Distributed Symmetry Breaking

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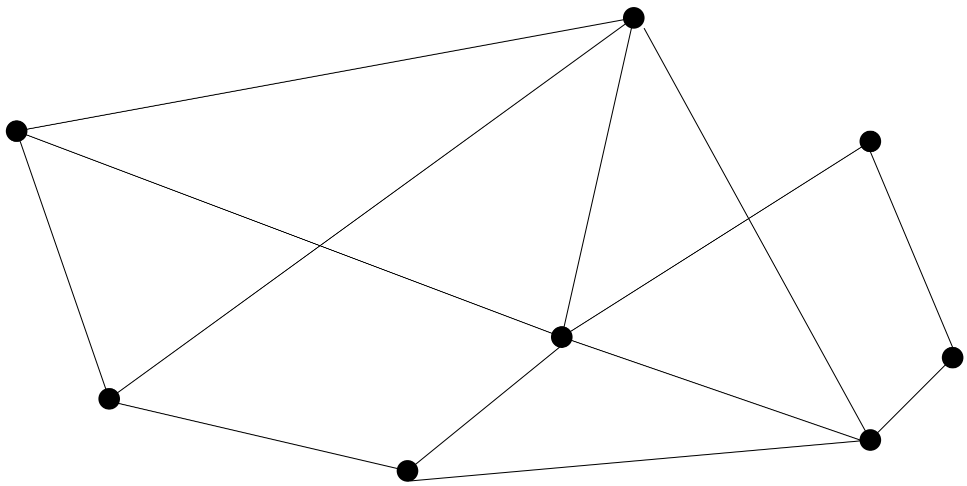
joint w/ L. Barenboim
(PODC'08, STOC'09, PODC'10,
PODC'11, J. ACM'11)

and w/ L. Barenboim, S. Pettie and
J. Schneider (FOCS'12)

and w/ S. Pettie and H. Su (SODA'15)

The Model



- Unweighted undirected graph $G = (V, E)$.
- Vertices host processors.
- Processors communicate over edges of G.
- Communication is synchronous, i.e., occurs in *discrete* rounds.

- Running time $=$ # rounds.

- All vertices wake up simultaneously.

- Vertices have unique Ids from $\{1, 2, \ldots, n\} = [n]$.

- Arbitrarily large messages are allowed, though short (of size $O(\log n)$) are preferred.

Coloring

- $\Delta = \Delta(G)$ - maximum degree of a vertex in G.

- $\varphi : V \to [k]$ is a k-*vertex-coloring* if $\forall e = (u, w) \in E, \quad \varphi(u) \neq \varphi(w)$.

- $\psi : E \to [t]$ is a t-*edge-coloring* if $\forall e, e'$ s.t. $e \cap e' \neq \emptyset, \psi(e) \neq \psi(e')$.

- In distributed setting, typically $k \geq \Delta + 1,\ t \geq 2\Delta - 1$.

- MIS U:
  (1) $\forall v, w \in U, \quad (v, w) \notin E$.
  (2) $\forall v \notin U, \quad \exists u \in U$ s.t. $(u, v) \in E$.

- MM M:
  (1) $\forall e, e' \in M, e \cap e' = \emptyset$.
  (2) $\forall e' \notin M, \exists e \in M$ s.t. $e \cap e' = \emptyset$.

- $(\Delta + 1)$-coloring in $O(n)$ rounds is easy.

  Color vertices one-by-one:
  For each new vertex v there are
  $\leq \Delta$ forbidden colors.

  Hence there is always an available color
  for v in $[\Delta + 1]$.

- MIS in $O(n)$ rounds is easy too.

  Initialize $U \leftarrow \emptyset$;
  Treat vertices one-by-one:
  For each new vertex v do:
  if $\Gamma(v) \cap U = \emptyset$ then
  v joins U;

- $(2\Delta - 1)$-edge-coloring reduces to
  $(\Delta + 1)$-vertex-coloring,
  MM and $(\Delta + 1)$-vertex-coloring
  reduce to MIS.

Elementary Color Reduction Technique

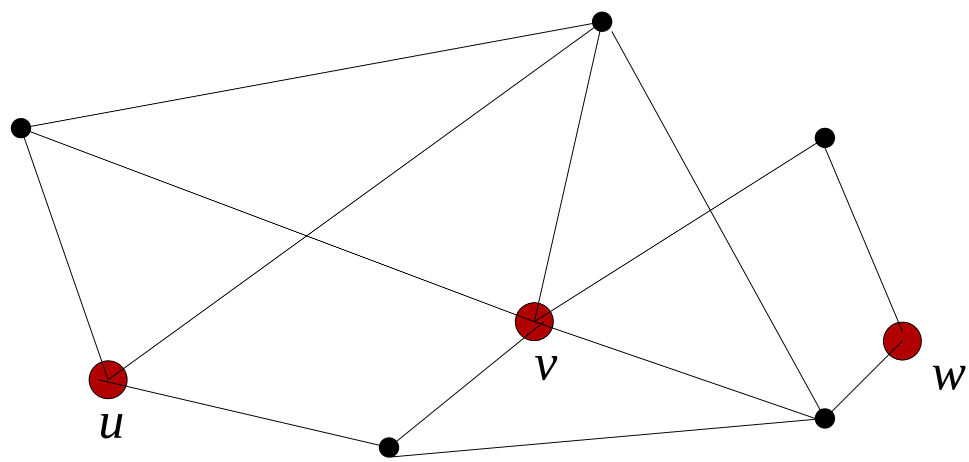
Given an α-coloring, $\alpha > \Delta + 1$,
eliminate one color class in each round.

Vertices of color α form an independent set.

Each of them recolors itself into an available color from $[\Delta + 1]$.

So in $\alpha - (\Delta + 1)$ rounds we get
a $(\Delta + 1)$-coloring.

Contunue with it for $\Delta + 1$ more rounds
to get an MIS.



Kuhn-Wattenhofer's (KW) Color Reduction Technique

$(\Delta+1)$-coloring in $O(\Delta \log \frac{\alpha}{\Delta+1}) + \log^* n$ time.
[Kuhn,Wattenhofer (PODC'06)]
[Szegedy,Vishwanathan (STOC'92)]

- Given an α-coloring φ,
  $\alpha = c \cdot (\Delta+1)$,
  c is a large integer power of 2.

- $\forall i \in [c]$, let

$$U_i = \{v \quad | \quad (i-1)\cdot(\Delta+1)+1 \leq \varphi(v) \leq i \cdot (\Delta+1)\}.$$

- Pair subgraphs $G(U_1)$ with $G(U_2)$,
  $G(U_3)$ with $G(U_4)$,...,
  $G(U_{c-1})$ with $G(U_c)$.

Consider $G(U_1 \cup U_2)$.
It is $2 \cdot (\Delta + 1)$-colored by φ.

- Reduce the $2(\Delta + 1)$-coloring of $G(U_1 \cup U_2)$ to get a $(\Delta + 1)$-coloring of $G(U_1 \cup U_2)$ in $2(\Delta + 1) - (\Delta + 1) = \Delta + 1$ rounds.

  In parallel, reduce the colorings of
  $G(U_3 \cup U_4), G(U_5 \cup U_6), \ldots$
  In $\Delta + 1$ rounds we get
  $\frac{1}{2}\alpha$-coloring of G.

- Keep halving the #colors
  by phases that last $\Delta + 1$ rounds each.

  In $\log \frac{\alpha}{\Delta+1}$ phases
  (i.e., in $O(\Delta \cdot \log \alpha / \Delta)$ time)
  we get $(\Delta + 1)$-coloring.

- [Linial (FOCS'87)]:
  $O(\Delta^2)$-coloring in $\log^* n$ time.

  In conjunction with the KW color reduction we get $O(\Delta \log \Delta) + \log^* n$ time for $(\Delta + 1)$-coloring.

- *Locally-iterative* means: in every round every vertex recolors itself based only on colors of its neighbors.

  [Szegedy,Vishwanathan (STOC'92)]:
  Any *locally-iterative* $(\Delta + 1)$-coloring algorithm requires $\Omega(\Delta \log \Delta)$ time.

  The $(\Delta + 1)$-coloring algorithms of Linial and of Kuhn and Wattenhofer can be casted as locally-iterative.

  So the KW is an optimal locally-iterative $(\Delta + 1)$-coloring algorithm.

Distributed Coloring - Known Randomized Results

- $(\Delta + 1)$-coloring, MIS and MM in $O(\log n)$ time.
  [Luby (STOC'85)],
  [Alon,Babai,Itai (J.Alg.'86)],
  [Israeli,Itai (IPL'86)].

  $(\Delta + 1)$-vertex-coloring in $O(\log \Delta + \sqrt{\log n})$ time.
  [Schneider,Wattenhofer (PODC'10)].

- $O(\Delta)$-vertex-coloring in $O(\sqrt{\log n})$ time
  [Kothapalli,Scheideler,Onus, Schindelhauer (IPDPS'06)].

- $O(\Delta + \log n)$-vertex-coloring in $O(\log \log n)$ time,
  and $O(\Delta \log^{(c)} n + \log^{1+1/c} n)$-coloring in $O(f(c)) = O(1)$ time.
  [Schneider,Wattenhofer (PODC'10)].

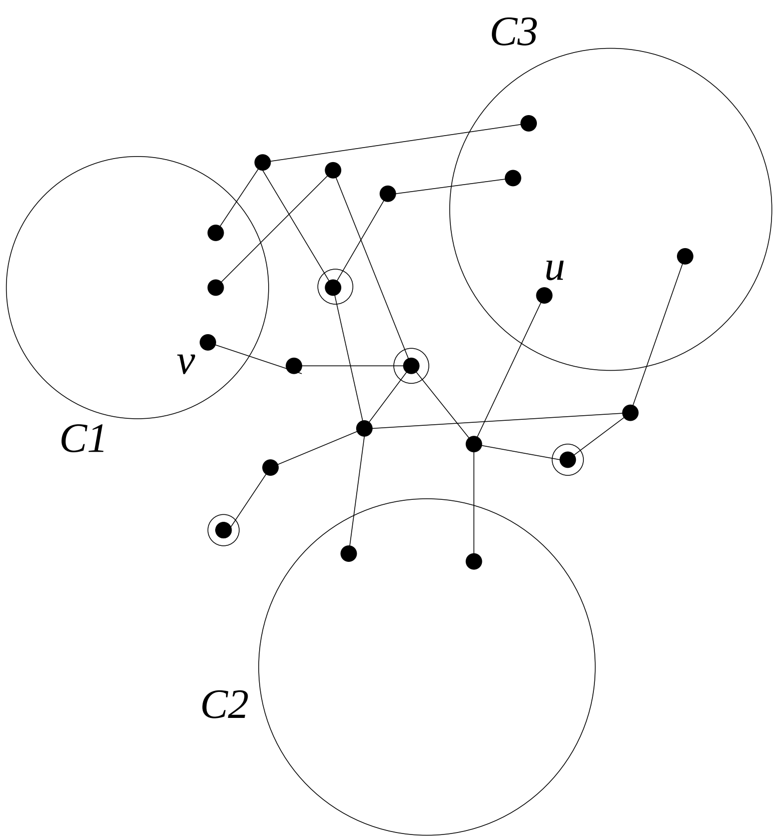
New Randomized Algorithms

[Barenboim, E., Pettie, Schneider (FOCS'12)]

- MM in $O(\log\Delta + \log^4 \log n)$ time.
- $(\Delta + 1)$-coloring in $O(\log\Delta) + exp\{O(\sqrt{\log\log n})\}$ time.
- $O(\Delta)$-coloring in $exp\{O(\sqrt{\log\log n})\}$ time.
- $\Delta^{1+\eta}$-coloring in $O(\log^2 \log n)$ time.
- MIS in $O(\log^2\Delta) + exp\{O(\sqrt{\log\log n})\}$ time.
- $\Delta^{1+\eta}$-edge-coloring in $O(\log\log n)$ time.

Basic Approach in BEPS's algorithms

- Do (roughly) $O(\log\Delta)$ "Luby steps" to break the graph into disconnected components of size $s \leq \text{polylog}(n)$.



$|C1|, |C2|, |C3| \leq s$.

- Use the state-of-the-art *deterministic* MIS algorithm for each component.

  It completes the MIS within additional $exp\{O(\sqrt{\log s})\} \leq exp\{O(\sqrt{\log\log n})\}$ time. [Panconesi,Srinivasan'92]

  Using randomized subroutine within components fails because the failure probability is $1/\text{poly}(s) \approx 1/\text{polylog}(n)$.

- Works similarly for $(\Delta + 1)$-vertex-coloring and MM problems.

  For MM the second step requires just $O(\log^4 s) = O(\log^4 \log n)$ time. [Hanckowiak,Karonski,Panconesi'99]

- Improved deterministic algorithms give rise to improved randomized ones!

New Randomized Algorithms (Cont.)

[E., Pettie, Su (SODA'15)]

$(2\Delta - 1)$-edge-coloring
in $exp\{O(\sqrt{\log\log n})\}$ time.

Establishes a *separation* between
MM and $(2\Delta - 1)$-edge-coloring.

MM requires $\Omega(\sqrt{\log n})$ time.

[Kuhn, Moscibroda, Wattenhofer'10].

Open if such a separation exists also between MIS and $(\Delta + 1)$-vertex-coloring.

Our algorithm (in SODA'15) gives actually $(\Delta + 1)$-vertex-coloring in $O(\log 1/\epsilon) + exp\{O(\sqrt{\log\log n})\}$ time for $(1-\epsilon)$-*locally-sparse* graphs.

Def: $G = (V, E)$ of max' degree Δ is $(1-\epsilon)$-*locally-sparse* if

$$\forall v,\ \ |E(\Gamma(v))| < (1-\epsilon)\binom{\Delta}{2}$$

Line graphs are $(1/2 + o(1))$-locally-sparse, implying our result for edge-coloring.

More Details about MIS Algorithm (BEPS)

Procedure Halve -
reduces the degree from Δ to $\Delta/2$.

Consists of two parts.

Part I: Runs κ Luby trials.
κ will be set
(1) either as $\kappa = O(\sqrt{\log n})$,
(2) or as $\kappa = O(\log \Delta)$.

The graph of remaining high-degree vertices breaks into connected components.

In case (1) these components have weak diameter $O(\sqrt{\log n})$.

In case (2) they have size $O(\Delta^4 \cdot \log n)$.

Procedure Halve (Cont.)

To complete computing the MIS
for high-degree vertices ($deg \geq \Delta/2$) we need

(1) $O(\sqrt{\log n})$ time in case (1) via brute-force.

(2) $exp\{O(\sqrt{\log O(\Delta^4 \log n)})\}$ time in case (2) via the state-of-the-art deterministic MIS algorithm due to [Panconesi,Srinivasan'92].

So Procedure Halve requires total

(1) $O(\sqrt{\log n})$ time in case (1).

(2) $O(\log \Delta) + exp\{O(\sqrt{\log \Delta + \log \log n})\} = exp\{O(\sqrt{\log \Delta + \log \log n})\}$ time in case (2).

Hence the entire running time is
$O(\sqrt{\log n} \cdot \log \Delta)$ in case (1),
and $O(\log \Delta \cdot exp\{O(\sqrt{\log \Delta + \log \log n})\})$
in case (2).

The latter expression is $exp\{O(\sqrt{\log \log n})\}$,
whenever $\Delta \le polylog(n)$.

A better bound
$O(\log^2 \Delta + exp\{O(\sqrt{\log \log n})\})$:
not in this talk.

Part I of Procedure Halve

Run κ Luby trials.

$I_1 \leftarrow \emptyset,\ A_1 \leftarrow V$.

I_i - the independent set before the beginning of iteration i.

$A_i \leftarrow V \setminus \widehat{\Gamma}(I_i)$ - active vertices before iteration i.

$I_{\kappa+1}$ - the independent set after all the κ Luby trials.

Part II works on $G(V \setminus \widehat{\Gamma}(I_{\kappa+1}))$.

This subgraph decomposes into components of either bounded weak diameter or size.

Procedure Halve: Part I

Run for κ iterations ($i = 1, 2, \ldots, \kappa$)
on the *entire* graph G, i.e.,
on both high-degree ($deg \geq \Delta/2$)
and low-degree ($deg < \Delta/2$) vertices.

(Assume the maximum degree is at most Δ.)

1. Each $v \in A_i$ selects itself with probability $\frac{1}{\Delta+1}$, i.a.r.

2. $I_{i+1} \leftarrow I_i \cup \{v \mid v \textit{ is the only vertex in } \hat{\Gamma}(v) \textit{ which selected itself}\}$

Procedure Halve: Analysis of Part I

$\Gamma_i(u)$ - set of active neighbors of a vertex u at the beginning of iteration i.

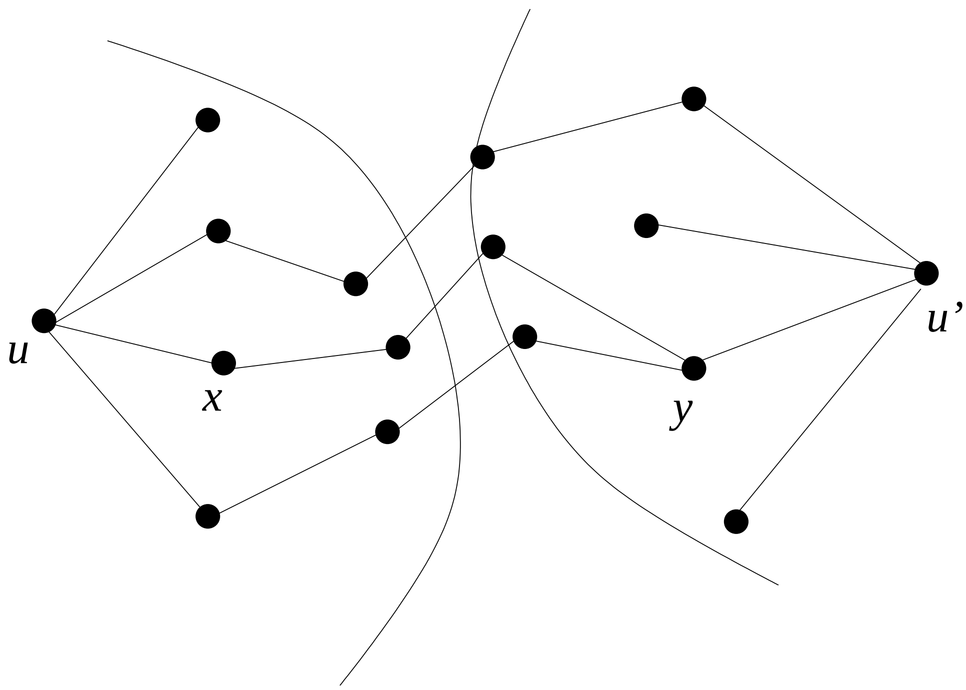
$deg_i(u) = |\Gamma_i(u)|$

Def: A subset $S \subseteq V$ is *well-separated* if $\forall\ u, u' \in S,\ u \neq u', dist_G(u, u') \geq 5$.

Set-Survival Lemma:
Let $S \subseteq A_i$ be a well-separated set of active vertices with $deg_i(u) \geq \Delta/2$ each.
Then

$$\mathbb{P}(S \subseteq A_{i+1}) \leq p^{|S|},$$

for some *constant* $0 < p < 1$.



Neighborhoods of neighbors are disjoint.

In a particular Luby trial,
survival of a vertex u depends only
upon coins tosses of its 2-neighborhood.

Hence vertices u, u' at distance ≥ 5
from one another are independent.

Proof of Set-Survival Lemma

Sufficient to prove

$$\mathbb{P}(u \in A_{i+1} \mid u \in A_i) \leq p \ .$$

Let $u = v_0, \{v_1, v_2, \ldots, v_d\} = \Gamma_i(u)$.

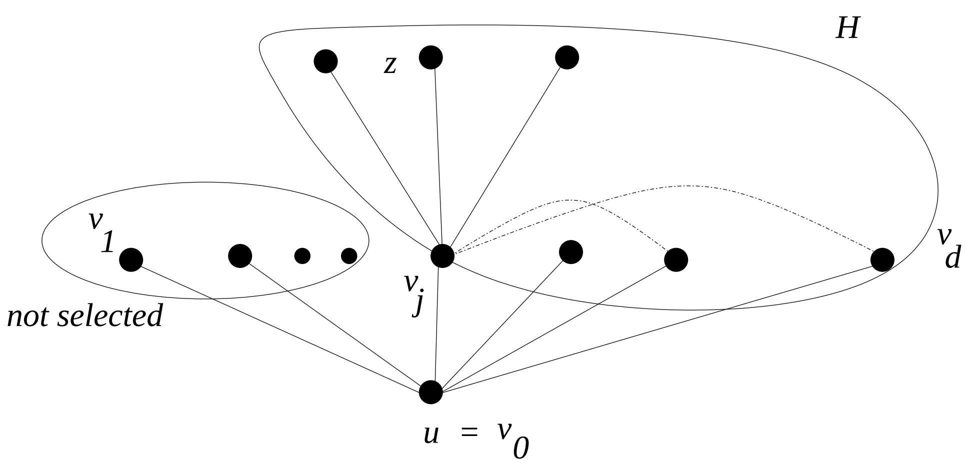
By Lemma's assumption,
$d = deg_i(u) \geq \mathit{\Delta}/2$.

$$\mathbb{P}(\textit{some vertex } \in \{v_0, \ldots, v_d\} \textit{ selects itself})$$
$$= 1 - (1 - \frac{1}{\Delta + 1})^{d+1}$$
$$\geq 1 - (1 - \frac{1}{\Delta + 1})^{\Delta/2+1} > 1 - e^{-1/2} \ .$$

Proof of Set-Survival Lemma (Cont.)

Conditioned upon the event
$\textit{some vertex} \in \{v_0, \ldots, v_d\}$ *selects itself*:

j - the smallest index such that
v_j selects itself.



$$H = \Gamma_i(v_j) \setminus \{v_0, \ldots, v_j\}$$

$$\begin{aligned}
&\mathbb{P}(v_j \ joins \ MIS \mid v_j \ selects \ itself, \\
&j \ is \ the \ smallest) = \\
&\mathbb{P}(\Gamma_i(v_j) \setminus \{v_0, \ldots, v_{j-1}\} \\
&\ did \ not \ select \ themselves) = \\
&= (1 - \frac{1}{\Delta + 1})^{|\Gamma_i(v_j) \setminus \{v_0, \ldots, v_{j-1}\}|} \geq \\
&\geq (1 - \frac{1}{\Delta + 1})^{\Delta} \ \geq \ e^{-1}.
\end{aligned}$$

$$\begin{aligned}
&\mathbb{P}(u \ becomes \ inactive) \ \geq \\
&\mathbb{P}(some \ v_j \ selects \ itself) \cdot \mathbb{P}(v_j \in I_{i+1} \mid v_j \ exists) \\
&\geq (1 - e^{-1/2}) \cdot e^{-1} \ .
\end{aligned}$$

$$\mathbb{P}(u \ survives) \ = \ 1 - (1 - e^{-1/2}) \cdot e^{-1} \ = \ p \ .$$

$0 < p < 1$

QED

Analysis of Proc. Halve: Long-Path Lemma, Notation

$$U = \{v \in A_{\kappa+1} \mid deg_{\kappa+1}(u) \geq \Delta/2\}$$

Vertices that survived all Luby trials and retained high degree.

Set $\kappa = c \cdot \sqrt{\log n}$ (i.e., case (1)).

$\mathcal{P}$ - the set of all paths (not necessarily shortest) which involve only vertices that *originally had high degree* between pairs u, v of *remote* vertices, i.e., s.t. $dist_G(u, v) \geq 5 \cdot \sqrt{\log n}$.

Long-Path Lemma

Lemma: Each $P \in \mathcal{P}$ contains a well-separated set $Q(P)$ of size $|Q(P)| = \sqrt{\log n}$.

Proof:

$P = (u = u_0, u_1, \ldots, u_\ell = v)$,

$\ell \geq 5\sqrt{\log n}$.

Set $q_1 \leftarrow u_0$.

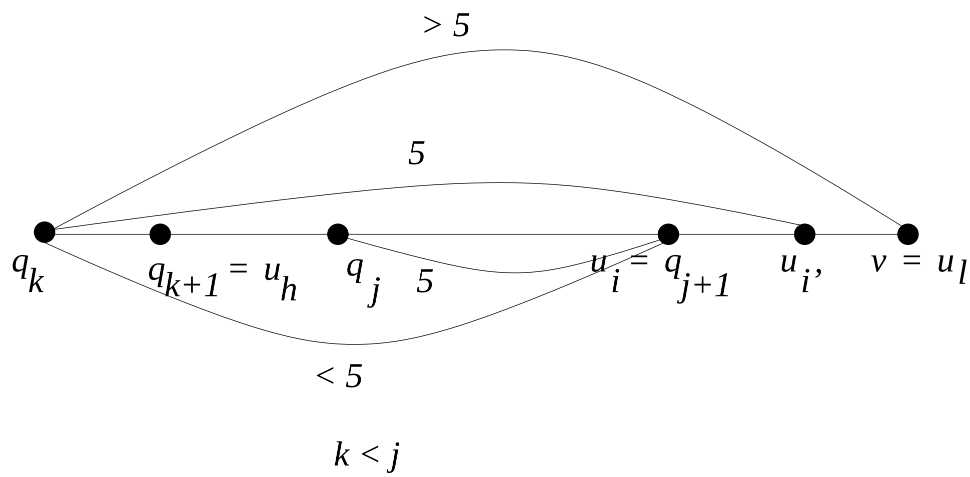
Suppose we have already built
$q_1, q_2, \ldots, q_j$, $j \geq 1$,
with pairwise distances ≥ 5, and
s.t. $dist_G(q_j, v) \geq 5$.

Proof of Long-Path Lemma: Cont.

$q_{j+1} \leftarrow u_i \in P$ with
the highest index s.t. $dist_G(q_j, u_i) = 5$.

(Exists, because $dist_G(q_j, v) \geq 5$.)

Claim: $\forall\ k,\ 1 \leq k \leq j$,
$dist_G(q_k, q_{j+1} = u_i) \geq 5$.



Proof (of Claim):

Otherwise $dist_G(q_k, u_i) < 5$,
but $dist_G(q_k, u_\ell) = dist_G(q_k, v) \geq 5$.

Hence $\exists i',\ i < i' \leq \ell$ s.t.
$dist_G(q_k, u_{i'}) = 5$, contradicting
the maximality of the index h,
$u_h = q_{k+1}$.

QED (Claim)

Proof of Long-Path Lemma: Cont. II

$dist_G(q_1, v) = \ell$, $dist_G(q_1, q_2) = 5$,

$$\begin{aligned} dist_G(q_2, v) + dist_G(q_1, q_2) &\geq \\ dist_G(q_1, v) &= 5\sqrt{\log n}. \end{aligned}$$

Hence

$dist_G(q_2, v) \geq 5(\sqrt{\log n} - 1)$,

$dist_G(q_3, v) \geq 5(\sqrt{\log n} - 2)$,

etc.

We obtain $\sqrt{\log n}$ vertices $q_1, q_2, \ldots, q_{\sqrt{\log n}}$ in a well-separated set.

QED (of Lemma)

Small Components' Lemma, Preliminaries

$\mathcal{Q} = \{Q(P) \mid P \in \mathcal{P}\}$

The set of well-separated subsets obtained from paths of high-degree vertices whose endpoints are at distance $\geq 5\sqrt{\log n}$ one from another in G.

$\mathcal{W}$ - the set of walks of length $5\sqrt{\log n}$.

Every $Q(P) \in \mathcal{Q}$ can be mapped into a walk $W(P) \in \mathcal{W}$.
(By connecting consecutive q_i, q_{i+1} in $Q(P)$ by an arbitrary 5-path.)

$$|\mathcal{Q}| \leq |\mathcal{W}|$$

Also,

$$|\mathcal{W}| \leq n \cdot \Delta^{5\sqrt{\log n}-1}.$$

(n possibilities to pick the first vertex, and Δ possibilities to pick each consecutive one.)

Hence

$$|\mathcal{Q}| \leq |\mathcal{W}| \leq n \cdot \Delta^{5\sqrt{\log n}-1}.$$

Small Components' Lemma

Lemma: After $\kappa = c\sqrt{\log n}$ Luby trials, the vertex set U (of vertices w with $deg_{\kappa+1}(w) \geq \Delta/2$) induces connected components of weak diameter $\leq 5 \cdot \sqrt{\log n}$, whp.

(Meaning that the maximum distance *in* G between every pair of vertices w, w' in the same component is $\leq 5\sqrt{\log n}$.)

Proof of Small Components' Lemma

u, v - a pair of high-degree vertices
with $dist_G(u, v) \geq 5\sqrt{\log n}$.

$P \in \mathcal{P}$ - some path between them that consists of only high-degree vertices.

$$\mathbb{P}(Q(P) \ survives \ a \ single \ iteration) \leq p^{|Q(P)|} \ .$$

The probability of the entire $Q(P)$ to survive κ iterations in all of which all its vertices keep being high-degree is

$$\leq p^{|Q(P)| \cdot \kappa} = exp\{-\Omega(\sqrt{\log n} \cdot c\sqrt{\log n})\} \ = \ n^{-c}$$

Proof of Small Components' Lemma, Cont.

Number of possible Q-sets is
$|\mathcal{Q}| \leq |\mathcal{W}| \leq n \cdot \Delta^{5\sqrt{\log n}}$.

For $\Delta \leq 2^{\sqrt{\log n}}$, we get a bound of
$n \cdot 2^{5 \log n} = n^6$.

Probability that some $Q(P)$ survives is
$\leq \frac{n^6}{n^c} = \frac{1}{n^{c-6}} = \frac{1}{poly(n)}$.

Hence, whp, every path $P \in \mathcal{P}$ which connects a pair u, v with $dist_G(u, v) \geq 5\sqrt{\log n}$ and such that all vertices of $V(P)$ are high-degree, *does not (entirely) survive*, i.e., $V(P) \not\subseteq U$.

Hence, whp, only vertices u, v with $dist_G(u, v) < 5\sqrt{\log n}$ may be connected *in* U.

For $\Delta \geq 2^{\sqrt{\log n}}$, the Luby's bound of $O(\log n)$ is better than the bound of $O(\log \Delta \sqrt{\log n})$.

So in both cases we have
MIS in $O(\log \Delta \sqrt{\log n})$ time.

In the paper we show a much better bound of $O(\log^2 \Delta) + exp(O(\sqrt{\log \log n}))$.

Lower Bounds vs. Upper Bounds

- $f(\Delta)$-coloring requires $\frac{1}{2}\log^* n$ time.

  [Linial (FOCS'87)]

  The upper bound (BEPS)
  for $(\Delta+1)$-coloring is
  $O(\log\Delta) + \exp\{O(\sqrt{\log\log n})\}$.

  Huge gap!

- Coloring Δ-regular trees in $o(\sqrt{\Delta})$ colors requires $\omega(\log_\Delta n)$ time.

  [Linial (FOCS'87)]

  One can color unoriented forests in
  Δ^ϵ colors within $O(\log_\Delta n)$ time,
  for an arbitrarily small $\epsilon > 0$.

  [Barenboim,E. (PODC'08)] (tight).

- $\Omega(\log \Delta)$ and $\Omega(\sqrt{\log n})$ time is required for MIS and MM.

  [Kuhn,Moscibroda,Wattenhofer], [(PODC'04), (ArXiv'10)]

  The upper bound (BEPS) for MM is $O(\log \Delta + \log^4 \log n)$.

  Tight for $\log^4 \log n \leq \log \Delta \leq \sqrt{\log n}$.

  For MIS the BEPS's upper bound is $O(\log^2 \Delta) + \exp\{O(\sqrt{\log \log n})\}$.

Known Deterministic Results

- $(\Delta+1)$-coloring and MIS in $O(\Delta^2 + \log^* n)$ time, and in $O(\Delta \log n)$ time.
  [Goldberg,Plotkin,Shannon'87]
  (based on [Cole,Vishkin'86])

- $O(\Delta^2)$-coloring in $\log^* n + O(1)$ time.
  [Linial'87]

  Asked: can one get much fewer than Δ^2 colors in time polylogarithmic in n?

- $(\Delta+1)$-coloring and MIS
  in $2^{O(\sqrt{\log n})}$ time. (Large messages)
  [Panconesi,Srinivasan'92], based on
  [Awerbuch,Goldberg,Luby,Plotkin'89]

- MM in $O(\log^4 n)$ time.
  [Hanckowiak,Karonski,Panconesi'99]

- $O(\Delta \cdot \log n)$-edge-coloring in $O(\log^4 n)$ time.

  [Czygrinow,Hanckowiak,Karonski (ESA'01)]

New Deterministic Results

- $(\Delta + 1)$-coloring and MIS in $O(\Delta) + \frac{1}{2}\log^* n$ time.

  [Barenboim,E. (ArXiv'08,STOC'09)], [Kuhn (SPAA'09)]

  Breaks the Szegedy-Vishwanathan's $\Omega(\Delta \log \Delta)$ barrier.

  Major Open Problem:
  The lower bound is only $\frac{1}{2} \cdot \log^* n$ ([Linial'87]),
  while the upper bound is $O(\Delta) + \frac{1}{2}\log^* n$.

- (1) $\Delta^{1+\eta}$-coloring in $O(\log \Delta \cdot \log n)$ time, for any $\eta > 0$.
  (2) $O(\Delta)$-coloring in $O(\Delta^{\epsilon} \cdot \log n)$ time, for any $\epsilon > 0$.

  [Barenboim,E. (PODC'10,J.ACM'11)]

  Answers Linial's question in the affirmative.

  (In polylogarithmic time one can get $\Delta \cdot 2^{O(\log \Delta / \log \log \Delta)}$-coloring.)

- (1) $\Delta^{1+\eta}$-edge-coloring in $O(\log \Delta + \log^* n)$ time, for any $\eta > 0$.

  (2) $O(\Delta)$-edge-coloring in $O(\Delta^{\epsilon} + \log^* n)$ time, for any $\epsilon > 0$.

  [Barenboim,E. (PODC'11)]

Basic Building Blocks for Further Progress

- **Defective coloring:**
  For $(\Delta + 1)$-coloring in $O(\Delta) + \log^* n$ time.
  [Barenboim,E. (STOC'09)],
  [Kuhn (SPAA'09)]

  Enables one to bypass the
  Szegedy-Vishwanathan's barrier
  of $\Omega(\Delta \log \Delta)$ for
  locally-iterative algorithms.

- **Arbdefective coloring:**
  For $\Delta^{1+\eta}$-coloring
  in $O(\log \Delta \cdot \log n)$ deterministic time.
  [Barenboim,E. (PODC'10,J.ACM'11)]

  Answering in the affirmative
  Linial's open question.

$(\Delta + 1)$-Coloring in $O(\Delta) + \log^* n$ Time (Defective Coloring)

[Burr,Jacobson'85],[Harary,Jones'86]
[Cowen,Cowen,Woodall'86]

Def: The *defect* of a vertex v wrt coloring φ is the number of neighbors $u \in \Gamma(v)$ with $\varphi(u) = \varphi(v)$.

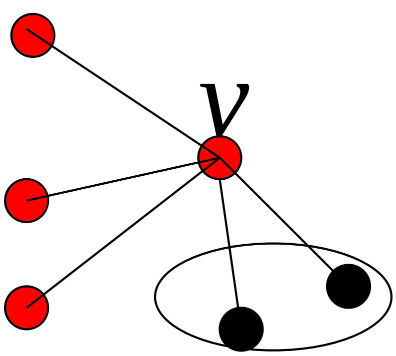
Def: The *defect* d of a k-coloring φ is the maximum defect of a vertex wrt φ. φ is called a d-*defective* k-*coloring*.

Thm: [Lovasz'66]
$\forall G$, $\forall p$ there exists a $\lfloor \Delta/p \rfloor$-defective p-coloring of G.

Proof of Lovasz's Thm

φ - an arbitrary p-coloring.
(Not necessarily legal or Δ/p-defective.)

while $\exists v$ with defect$(v) > \Delta/p$ *do*
{
$\varphi(v) \leftarrow$ the color used by
min. #neighbors of v;
}



Delta = 5,
p = 2,
there exists
a color used
by 2 < 5/2
neighbors

ME_i - the total #monochromatic edges before iteration i starts.

$$\mathsf{ME}_{i+1} = \mathsf{ME}_i - \mathsf{defect}(v) + \lfloor \frac{\Delta}{p} \rfloor < \mathsf{ME}_i.$$

But $0 \leq ME_i \leq |E|$, and so within a finite number of iterations this process terminates.

Distributed Counterparts of Lovasz's Theorem

Thm: [Barenboim,E. (STOC'09)]
$\forall G$, $\forall p$ $\lfloor \Delta/p \rfloor$-defective $O(p^2)$-coloring of G can be computed in $O(\Delta^\epsilon) + \frac{1}{2}\log^* n$ time, $\forall \epsilon > 0$.

Thm: [Kuhn (SPAA'09)]
$\forall G$, $\forall p$ $\lfloor \Delta/p \rfloor$-defective $O(p^2)$-coloring of G can be computed in $O(\log^* \Delta) + \frac{1}{2}\log^* n$ time.

Open: can one efficiently achieve a linear (in Δ) product of defect and #colors?

Partial answer: for *edge*-coloring it is possible.
Also, for vertex-coloring of graphs with bounded independence.
[Barenboim,E. (PODC'11)]

$(\Delta + 1)$-Coloring Algorithm

- Compute $O\left(\frac{\Delta}{\log \Delta}\right)$-defective $\log^2 \Delta$-coloring of G in $o(\Delta) + O(\log^* n)$ time.
  $(p = \log \Delta)$

- Each color class induces a subgraph with maximum degree $\Delta' = O\left(\frac{\Delta}{\log \Delta}\right)$.

  Subgraphs are vertex-disjoint.

- In parallel, compute $(\Delta' + 1)$-coloring in each of the $\log^2 \Delta$ subgraphs in $O(\Delta' \log \Delta' + \log^* n) = O(\Delta + \log^* n)$ time, using KW algorithm.

- Overall we get $O((\Delta' + 1) \log^2 \Delta) = O(\Delta \log \Delta)$-coloring φ of the entire original graph.

  (Using distinct palettes.)

- Invoke KW iterative procedure.

  Given α-coloring it returns
  $(\Delta + 1)$-coloring in $O(\Delta \cdot \log \frac{\alpha}{\Delta})$ time.

  For $\alpha = \Delta \log \Delta$,
  the time is $O(\Delta \log \log \Delta)$.

  Overall running time is
  $O((\Delta + 1) \cdot \log \log \Delta + \log^* n) + o(\Delta)$.

This is a self-improving scheme!

Now we have $(\Delta + 1)$-coloring algorithm that runs in $O(\Delta \log \log \Delta + \log^* n)$ time.

- Compute $O\left(\frac{\Delta}{\log \log \Delta}\right)$-defective
  $(\log \log \Delta)^2$-coloring
  in $o(\Delta) + O(\log^* n)$ time.

- $\Delta' = \frac{\Delta}{\log \log \Delta}$.

Compute $(\Delta' + 1)$-coloring of each subgraph in
$O(\Delta' \log\log \Delta' + \log^* n) = O(\Delta + \log^* n)$ time.

- Combine these colorings into an $O(\Delta \log\log \Delta)$-coloring of G (in zero time).

- Reduce the $O(\Delta \cdot \log\log \Delta)$-coloring via KW iterative procedure into a $(\Delta + 1)$-coloring within $O(\Delta \cdot \log^{(3)} \Delta + \log^* n)$ additional time.

  Overall we get $(\Delta + 1)$-coloring in $O(\Delta \cdot \log^{(3)} \Delta + \log^* n)$ time.

$$\Downarrow$$

Repeating this argument $\log^* \Delta$ times we get $(\Delta + 1)$-coloring in $O(\Delta + \log^* n)$ time.

A tradeoff (an application)

$\forall t$, $O(\Delta \cdot t)$-coloring in $O(\Delta/t + \log^* n)$ time. (Interpolates between Linial's $O(\Delta^2)$-coloring in $\log^* n$ time, and our $(\Delta+1)$-coloring in $O(\Delta + \log^* n)$ time.)

- Compute (Δ/t)-defective $O(t^2)$-coloring in $O(\log^* n)$ time.

- We get $O(t^2)$ vertex-disjoint subgraphs, each with $\Delta' \leq \Delta/t$.

  Compute $(\Delta'+1)$-coloring of each, in parallel, in $O(\Delta' + \log^* n) = O(\Delta/t + \log^* n)$, using the last result for $(\Delta'+1)$-coloring.

- Combine the colorings in zero time to get $O(t^2 \cdot \Delta') = O(\Delta \cdot t)$-coloring, in total $O(\Delta/t + \log^* n)$ time.

Open Questions

1. A $(\Delta + 1)$-coloring or an MIS in deterministic polylogarithmic time?

   Or at least $O(\Delta)$-coloring.

   Currently we have $\Delta \cdot 2^{O(\frac{\log \Delta}{\log \log \Delta})}$-coloring.

2. A $\Delta^{2-\epsilon}$-coloring in sublogarithmic deterministic time?

3. A $(\Delta + 1)$-coloring in $o(\Delta) + \frac{1}{2} \log^* n$ time? Or a lower bound?

   Currently we have $O(\Delta) + \frac{1}{2} \log^* n$ time.

4. Δ/p-defective $O(p)$-coloring in deterministic polylogarithmic time? (Known for edge-coloring, and for vertex-coloring of graphs with bounded neighborhood independence.)

5. $(2a+1)$-coloring faster than in $O(a^2 \log n)$ time?
$(2+\eta) \cdot a$-coloring faster than in $O(a \log n)$ time?

We know
$(2+\eta)^{1/\epsilon} a$-coloring in $O(a^{\epsilon} \cdot \log n)$ time, and $a^{1+\eta}$-coloring in $O(\log a \cdot \log n)$ time.

There is also a *lower bound* of $\Omega\left(\frac{\log n}{\log a}\right)$ for $O(a^2)$-coloring.

So unlike graphs with bounded degree, for graphs of bounded arboricity one cannot hope for sublogarithmic time.

6. MIS or MM in
   randomized $o(\log n)$ time,
   for *all* values of Δ (or a)?

7. Randomized MIS in planar graphs
   in $o(\log^{2/3} n)$ time?
   Or a lower bound?

Thank you!!