Distributed Symmetry Breaking

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joint w/ L. Barenboim
(PODC’08, STOC’09, PODC’10, PODC’11, J. ACM’11)

and w/ L. Barenboim, S. Pettie and J. Schneider (FOCS’12)

and w/ S. Pettie and H. Su (SODA’15)
The Model

- Unweighted undirected graph $G = (V, E)$.
- Vertices host processors.
- Processors communicate over edges of $G$.
- Communication is synchronous, i.e., occurs in *discrete* rounds.
• Running time = \# rounds.

• All vertices wake up simultaneously.

• Vertices have unique Ids from \{1, 2, \ldots, n\} = [n].

• Arbitrarily large messages are allowed, though short (of size \(O(\log n)\)) are preferred.
Coloring

- $\Delta = \Delta(G)$ - maximum degree of a vertex in $G$.

- $\varphi : V \to [k]$ is a $k$-vertex-coloring if $\forall e = (u, w) \in E$, $\varphi(u) \neq \varphi(w)$.

- $\psi : E \to [t]$ is a $t$-edge-coloring if $\forall e, e' \text{ s.t. } e \cap e' \neq \emptyset, \psi(e) \neq \psi(e')$.

- In distributed setting, typically $k \geq \Delta + 1, \ t \geq 2\Delta - 1$.

- **MIS** $U$:
  1. $\forall v, w \in U, \ (v, w) \notin E$.
  2. $\forall v \notin U, \ \exists u \in U \text{ s.t. } (u, v) \in E$.

- **MM** $M$:
  1. $\forall e, e' \in M, \ e \cap e' = \emptyset$.
  2. $\forall e' \notin M, \ \exists e \in M \text{ s.t. } e \cap e' = \emptyset$. 
• \((\Delta + 1)\)-coloring in \(O(n)\) rounds is easy.

  Color vertices one-by-one:
  For each new vertex \(v\) there are \(\leq \Delta\) forbidden colors.

  Hence there is always an available color for \(v\) in \([\Delta + 1]\).

• MIS in \(O(n)\) rounds is easy too.

  Initialize \(U \leftarrow \emptyset\);
  Treat vertices one-by-one:
  For each new vertex \(v\) do:
  if \(\Gamma(v) \cap U = \emptyset\) then
  \(v\) joins \(U\);

• \((2\Delta - 1)\)-edge-coloring reduces to \((\Delta + 1)\)-vertex-coloring,
  MM and \((\Delta + 1)\)-vertex-coloring reduce to MIS.
Elementary Color Reduction Technique

Given an $\alpha$-coloring, $\alpha > \Delta + 1$, eliminate one color class in each round.

Vertices of color $\alpha$ form an independent set.

Each of them recolors itself into an available color from $[\Delta + 1]$.

So in $\alpha - (\Delta + 1)$ rounds we get a $(\Delta + 1)$-coloring.

Continue with it for $\Delta + 1$ more rounds to get an MIS.
Kuhn-Wattenhofer’s (KW) Color Reduction Technique

$(\Delta + 1)$-coloring in $O(\Delta \log \frac{\alpha}{\Delta+1}) + \log^* n$ time.

[Kuhn, Wattenhofer (PODC’06)]
[ Szegedy, Vishwanathan (STOC’92)]

• Given an $\alpha$-coloring $\varphi$,
  $\alpha = c \cdot (\Delta + 1),$
  $c$ is a large integer power of 2.

• $\forall i \in [c], \text{ let}$
  $U_i = \{v \mid (i - 1) \cdot (\Delta + 1) + 1 \leq \varphi(v) \leq i \cdot (\Delta + 1)\}.$

• Pair subgraphs $G(U_1)$ with $G(U_2)$,
  $G(U_3)$ with $G(U_4)$, ..., 
  $G(U_{c-1})$ with $G(U_c)$. 
Consider $G(U_1 \cup U_2)$. It is $2 \cdot (\Delta + 1)$-colored by $\varphi$.

- Reduce the $2(\Delta + 1)$-coloring of $G(U_1 \cup U_2)$ to get a $(\Delta + 1)$-coloring of $G(U_1 \cup U_2)$ in $2(\Delta + 1) - (\Delta + 1) = \Delta + 1$ rounds.

In parallel, reduce the colorings of $G(U_3 \cup U_4), G(U_5 \cup U_6), \ldots$ In $\Delta + 1$ rounds we get $\frac{1}{2}\alpha$-coloring of $G$.

- Keep halving the #colors by phases that last $\Delta + 1$ rounds each.

In $\log \frac{\alpha}{\Delta + 1}$ phases (i.e., in $O(\Delta \cdot \log \alpha/\Delta)$ time) we get $(\Delta + 1)$-coloring.
• [Linial (FOCS’87)]:
\[O(\Delta^2)\text{-coloring in } \log^* n \text{ time.}\]

In conjunction with the KW color reduction we get \[O(\Delta \log \Delta) + \log^* n \text{ time for } (\Delta + 1)\text{-coloring.}\]

• \textit{Locally-iterative} means: in every round every vertex recolors itself based only on colors of its neighbors.

[Szegedy, Vishwanathan (STOC’92)]:
\textit{Any locally-iterative} \((\Delta + 1)\text{-coloring algorithm requires } \Omega(\Delta \log \Delta) \text{ time.}\]

The \((\Delta + 1)\text{-coloring algorithms of Linial and of Kuhn and Wattenhofer can be casted as locally-iterative.}\]

So the KW is an optimal locally-iterative \((\Delta + 1)\text{-coloring algorithm.}\]
Distributed Coloring -
Known Randomized Results

- \((\Delta + 1)\)-coloring, MIS and MM in \(O(\log n)\) time.
  [Luby (STOC’85)],
  [Alon,Babai,Itai (J.Alg.’86)],
  [Israeli,Itai (IPL’86)].

- \((\Delta + 1)\)-vertex-coloring in \(O(\log \Delta + \sqrt{\log n})\) time.
  [Schneider,Wattenhofer (PODC’10)].

- \(O(\Delta)\)-vertex-coloring in \(O(\sqrt{\log n})\) time
  [Kothapalli,Scheideler,Onus,
  Schindelhauer (IPDPS’06)].

- \(O(\Delta + \log n)\)-vertex-coloring
  in \(O(\log \log n)\) time,
  and \(O(\Delta \log^{c} n + \log^{1+1/c} n)\)-coloring in
  \(O(f(c)) = O(1)\) time.
  [Schneider,Wattenhofer (PODC’10)].
New Randomized Algorithms

[Barenboim, E., Pettie, Schneider (FOCS’12)]

- MM in $O(\log \Delta + \log^4 \log n)$ time.

- $(\Delta + 1)$-coloring in $O(\log \Delta) + \exp\{O(\sqrt{\log \log n})\}$ time.

- $O(\Delta)$-coloring in $\exp\{O(\sqrt{\log \log n})\}$ time.

- $\Delta^{1+\eta}$-coloring in $O(\log^2 \log n)$ time.

- MIS in $O(\log^2 \Delta) + \exp\{O(\sqrt{\log \log n})\}$ time.

- $\Delta^{1+\eta}$-edge-coloring in $O(\log \log n)$ time.
Basic Approach in BEPS's algorithms

- Do (roughly) $O(\log \Delta)$ "Luby steps" to break the graph into disconnected components of size $s \leq \text{polylog}(n)$.

\[|C_1|, |C_2|, |C_3| \leq s.\]
• Use the state-of-the-art deterministic MIS algorithm for each component.

It completes the MIS within additional
\[ \exp\{O(\sqrt{\log s})\} \leq \exp\{O(\sqrt{\log \log n})\} \] time. [Panconesi, Srinivasan’92]

Using randomized subroutine within components fails because the failure probability is \( 1/\text{poly}(s) \approx 1/\text{polylog}(n) \).

• Works similarly for \((\Delta + 1)\)-vertex-coloring and MM problems.

For MM the second step requires just \( O(\log^4 s) = O(\log^4 \log n) \) time. [Hanckowiak, Karonski, Panconesi’99]

• Improved deterministic algorithms give rise to improved randomized ones!
New Randomized Algorithms (Cont.)

[E., Pettie, Su (SODA’15)]

\((2\Delta - 1)\)-edge-coloring in \(\exp\{O(\sqrt{\log \log n})\}\) time.

Establishes a separation between MM and \((2\Delta - 1)\)-edge-coloring.

MM requires \(\Omega(\sqrt{\log n})\) time.

[Kuhn, Moscibroda, Wattenhofer’10].
Open if such a separation exists also between MIS and \((\Delta + 1)\)-vertex-coloring.

Our algorithm (in SODA’15) gives actually \((\Delta + 1)\)-vertex-coloring in \(O(\log 1/\epsilon) + \exp\{O(\sqrt{\log \log n})\}\) time for \((1 - \epsilon)\)-locally-sparse graphs.

**Def:** \(G = (V, E)\) of max’ degree \(\Delta\) is \((1 - \epsilon)\)-locally-sparse if

\[
\forall v, \quad |E(\Gamma(v))| < (1 - \epsilon)\binom{\Delta}{2}
\]

Line graphs are \((1/2 + o(1))\)-locally-sparse, implying our result for edge-coloring.
More Details about MIS Algorithm (BEPS)

Procedure Halve - reduces the degree from $\Delta$ to $\Delta/2$.

Consists of two parts.

Part I: Runs $\kappa$ Luby trials. $\kappa$ will be set
(1) either as $\kappa = O(\sqrt{\log n})$,
(2) or as $\kappa = O(\log \Delta)$.

The graph of remaining high-degree vertices breaks into connected components.

In case (1) these components have weak diameter $O(\sqrt{\log n})$.

In case (2) they have size $O(\Delta^4 \cdot \log n)$. 

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Procedure Halve (Cont.)

To complete computing the MIS for high-degree vertices (\(deg \geq \Delta/2\)) we need

(1) \(O(\sqrt{\log n})\) time in case (1) via brute-force.

(2) \(\exp\{O(\sqrt{\log O(\Delta^4 \log n)})\}\) time in case (2) via the state-of-the-art deterministic MIS algorithm due to [Panconesi, Srinivasan’92].

So Procedure Halve requires total

(1) \(O(\sqrt{\log n})\) time in case (1).

(2) \(O(\log \Delta) + \exp\{O(\sqrt{\log \Delta + \log \log n})\} = \exp\{O(\sqrt{\log \Delta + \log \log n})\}\) time in case (2).
Hence the entire running time is $O(\sqrt{\log n} \cdot \log \Delta)$ in case (1),
and $O(\log \Delta \cdot \exp\{O(\sqrt{\log \Delta + \log \log n})\})$
in case (2).

The latter expression is $\exp\{O(\sqrt{\log \log n})\}$,
whenever $\Delta \leq \text{polylog}(n)$.

A better bound
$O(\log^2 \Delta + \exp\{O(\sqrt{\log \log n})\})$:
not in this talk.
Part I of Procedure Halve

Run $\kappa$ Luby trials.

$I_1 \leftarrow \emptyset, \ A_1 \leftarrow V$.

$I_i$ - the independent set before the beginning of iteration $i$.

$A_i \leftarrow V \setminus \hat{\Gamma}(I_i)$ - active vertices before iteration $i$.

$I_{\kappa+1}$ - the independent set after all the $\kappa$ Luby trials.

Part II works on $G(V \setminus \hat{\Gamma}(I_{\kappa+1}))$.

This subgraph decomposes into components of either bounded weak diameter or size.
Procedure Halve: Part I

Run for $\kappa$ iterations ($i = 1, 2, \ldots, \kappa$) on the entire graph $G$, i.e.,
on both high-degree ($\text{deg} \geq \Delta/2$) and low-degree ($\text{deg} < \Delta/2$) vertices.

(Assume the maximum degree is at most $\Delta$.)

1. Each $v \in A_i$ selects itself
   with probability $\frac{1}{\Delta+1}$, i.a.r.

2. $I_{i+1} \leftarrow I_i \cup \{v \mid v \text{ is the only vertex in } \hat{\Gamma}(v) \text{ which selected itself} \}$
Procedure Halve: Analysis of Part I

$\Gamma_i(u)$ - set of active neighbors of a vertex $u$ at the beginning of iteration $i$.

$\text{deg}_i(u) = |\Gamma_i(u)|$

**Def:** A subset $S \subseteq V$ is well-separated if $\forall u, u' \in S, u \neq u', \text{dist}_G(u, u') \geq 5$.

**Set-Survival Lemma:**
Let $S \subseteq A_i$ be a well-separated set of active vertices with $\text{deg}_i(u) \geq \Delta/2$ each. Then

$$\mathbb{P}(S \subseteq A_{i+1}) \leq p^{|S|},$$

for some constant $0 < p < 1$. 
Neighborhoods of neighbors are disjoint.

In a particular Luby trial, survival of a vertex $u$ depends only upon coins tosses of its 2-neighborhood.

Hence vertices $u, u'$ at distance $\geq 5$ from one another are independent.
Proof of Set-Survival Lemma

Sufficient to prove

$$\text{IP}(u \in A_{i+1} \mid u \in A_i) \leq p.$$ 

Let \( u = v_0, \{v_1, v_2, \ldots, v_d\} = \Gamma_i(u). \)

By Lemma’s assumption,

\[ d = \text{deg}_i(u) \geq \Delta/2. \]

\[
\text{IP}(\text{some vertex } \in \{v_0, \ldots, v_d\} \text{ selects itself}) \\
= 1 - \left(1 - \frac{1}{\Delta + 1}\right)^{d+1} \\
\geq 1 - \left(1 - \frac{1}{\Delta + 1}\right)^{\Delta/2+1} > 1 - e^{-1/2}.
\]
Proof of Set-Survival Lemma (Cont.)

Conditioned upon the event

some vertex \( \in \{v_0, \ldots, v_d\} \) selects itself:

\( j \) - the smallest index such that \( v_j \) selects itself.

\[
H = \Gamma_i(v_j) \setminus \{v_0, \ldots, v_j\}
\]
\[ \text{IP}(v_j \text{ joins } MIS \mid v_j \text{ selects itself, } j \text{ is the smallest}) = \]
\[ \text{IP}(\Gamma_i(v_j) \setminus \{v_0, \ldots, v_{j-1}\} \text{ did not select themselves}) = \]
\[ = (1 - \frac{1}{\Delta + 1})|\Gamma_i(v_j)\setminus\{v_0,\ldots,v_{j-1}\}| \geq \]
\[ \geq (1 - \frac{1}{\Delta + 1})^\Delta \geq e^{-1}. \]

\[ \text{IP}(u \text{ becomes inactive}) \geq \]
\[ \text{IP}(\text{some } v_j \text{ selects itself}) \cdot \text{IP}(v_j \in I_{i+1} \mid v_j \text{ exists}) \]
\[ \geq (1 - e^{-1/2}) \cdot e^{-1}. \]

\[ \text{IP}(u \text{ survives}) = 1 - (1 - e^{-1/2}) \cdot e^{-1} = p. \]

\[ 0 < p < 1. \]

\textbf{QED}
Analysis of Proc. Halve: Long-Path Lemma, Notation

\[ U = \{ v \in A_{\kappa+1} \mid \deg_{\kappa+1}(u) \geq \Delta/2 \} \]

Vertices that survived all Luby trials and retained high degree.

Set \( \kappa = c \cdot \sqrt{\log n} \) (i.e., case (1)).

\( \mathcal{P} \) - the set of all paths (not necessarily shortest) which involve only vertices that \textit{originally had high degree} between pairs \( u, v \) of \textit{remote} vertices, i.e., s.t. \( \text{dist}_G(u, v) \geq 5 \cdot \sqrt{\log n} \).
Long-Path Lemma

Lemma: Each $P \in \mathcal{P}$ contains a well-separated set $Q(P)$ of size $|Q(P)| = \sqrt{\log n}$.

Proof:

$P = (u = u_0, u_1, \ldots, u_\ell = v)$,

$\ell \geq 5\sqrt{\log n}$.

Set $q_1 \leftarrow u_0$.

Suppose we have already built $q_1, q_2, \ldots, q_j$, $j \geq 1$, with pairwise distances $\geq 5$, and s.t. $\text{dist}_G(q_j, v) \geq 5$. 

Proof of Long-Path Lemma: Cont.

\( q_{j+1} \leftarrow u_i \in P \) with the highest index s.t. \( \text{dist}_G(q_j, u_i) = 5 \).

(Exists, because \( \text{dist}_G(q_j, v) \geq 5 \).)

**Claim:** \( \forall \ k, \ 1 \leq k \leq j, \text{ \ \ \ \ \ } \text{dist}_G(q_k, q_{j+1} = u_i) \geq 5 \).
Proof (of Claim):

Otherwise $\text{dist}_G(q_k, u_i) < 5$, but $\text{dist}_G(q_k, u_\ell) = \text{dist}_G(q_k, v) \geq 5$.

Hence $\exists i', i < i' \leq \ell$ s.t. $\text{dist}_G(q_k, u_{i'}) = 5$, contradicting the maximality of the index $h$, $u_h = q_{k+1}$.

QED (Claim)
Proof of Long-Path Lemma: Cont. II

\[ \text{dist}_G(q_1, v) = \ell, \quad \text{dist}_G(q_1, q_2) = 5, \]

\[ \text{dist}_G(q_2, v) + \text{dist}_G(q_1, q_2) \geq \text{dist}_G(q_1, v) = 5\sqrt{\log n}. \]

Hence

\[ \text{dist}_G(q_2, v) \geq 5(\sqrt{\log n} - 1), \]

\[ \text{dist}_G(q_3, v) \geq 5(\sqrt{\log n} - 2), \]

etc.

We obtain \( \sqrt{\log n} \) vertices \( q_1, q_2, \ldots, q_{\sqrt{\log n}} \) in a well-separated set.

QED (of Lemma)
Small Components’ Lemma, Preliminaries

\[ Q = \{ Q(P) \mid P \in \mathcal{P} \} \]

The set of well-separated subsets obtained from paths of high-degree vertices whose endpoints are at distance \( \geq 5\sqrt{\log n} \) one from another in \( G \).

\( \mathcal{W} \) - the set of walks of length \( 5\sqrt{\log n} \).

Every \( Q(P) \in Q \) can be mapped into a walk \( W(P) \in \mathcal{W} \).

(By connecting consecutive \( q_i, q_{i+1} \) in \( Q(P) \) by an arbitrary 5-path.)

\[ |Q| \leq |\mathcal{W}| \]
Also, 

$$|W| \leq n \cdot \Delta^{5\sqrt{\log n}} - 1.$$ 

($n$ possibilities to pick the first vertex, and $\Delta$ possibilities to pick each consecutive one.)

Hence 

$$|Q| \leq |W| \leq n \cdot \Delta^{5\sqrt{\log n}} - 1.$$
Small Components’ Lemma

**Lemma:** After $\kappa = c\sqrt{\log n}$ Luby trials, the vertex set $U$ (of vertices $w$ with $\deg_{\kappa+1}(w) \geq \Delta/2$) induces connected components of weak diameter $\leq 5\sqrt{\log n}$, whp.

(Meaning that the maximum distance *in* $G$ between every pair of vertices $w, w'$ in the same component is $\leq 5\sqrt{\log n}$.)
Proof of Small Components’ Lemma

$u, v$ - a pair of high-degree vertices with $\text{dist}_G(u,v) \geq 5 \sqrt{\log n}$.

$P \in \mathcal{P}$ - some path between them that consists of only high-degree vertices.

$\Pr(Q(P) \text{ survives a single iteration}) \leq p^{|Q(P)|}$.

The probability of the entire $Q(P)$ to survive $\kappa$ iterations in all of which all its vertices keep being high-degree is

$\leq p^{|Q(P)| \cdot \kappa} = \exp\{-\Omega(\sqrt{\log n} \cdot c \sqrt{\log n})\} = n^{-c}$
Proof of Small Components’ Lemma, Cont.

Number of possible $Q$-sets is
$$|Q| \leq |W| \leq n \cdot \Delta^{5\sqrt{\log n}}.$$  

For $\Delta \leq 2^{\sqrt{\log n}}$, we get a bound of
$$n \cdot 2^{5\log n} = n^6.$$

Probability that some $Q(P)$ survives is
$$\leq \frac{n^6}{n^c} = \frac{1}{n^{c-6}} = \frac{1}{\text{poly}(n)}.$$

Hence, whp, every path $P \in \mathcal{P}$ which connects a pair $u, v$ with $\text{dist}_G(u, v) \geq 5\sqrt{\log n}$ and such that all vertices of $V(P)$ are high-degree,

*does not (entirely) survive*, i.e., $V(P) \not\subseteq U$.

Hence, whp, only vertices $u, v$ with
$\text{dist}_G(u, v) < 5\sqrt{\log n}$ may be connected *in* $U$. 
For $\Delta \geq 2^{\sqrt{\log n}}$, the Luby’s bound of $O(\log n)$ is better than the bound of $O(\log \Delta \sqrt{\log n})$.

So in both cases we have $\text{MIS in } O(\log \Delta \sqrt{\log n})$ time.

In the paper we show a much better bound of $O(\log^2 \Delta) + \exp(O(\sqrt{\log \log n}))$. 
Lower Bounds vs. Upper Bounds

• \( f(\Delta) \)-coloring requires \( \frac{1}{2} \log^* n \) time.

[Linial (FOCS’87)]

The upper bound (BEPS) for \((\Delta + 1)\)-coloring is
\[ O(\log \Delta) + \exp\{O(\sqrt{\log \log n})\}. \]

Huge gap!

• Coloring \( \Delta \)-regular trees in \( o(\sqrt{\Delta}) \) colors
requires \( \omega(\log_\Delta n) \) time.

[Linial (FOCS’87)]

One can color unoriented forests in \( \Delta^{\epsilon} \) colors within \( O(\log_\Delta n) \) time,
for an arbitrarily small \( \epsilon > 0 \).

[Barenboim,E. (PODC’08)] (tight).
• $\Omega(\log \Delta)$ and $\Omega(\sqrt{\log n})$ time is required for MIS and MM.

[Kuhn,Moscibroda,Wattenhofer], [(PODC'04), (ArXiv'10)]

The upper bound (BEPS) for MM is $O(\log \Delta + \log^4 \log n)$.

Tight for $\log^4 \log n \leq \log \Delta \leq \sqrt{\log n}$.

For MIS the BEPS’s upper bound is $O(\log^2 \Delta) + \exp\{O(\sqrt{\log \log n})\}$. 
Known Deterministic Results

• \((\Delta + 1)\)-coloring and MIS in \(O(\Delta^2 + \log^* n)\) time, and in \(O(\Delta \log n)\) time.
  [Goldberg, Plotkin, Shannon’87]
  (based on [Cole, Vishkin’86])

• \(O(\Delta^2)\)-coloring in \(\log^* n + O(1)\) time.
  [Linial’87]

  Asked: can one get much fewer than \(\Delta^2\) colors in time polylogarithmic in \(n\)?

• \((\Delta + 1)\)-coloring and MIS in \(2^{O(\sqrt{\log n})}\) time. (Large messages)
  [Panconesi, Srinivasan’92], based on
  [Awerbuch, Goldberg, Luby, Plotkin’89]
• **MM in** $O(\log^4 n)$ **time.**
  [Hanckowiak, Karonski, Panconesi’99]

• **$O(\Delta \cdot \log n)$-edge-coloring in** $O(\log^4 n)$ **time.**
  [Czygrinow, Hanckowiak, Karonski (ESA’01)]
New Deterministic Results

• \((\Delta + 1)\)-coloring and MIS in \(O(\Delta) + \frac{1}{2} \log^* n\) time.

[Barenboim, E. (ArXiv’08, STOC’09)], [Kuhn (SPAA’09)]

Breaks the Szegedy-Vishwanathan’s \(\Omega(\Delta \log \Delta)\) barrier.

Major Open Problem:
The lower bound is only \(\frac{1}{2} \cdot \log^* n\) ([Linial’87]), while the upper bound is \(O(\Delta) + \frac{1}{2} \log^* n\).
• (1) $\Delta^{1+\eta}$-coloring in $O(\log \Delta \cdot \log n)$ time, for any $\eta > 0$.
(2) $O(\Delta)$-coloring in $O(\Delta^\epsilon \cdot \log n)$ time, for any $\epsilon > 0$.

[Barenboim, E. (PODC’10, J.ACM’11)]

Answers Linial’s question in the affirmative.

(In polylogarithmic time one can get $\Delta \cdot 2^{O(\log \Delta / \log \log \Delta)}$-coloring.)

• (1) $\Delta^{1+\eta}$-edge-coloring in $O(\log \Delta + \log^* n)$ time, for any $\eta > 0$.
(2) $O(\Delta)$-edge-coloring in $O(\Delta^\epsilon + \log^* n)$ time, for any $\epsilon > 0$.

[Barenboim, E. (PODC’11)]
Basic Building Blocks for Further Progress

- **Defective coloring:**
  For \((\Delta + 1)\)-coloring in \(O(\Delta) + \log^* n\) time.
  [Barenboim, E. (STOC’09)],
  [Kuhn (SPAA’09)]

  Enables one to bypass the Szegedy-Vishwanathan’s barrier of \(\Omega(\Delta \log \Delta)\) for locally-iterative algorithms.

- **Arbdefective coloring:**
  For \(\Delta^{1+\eta}\)-coloring in \(O(\log \Delta \cdot \log n)\) deterministic time.
  [Barenboim, E. (PODC’10, J.ACM’11)]

  Answering in the affirmative Linial’s open question.
(\(\Delta + 1\))-Coloring in \(O(\Delta) + \log^* n\) Time
(Defective Coloring)

[Burr, Jacobson’85], [Harary, Jones’86], [Cowen, Cowen, Woodall’86]

**Def:** The *defect* of a vertex \(v\) wrt coloring \(\varphi\) is the number of neighbors \(u \in \Gamma(v)\) with \(\varphi(u) = \varphi(v)\).

**Def:** The *defect* \(d\) of a \(k\)-coloring \(\varphi\) is the maximum defect of a vertex wrt \(\varphi\). \(\varphi\) is called a *\(d\)-defective \(k\)-coloring*.

**Thm:** [Lovasz’66]
\(\forall G, \forall p\) there exists a \([\Delta/p]\)-defective \(p\)-coloring of \(G\).
Proof of Lovasz’s Thm

φ - an arbitrary $p$-coloring.
(Not necessarily legal or $\Delta/p$-defective.)

\[ \text{while } \exists v \text{ with } \text{defect}(v) > \Delta/p \text{ do} \]
\[ \{
\]
\[ \varphi(v) \leftarrow \text{the color used by}
\]
\[ \text{min. } \#\text{neighbors of } v; \]
\[ \} \]

\[ \Delta = 5, \]
\[ p = 2, \]
\[ \text{there exists a color used by } 2 < 5/2 \]
\[ \text{neighbors} \]
\( \text{ME}_i \) - the total \#monochromatic edges before iteration \( i \) starts.

\[ \text{ME}_{i+1} = \text{ME}_i - \text{defect}(v) + \left\lfloor \frac{\Delta}{p} \right\rfloor < \text{ME}_i. \]

But \( 0 \leq ME_i \leq |E| \), and so within a finite number of iterations this process terminates.
Distributed Counterparts of Lovasz’s Theorem

**Thm:** [Barenboim,E. (STOC’09)]
\[\forall G, \forall p \left\lfloor \Delta/p \right\rfloor\text{-defective } O(p^2)\text{-coloring of } G\]
can be computed in \(O(\Delta^\epsilon) + \frac{1}{2} \log^* n\) time, \(\forall \epsilon > 0\).

**Thm:** [Kuhn (SPAA’09)]
\[\forall G, \forall p \left\lfloor \Delta/p \right\rfloor\text{-defective } O(p^2)\text{-coloring of } G\]
can be computed in \(O(\log^* \Delta) + \frac{1}{2} \log^* n\) time.

**Open:** can one efficiently achieve a linear (in \(\Delta\)) product of defect and \#colors?

**Partial answer:** for edge-coloring it is possible.
Also, for vertex-coloring of graphs with bounded independence.
[Barenboim,E. (PODC’11)]
(Δ + 1)-Coloring Algorithm

- Compute $O\left(\frac{\Delta}{\log \Delta}\right)$-defective $\log^2 \Delta$-coloring of $G$ in $o(\Delta) + O(\log^* n)$ time. ($p = \log \Delta$)

- Each color class induces a subgraph with maximum degree $\Delta' = O\left(\frac{\Delta}{\log \Delta}\right)$. Subgraphs are vertex-disjoint.

- In parallel, compute $(\Delta' + 1)$-coloring in each of the $\log^2 \Delta$ subgraphs in $O(\Delta' \log \Delta' + \log^* n) = O(\Delta + \log^* n)$ time, using KW algorithm.

- Overall we get $O((\Delta' + 1) \log^2 \Delta) = O(\Delta \log \Delta)$-coloring $\varphi$ of the entire original graph.

(Using distinct palettes.)
Invoke KW iterative procedure.

Given $\alpha$-coloring it returns $(\Delta + 1)$-coloring in $O(\Delta \cdot \log \frac{\alpha}{\Delta})$ time.

For $\alpha = \Delta \log \Delta$, the time is $O(\Delta \log \log \Delta)$.

Overall running time is $O((\Delta + 1) \cdot \log \log \Delta + \log^* n) + o(\Delta)$.

This is a self-improving scheme!

Now we have $(\Delta + 1)$-coloring algorithm that runs in $O(\Delta \log \log \Delta + \log^* n)$ time.

- Compute $O\left(\frac{\Delta}{\log \log \Delta}\right)$-defective ($\log \log \Delta)^2$-coloring in $o(\Delta) + O(\log^* n)$ time.

- $\Delta' = \frac{\Delta}{\log \log \Delta}$. 
Compute \((\Delta' + 1)\)-coloring of each subgraph in \(O(\Delta' \log \log \Delta' + \log^* n) = O(\Delta + \log^* n)\) time.

• Combine these colorings into an \(O(\Delta \log \log \Delta)\)-coloring of \(G\) (in zero time).

• Reduce the \(O(\Delta \cdot \log \log \Delta)\)-coloring via KW iterative procedure into a \((\Delta + 1)\)-coloring within \(O(\Delta \cdot \log^{(3)} \Delta + \log^* n)\) additional time.

Overall we get \((\Delta + 1)\)-coloring in \(O(\Delta \cdot \log^{(3)} \Delta + \log^* n)\) time.

\[\Downarrow\]

Repeating this argument \(\log^* \Delta\) times we get \((\Delta + 1)\)-coloring in \(O(\Delta + \log^* n)\) time.
A tradeoff (an application)

∀\(t\), \(O(\Delta \cdot t)\)-coloring in \(O(\Delta/t + \log^* n)\) time. (Interpolates between Linial’s \(O(\Delta^2)\)-coloring in \(\log^* n\) time, and our \((\Delta + 1)\)-coloring in \(O(\Delta + \log^* n)\) time.)

- Compute \((\Delta/t)\)-defective \(O(t^2)\)-coloring in \(O(\log^* n)\) time.

- We get \(O(t^2)\) vertex-disjoint subgraphs, each with \(\Delta' \leq \Delta/t\).

Compute \((\Delta' + 1)\)-coloring of each, in parallel, in \(O(\Delta' + \log^* n) = O(\Delta/t + \log^* n)\), using the last result for \((\Delta' + 1)\)-coloring.

- Combine the colorings in zero time to get \(O(t^2 \cdot \Delta') = O(\Delta \cdot t)\)-coloring, in total \(O(\Delta/t + \log^* n)\) time.
Open Questions

1. A \((\Delta + 1)\)-coloring or an MIS in deterministic polylogarithmic time?

Or at least \(O(\Delta)\)-coloring.

Currently we have \(\Delta \cdot 2^{O\left(\frac{\log \Delta}{\log \log \Delta}\right)}\)-coloring.

2. A \(\Delta^{2-\epsilon}\)-coloring in sublogarithmic deterministic time?

3. A \((\Delta + 1)\)-coloring in \(o(\Delta) + \frac{1}{2} \log^* n\) time?

Or a lower bound?

Currently we have \(O(\Delta) + \frac{1}{2} \log^* n\) time.
4. $\Delta/p$-defective $O(p)$-coloring in deterministic polylogarithmic time? (Known for edge-coloring, and for vertex-coloring of graphs with bounded neighborhood independence.)

5. $(2a + 1)$-coloring faster than in $O(a^2 \log n)$ time?
$(2 + \eta) \cdot a$-coloring faster than in $O(a \log n)$ time?

We know
$(2 + \eta)^{1/\epsilon}a$-coloring in $O(a^\epsilon \cdot \log n)$ time, and $a^{1+\eta}$-coloring in $O(\log a \cdot \log n)$ time.

There is also a lower bound of $\Omega \left( \frac{\log n}{\log a} \right)$ for $O(a^2)$-coloring.

So unlike graphs with bounded degree, for graphs of bounded arboricity one cannot hope for sublogarithmic time.
6. MIS or MM in randomized $o(\log n)$ time, for all values of $\Delta$ (or $a$)?

7. Randomized MIS in planar graphs in $o(\log^{2/3} n)$ time? Or a lower bound?
Thank you!!