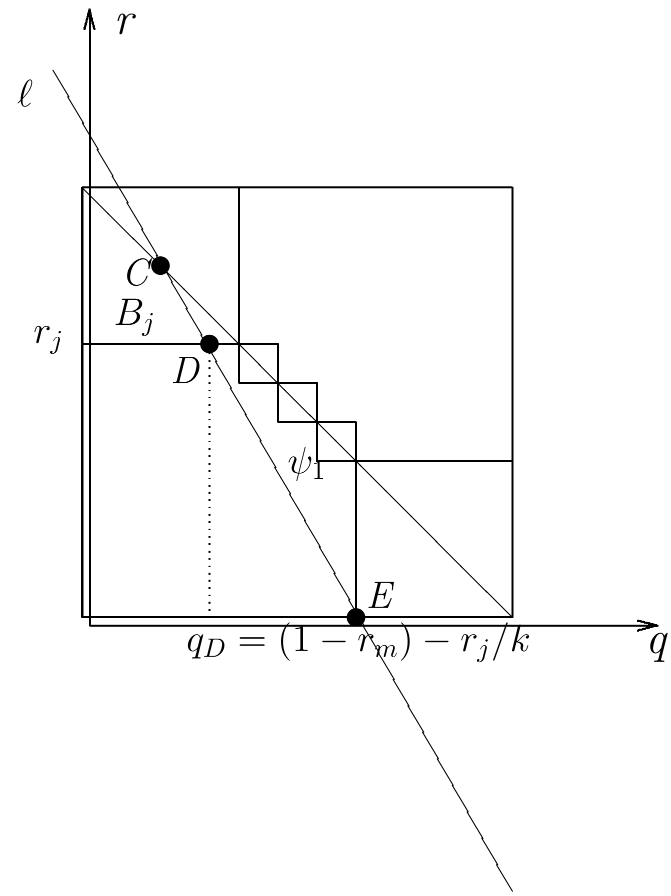

Figure 5: The line ℓ contains the segment ψ_{h-1}, and intersects the bottom edge B_j of the square S_j. The squares of the partition, the anti-diagonal and the line ℓ are all depicted by solid lines, and the dotted line is used to connect the point D with its projection on the axis q.

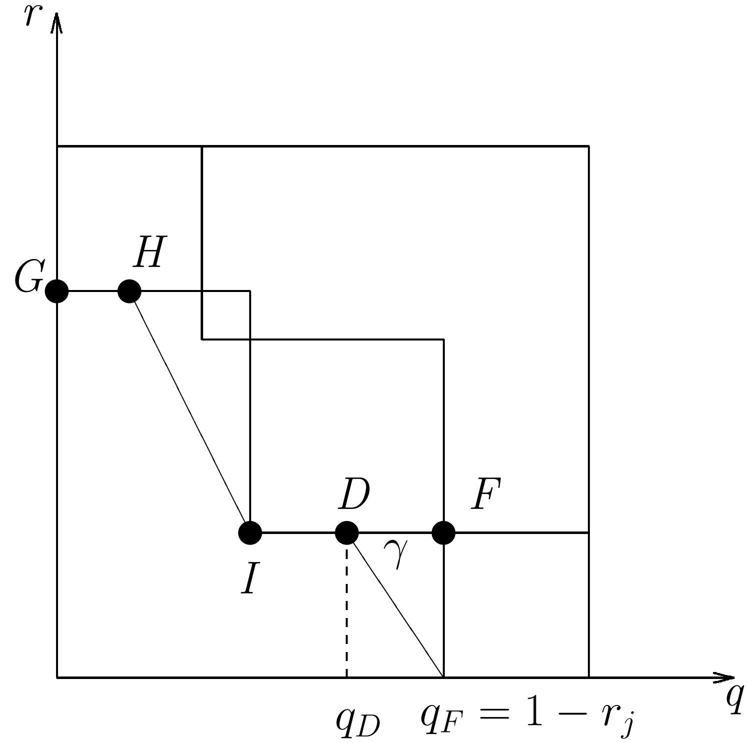

Figure 6: The piecewise linear curve $GHID$ is $\tilde{\varphi}_{m+1}$, and the curve $GHIF$ is φ_j. The segment DF is γ.

restriction of φ_j to the range $[0, q_D]$. The curve φ_j is defined in the domain $[0, 1 - r_j]$. Let γ be the restriction of φ_j to the complementary domain $[q_D, 1 - r_j] = [1 - r_m - r_j/k, 1 - r_j]$. See Figure 6 for an illustration.

By definition of the functional L,

$$L(\varphi_j) = L(\tilde{\varphi}_{m+1}) + L(\gamma) \ .$$

By (7),

$$L(\gamma) \ = \ \sqrt{k}\sqrt{\frac{1}{r_{j-1} - r_j}}(1 - r_j - q_D) \ = \ \sqrt{k}\sqrt{\frac{1}{r_{j-1} - r_j}}(r_m - \frac{k-1}{k}r_j) \ .$$

The segment γ is contained in the bottom edge B_j of the square S_j. The length of γ is $r_m - \frac{k-1}{k}r_j$, and the length of B_j is $r_{j-1} - r_j$. It follows that $r_{j-1} - r_j \leq r_m - \frac{k-1}{k}r_j$, and so

$$L(\gamma) \leq \sqrt{k \cdot \left(r_m - \frac{k-1}{k}r_j\right)}. \tag{11}$$

We conclude that

$$L(\tilde{\varphi}_{m+1}) \ \geq \ L(\varphi_j) - \sqrt{k\left(r_m - \frac{k-1}{k}r_j\right)} \ .$$

To estimate $L(\varphi_j)$, consider the affine map $U : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$U(q, r) = ((1 - r_j)q, (1 - r_j)r + r_j) \ .$$

This map contracts the plane by a factor of $1 - r_j$ around the fixed point $(0, 1)$. Consider the partition $\bar{r}' = (1 = 1 - \frac{1-r_0}{1-r_j}, 1 - \frac{1-r_1}{1-r_j}, \ldots, 1 - \frac{1-r_{j-1}}{1-r_j}, 1 - \frac{1-r_j}{1-r_j} = 0)$ of size $j < m + 1$. This partition determines the curve $\hat{\varphi} = \varphi_{(\bar{r}', j)}$. By the definition of the functional L (see (6) and (7)), $L(U(\varphi)) = \sqrt{1 - r_j} \cdot L(\varphi)$ for any curve φ. Since $j < m + 1$, by the induction hypothesis, $L(\hat{\varphi}) \geq \sqrt{k-1}$. Therefore,

$$L(\varphi_j) \ \geq \ \sqrt{(k-1)(1 - r_j)} \ .$$

By (11),

$$L(\tilde{\varphi}_{m+1}) \ = \ L(\varphi_j) - L(\gamma) \ \geq \ \sqrt{(k-1)(1 - r_j)} - \sqrt{k(r_m - \frac{k-1}{k}r_j)} \ .$$

Consequently,

$$L(\varphi_{m+1}) \ = \ L(\tilde{\varphi}_{m+1}) + L(\psi_h) \ \geq \ \left(\sqrt{(k-1)(1 - r_j)} - \sqrt{k(r_m - \frac{k-1}{k}r_j)}\right) + \sqrt{kr_m} \ .$$

Let $a = r_j/r_m$ be the ratio between r_j and r_m. By (10),

$$1 \leq a \leq \frac{k}{k-1}. \tag{12}$$

It follows that

$$L(\varphi_{m+1}) \ \geq \ \sqrt{k-1}\sqrt{1-ar_m} - \sqrt{k}\left(\sqrt{1-\frac{k-1}{k}a}-1\right)\sqrt{r_m}\ .$$

Let $g(r_m, a)$ denote the right-hand side. Next, we prove that for all a and r_m, $1 \leq a \leq \frac{k}{k-1}$ and $0 \leq r_m \leq \frac{k-1}{k}$,

$$g(r_m, a) \ \geq \ \sqrt{k-1}\ . \tag{13}$$

Obviously, this will complete the proof. Differentiating the function $g(r_m, a)$ with respect to the variable r_m we get

$$\frac{\partial g}{\partial(r_m)}(r_m, a) \ = \ (-a)\frac{\sqrt{k-1}}{2\sqrt{1-ar_m}} - \sqrt{k}\left(\sqrt{1-\frac{k-1}{k}a}-1\right)\frac{1}{2\sqrt{r_m}}\ .$$

The equality $\frac{\partial g}{\partial(r_m)}(r_m, a) = 0$ holds when

$$\sqrt{k}\left(1-\sqrt{1-\frac{k-1}{k}a}\right)(\sqrt{1-ar_m}) = a\sqrt{k-1}\sqrt{r_m}\ .$$

Since $r_m < \frac{k-1}{k}$ and $1 \leq a \leq \frac{k}{k-1}$, both sides are non-negative, and thus squaring both sides results in the following equivalent equation.

$$k\left(1-\sqrt{1-\frac{k-1}{k}a}\right)^2(1-ar_m) = a^2(k-1)r_m\ . \tag{14}$$

Fix a and consider (14) as an equation in the single variable r_m. This is clearly a linear equation. The free coefficient of this equation is positive, and thus this equation has at most one solution. Since $g(0, a) = g(\frac{k-1}{k}, a) = \sqrt{k-1}$ for all values of a, by the mean value theorem this equation has exactly one solution. Hence the function $g_a(r_m) = g(r_m, a)$ has a unique extremum in the interval $0 \leq r_m \leq \frac{k-1}{k}$. Moreover, since $\lim_{r_m\to 0}\frac{\partial g}{\partial(r_m)}(0, a) = \infty$ it follows that this extremum is a maximum. Consequently, for all values of a, $1 \leq a \leq \frac{k}{k-1}$, and $r_m < \frac{k-1}{k}$, it holds that $g(r_m, a) \geq g(0, a) = \sqrt{k-1}$. ∎

References

[1] Bachmat, E., Berend, D., Sapir, L., Skiena, S., N. Stolyarov. 2006. Analysis of airplane boarding, spacetime geometry and random matrix theory. *Journal of physics A: Mathematical and general*, 39, pages L453-L459.

[2] Bachmat, E., Berend, D., Sapir, L., Skiena, S., N. Stolyarov. 2005. Analysis of airplane boarding times, submitted. Available at www.cs.bgu.ac.il/ẽbachmat.

[3] Ferrari, P., K. Nagel. 2005. Robustness of efficient passenger boarding in airplanes, *Transportation Research Board Annual Meeting*, paper number 05-0405, Washington D.C., Available at www.trb.org.

[4] Marelli, S., G. Mattocks, R. Merry. 1998. The role of computer simulation in reducing airplane turn time, *Boeing Aero Magazine*, Issue 1.

[5] Van den Briel, M., J. Villalobos, G. Hogg. 2003. The Aircraft Boarding Problem. *Proc. of the 12'th Industrial Eng. Res. Conf., IERC*, CD ROM only, article number 2153.

[6] Van den Briel, M., J. Villalobos, G. Hogg, T. Lindemann, A.V. Mule. 2005. America West develops efficient boarding strategies, *Interfaces*, 35, 191-201.

[7] Van Landeghem, H., A. Beuselinck. 2002. Reducing passenger boarding time in airplanes: A simulation approach. *European J. of Operations Research*, 142, 294-308.