An Improved Construction of Progression-Free Sets

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The Problem

Numbers i, j, ℓ form an *arithmetic triple* if $i = \frac{j+\ell}{2}$ or $j = \frac{i+\ell}{2}$ or $\ell = \frac{i+j}{2}$.

A subset $S \subseteq \{1, 2, \ldots, n\}$ is *progression-free* if it contains no arithmetic triple.

$\nu(n)$ - the largest size of a progression-free subset $S \subseteq \{1, 2, \ldots, n\}$.

$$\nu(n) \;=\; ?$$

Previous Research - Lower Bounds

[van der Waerden, 1927]

Closely related problem.

$\forall\ k, \ell \quad \exists\ n = n(k, \ell)$ such that
$\forall$ partition $S_1 \cup S_2 \cup \ldots \cup S_k = \{1, 2, \ldots, n\}$
$\exists\ S_i$ that contains an arithmetic progression of length ℓ.

[Erdos,Turan, 1936]

Introduced $\nu(n)$ and showed that

$$\nu(n) = \Omega(n^{\log_3 2})$$

Previous Research - Lower Bounds (Cont.)

[Salem,Spencer, 1942]

$$\nu(n) = \Omega\left(\frac{n}{2^{\frac{c\log n}{\log\log n}}}\right),$$

for a constant $c > 0$.

[Behrend, 1946]

$$\nu(n) = \Omega\left(\frac{n}{2^{2\sqrt{2}\sqrt{\log_2 n}} \cdot \log^{1/4} n}\right)$$

No improvement since then!

Previous Research

Upper Bounds:

[Roth, 1953]

$\nu(n) = O(\frac{n}{\log\log n})$

[Bourgain, 1999,2007]

$\nu(n) = O(\frac{n}{\log^{2/3} n}(\log\log n)^2)$

Related work:

[Rankin, 1960] - excluding arithmetic progressions of length $k \geq 4$.

[Ruzsa, 1993], [Shapira, 2006], [Koester, 2008] - excluding more general subsequences.

[Szemeredi, 1975] - any dense subset of integers contains an arithmetic progression of any length.

[Green,Tao, 2004] - primes contain an arithmetic progression of any length.

Our Result

$$\nu(n) \;=\; \Omega\left(\frac{n}{2^{2\sqrt{2}\sqrt{\log_2 n}}} \cdot \log^{1/4} n\right)$$

Improves the result of Behrend
by a factor of $\Theta(\sqrt{\log n})$.

Proves that

- Behrend's construction is *not optimal*.

- $\frac{n}{2^{2\sqrt{2}\sqrt{\log_2 n}}}$ is *not the right expression*.

Consequent Work

- [Green,Wolf] Arxiv, Oct. 2008
  "A Note on Elkin's Improvement of Behrend's Construction"

  Alternative (much shorter) proof of our result.

  Citing [Green,Wolf]:

  "The only advantage of our approach is its brevity: it is based on ideas morally close to those of Elkin, and moreover, his argument is more constructive than ours."

- [O'Bryant] Arxiv, Nov. 2008

  Combines our (and Green-Wolf's) technique with that of [Rankin,60].

  Gets slightly improved bounds on sets without k-term arithmetic progressions, for $k \geq 4$.

  Was anticipated in [Elkin] Arxiv, Jan. 2008.

The Geometric Application

Def: $U \subseteq \mathbb{R}^k$ is a *convexly independent set (CIS)* if $ConvHull(U) = U$.

$C_k(R)$ - maximum size of a CIS of integer k-vectors of norm $\leq R$.

[Jarnik,1925] $C_2(R) = \Theta(R^{2/3})$.

[Arnold,80],[Balog,Barany,91]

More precise estimates for $C_2(R)$.

[Andrews,63] $C_k(R) = O(R^{k-2+\frac{2}{k+1}}), \quad \forall k \geq 3$.

[Barany,Larman, 98]

$$C_k(R) = \Omega(R^{k-2+\frac{2}{k+1}}), \quad \forall k \geq 3.$$

Quite elaborate proof - uses results from the approximation theory of smooth convex bodies by polytopes, and Khintchine's Flatness Theorem.
Not constructive!

Our result: a constructive algorithmic proof that

$\forall k \geq 5$: $C_k(R) = \Omega(R^{k-2+\frac{2}{k+1}})$

$k = 4$: $C_4(R) = \Omega\left(\frac{R^{12/5}}{(\log\log R)^{2/5}}\right)$

$k = 3$: $C_3(R) = \Omega(R^{3/2-\epsilon}), \quad \forall \epsilon > 0$

Elementary, self-contained, except for standard estimates on discrepancy between volume and number of integer points in large balls.

Overview of Behrend's Construction

k - a positive integer parameter.

$y = \frac{n^{1/k}}{2}$.

Select an integer point v
uniformly at random from $C = \{0, 1, \ldots, y-1\}^k$.

$Z = ||v||^2$

$\mu_Z \approx \frac{k}{3} \cdot y^2$

$\sigma_Z = \Theta(\sqrt{k} \cdot y^2)$

Behrend's Construction (Cont.)

By Chebyshev inequality

$$\mathbb{P}(|Z - \mu_Z| > 2\sigma_z) \;\leq\; \frac{1}{4}$$

$$\Downarrow$$

$\geq \frac{3}{4} \cdot |C|$ of all vectors of C
belong to the (thick) annulus $\tilde{\mathcal{S}}$ given by

$$\tilde{\mathcal{S}} = \{v \;:\; \mu_Z - 2\sigma_Z \leq ||v||^2 \leq \mu_Z + 2\sigma_Z\}.$$

Let $K = C \cap \tilde{\mathcal{S}}$.
(Discrete cube intersected with
the thick annulus.)

$$|K| \geq \frac{3}{4} \cdot |C| = \frac{3}{4} y^k.$$

Since every $v \in K$ is an integer point,
$||v||^2$ may accept $\leq 4\sigma_Z + 1$ values
between $\mu_Z - 2\sigma_Z$ and $\mu_Z + 2\sigma_Z$.

By PHP, $\exists T$ such that

$$\mu_Z - 2\sigma_Z \leq T \leq \mu_Z + 2\sigma_Z$$

and $\geq \frac{|K|}{4\sigma_Z+1} \geq \frac{3}{4} \cdot \frac{1}{4\sigma_Z+1} \cdot |C| = \frac{3}{4} \cdot \frac{y^k}{4\sigma_Z+1} = \Omega(\frac{y^k}{\sigma_Z})$

vectors $v \in K$ satisfy $||v||^2 = T$.

$\mathcal{S}$ - the set of integer points of C
that belong to the sphere P
of squared norm T.

Since $\sigma_Z = \Theta(\sqrt{k} \cdot y^2)$,

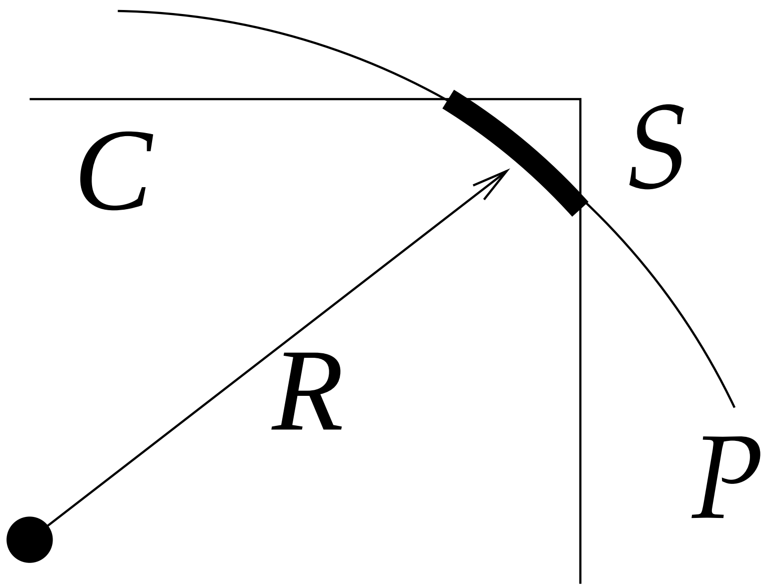
$$|\mathcal{S}| \;=\; \Omega(\frac{y^k}{\sigma_Z}) \;=\; \Omega(\frac{1}{\sqrt{k} \cdot y^2} \cdot y^k) \;=\; \Omega(\frac{y^{k-2}}{\sqrt{k}}) \;.$$

$R = \sqrt{T} \approx \sqrt{\mu_Z \pm 2\sigma_Z} \approx \sqrt{\frac{k}{3}} \cdot y$.

Set $k = \sqrt{2 \cdot \log_2 n}$.

$$\Downarrow$$

$$|\mathcal{S}| = \Omega(\frac{y^{k-2}}{\sqrt{k}}) = \Omega\left(\frac{n}{2^{2\sqrt{2}\sqrt{\log_2 n}} \cdot \log^{1/4} n}\right).$$



Via Freiman isomorphism, construct a progression-free $S \subseteq \{1, 2, \ldots, n\}$, with $|S| = |\mathcal{S}|$. Hence

$$|S| = \Omega\left(\frac{y^{k-2}}{\sqrt{k}}\right) = \Omega\left(\frac{n}{2^{2\sqrt{2}\sqrt{\log_2 n}} \cdot \log^{1/4} n}\right).$$

For Freiman isomorphism to work it is required that $\mathcal{S}$ will be a CIS.

Behrend Construction versus Our Construction

Behrend: Points are on a *sphere*.

Our construction: Replace the sphere by a *thin annulus*.

A hurdle: A set $\mathcal{Y}$ of integer points in an annulus is *not* necessarily convexly independent.

Our solution: Construct a large convexly independent subset (CIS) $Q = Ext(\mathcal{Y})$ of $\mathcal{Y}$.

The annulus is sufficiently thin so that

$$|Q| \geq \frac{|\mathcal{Y}|}{2} \ .$$

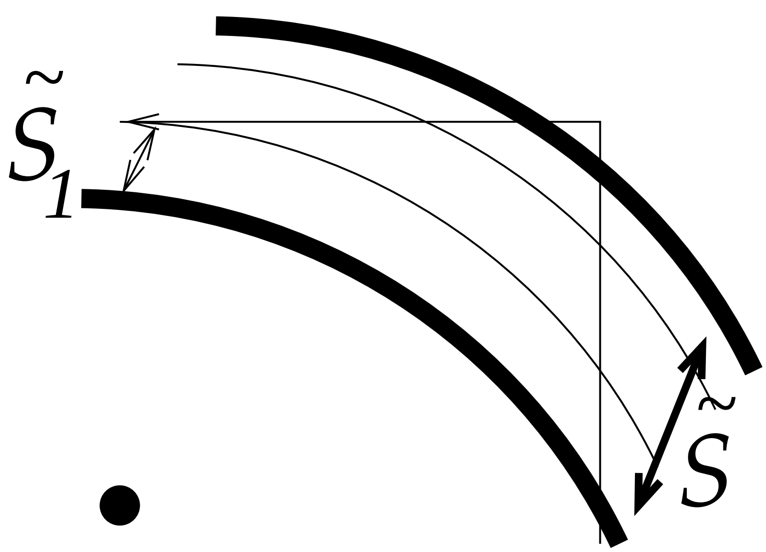
Constructing the Thin Annulus

The rv Z: the squared norm of a point selected uniformly at random from $C = \{0, 1, \ldots, y-1\}^k$.

The (thick) annulus $\tilde{\mathcal{S}}$: integer points of squared norm between $\mu_Z - 2 \cdot \sigma_Z$ and $\mu_Z + 2 \cdot \sigma_Z$.

$$|\tilde{\mathcal{S}}| \;=\; \Omega(y^k).$$

Partition $\tilde{\mathcal{S}}$ into disjoint thin annuli $\tilde{\mathcal{S}}_1, \tilde{\mathcal{S}}_2, \ldots$ of (squared) width $g = \epsilon \cdot k, \quad \epsilon > 0$.



Constructing the Annulus (Cont.)

Each $\tilde{\mathcal{S}}_i$ has (squared) width g:

$$\tilde{\mathcal{S}}_1 = \{v : \mu_Z - 2\sigma_Z \leq ||v||^2 < \mu_Z - 2\sigma_Z + g\}$$
$$\tilde{\mathcal{S}}_2 = \{v : \mu_Z - 2\sigma_Z + g \leq ||v||^2 < \mu_Z - 2\sigma_Z + 2g\}$$

$$\vdots$$

$\#$annuli $\approx \frac{4\sigma_Z}{g}$

Pick $\tilde{\mathcal{S}}_i$ that contains the largest number of integer points of $\tilde{\mathcal{S}}$.

By PHP, $\exists i$ such that

$$\begin{aligned} |\tilde{\mathcal{S}}_i| &= \Omega(\frac{|\tilde{\mathcal{S}}|}{4\sigma_Z/g}) = \Omega(\frac{y^k}{\sigma_Z} \cdot g) \\ &= \Omega(\frac{y^k}{\sqrt{k} \cdot y^2} \cdot \epsilon k) = \Omega(y^{k-2}\sqrt{k}). \end{aligned}$$

- $\sqrt{k}$ is in the numerator!

$$\Downarrow$$

The factor of $k = \Theta(\sqrt{\log n})$ is gained!

- But $\tilde{\mathcal{S}}_i = \hat{\mathcal{S}}$ is *not* convexly independent.

- We show that $\hat{\mathcal{S}}$ contains a CIS $\check{\mathcal{S}}$ such that

$$\begin{aligned} |\check{\mathcal{S}}| &\geq \frac{|\hat{\mathcal{S}}|}{2} = \Omega(y^{k-2} \cdot \sqrt{k}) \\ &= \Omega\left(\frac{n}{2^{2\sqrt{2}\sqrt{\log_2 n}}} \cdot \log^{1/4} n\right). \end{aligned}$$

- Via Freiman isomorphism, this CIS translates into a progression-free set $\check{S}$, $|\check{S}| = |\check{\mathcal{S}}|$.

It is left to construct a CIS $\check{\mathcal{S}}$

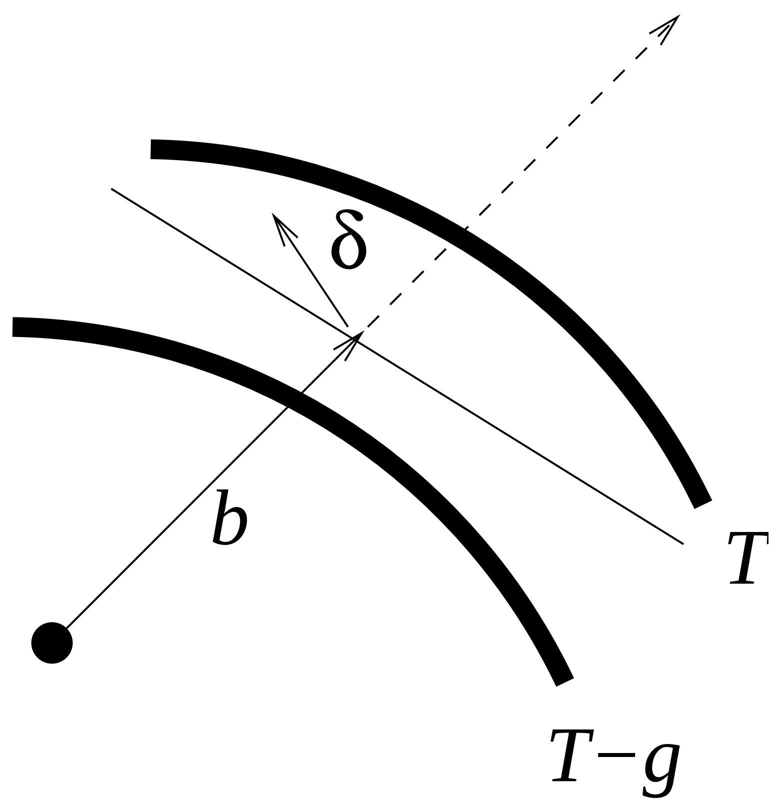
of size $|\check{\mathcal{S}}| \geq \frac{|\hat{\mathcal{S}}|}{2}$.

A Key Lemma

$B = B(\sqrt{T})$ - the set of integer points of the k-ball of squared radius T, centered at the origin.

$Ext(B)$ - the exterior set of B.

Lemma: Let $b \in B \setminus Ext(B)$ such that $T - g \leq ||b||^2 \leq T$.
Then $\exists$ integer vector $\delta \neq 0$ such that $0 \leq \langle b, \delta \rangle \leq g$ and $0 < ||\delta||^2 \leq g$.



Wiping off Points

By the Lemma, there is a collection of hyperplanes $\mathcal{H}(\delta, h) = \{b \mid \langle b, \delta \rangle = h\}$, integer h, δ, $0 \le h \le g$, $\quad 0 < ||\delta||^2 \le g$, that contains all points b of $B \setminus Ext(B)$ that are "close" to the surface of B.

("Close" means $T - g \le ||b||^2 \le T$.)

Intuition: There are "few" hyperplanes, and each wipes off only a small number of points.

So we are left with a large CIS.

The set of points of $\hat{\mathcal{S}}$ wiped off by a hyperplane $\mathcal{H}(\delta, h)$ has dimension $k - 1$, while $\hat{\mathcal{S}}$ has dimension k.

The "length" of $\hat{\mathcal{S}}$ in each dimension is $\approx R = \sqrt{T} \approx y$. Thus each hyperplane wipes off $\le \frac{1}{y}$-fraction of integer points of $\hat{\mathcal{S}}$.

$\#$ hyperplanes $\leq (\#h) \cdot (\#\delta)$

$(\#h) = g + 1$.

$(\#\delta) \leq 2^{\eta \cdot k}$,
for an arbitrarily small $\eta = \eta(\epsilon) > 0$, $g = \epsilon \cdot k$.

$\leq \left(\frac{2^{\eta(\epsilon) \cdot k}(g+1)}{y} \right)$-fraction of points of $\hat{\mathcal{S}}$
is wiped off by *one of the hyperplanes*.

$g \leq k = \Theta(\log y)$.

Set $\epsilon > 0$ to be sufficiently small.
$\eta > 0$ becomes as small as we wish.

$$\Downarrow$$

$$\frac{2^{\eta(\epsilon)\cdot k}(g+1)}{y} \leq \frac{1}{2}$$

Thus $\leq \frac{1}{2}$-fraction of points is wiped off.

$$\Downarrow$$

$$|Ext(\hat{\mathcal{S}})| \;\geq\; \frac{1}{2}|\hat{\mathcal{S}}|$$

$Ext(\hat{\mathcal{S}})$ is a CIS.

Estimating $(\#\delta)$

δ is an integer k-vector
s.t. $0 < ||\delta||^2 \leq g$.

We prove that $(\#\delta) \leq 2^{\eta \cdot k}$

Proof:
$(\#\delta) \approx \ Vol(k\text{-ball of squared radius } g)$

$(\#\delta) \approx \ \beta_k \cdot g^{k/2}$,
where $\beta_k = Vol(\text{unit-radius } k\text{-ball}) = \frac{\pi^{k/2}}{\Gamma(k/2+1)}$.

$$(\#\delta) \approx \frac{(\epsilon \cdot k)^{k/2} (\pi e)^{k/2}}{(k/2)^{k/2}} \approx (2\pi e \cdot \epsilon)^{k/2} = 2^{\eta(\epsilon) \cdot k}.$$

QED

A Hurdle

To prove that each hyperplane wipes off $\leq \frac{1}{y}$-fraction of $\hat{\mathcal{S}}$.

Not obvious because:

$\hat{\mathcal{S}}$ is the set of integer points
of a *very thin* annulus $\mathcal{A}$
($\mathcal{A} = \{b \in \mathbb{R}^k : T - g \leq ||b||^2 \leq T\}$)
intersected with the discrete cube C
in a *very high dimension*.

Its squared radius is $\frac{k}{3}y^2$.

Its squared width is $g = \epsilon \cdot k$.

Its dimension is $k = 2\log y$.

(The dimension grows *logarithmically* with the radius.)

We end up proving a weaker statement:
each hyperplane wipes off
$\frac{2^{c \cdot k}}{y}$-fraction of the set of integer points of $\hat{\mathcal{S}}$
for a *sufficiently* small $c > 0$. $(0 < c < 1/2)$

(Not for an *arbitrarily* small $c > 0$.)

Sufficient for our purposes.

The Proof of the Lemma

Lemma: Let $b \in B \setminus Ext(B)$ such that
$T - g \leq ||b||^2 \leq T$.
Then $\exists \delta \neq 0$ such that $0 \leq \langle b, \delta \rangle \leq g$
and $0 < ||\delta||^2 \leq g$.

Proof:
$\exists a, c \in B$ (integer points)
$\exists p, \quad 0 < p < 1$ such that
$b = p \cdot a + (1 - p) \cdot c$.

$||a||^2, ||c||^2 \leq T$.

If $\langle a, b \rangle, \langle c, b \rangle < ||b||^2$
then $\langle b, b \rangle = p\langle a, b \rangle + (1 - p)\langle c, b \rangle < ||b||^2$,
contradiction.

Hence wlog $\langle a, b \rangle \geq ||b||^2$.

$$\Downarrow$$

$\langle a - b, b \rangle \geq 0$.

Set $\delta = a - b$.
Hence $\langle b, \delta \rangle \geq 0$.

δ is an integer vector.

Since $0 < p < 1$, $\delta \neq 0$.

$$T \geq ||a||^2 = ||b + \delta||^2 = ||b||^2 + 2\langle b, \delta \rangle + ||\delta||^2.$$

But $||b||^2 \geq T - g$.

$$\Downarrow$$

$2\langle b, \delta \rangle + ||\delta||^2 \leq g$.

Since $\langle b, \delta \rangle \geq 0$,
it follows that $\langle b, \delta \rangle, ||\delta||^2 \leq g$.

QED

Counting Points Wiped off by a Hyperplane

$\mathcal{A} = \{b \in \mathbb{R}^k : T - g \leq ||b||^2 \leq T\}$
(continuous annulus)

$\mathcal{C} = [0, y-1]^k$ (continuous hypercube)

$\hat{\mathcal{S}} = \mathcal{A} \cap C$ (continuous annulus
intersected with *discrete* hypercube)

$\mathcal{H} = \mathcal{H}(\delta, h) = \{b \mid \langle b, \delta \rangle = h\}$
(we fix δ and h;
$\mathcal{H}$ is a fixed hyperplane)

$\hat{W} = \hat{W}(\delta, h) = \hat{\mathcal{S}} \cap \mathcal{H} = (\mathcal{A} \cap C) \cap \mathcal{H}$

Our goal is to estimate $|\hat{W}|$, i.e.,
to show that $\frac{|\hat{W}|}{|\hat{\mathcal{S}}|} \leq \frac{2^{c \cdot k}}{y}$,
for an appropriate sufficiently
small constant c, $0 < c < 1/2$.

($y \approx 2^{k/2}$; so c should be $< 1/2$.)

Counting (Cont.)

$\mathcal{A}' = \mathcal{A} \cap \mathcal{H}$

$\mathcal{A}'$ is a $(k-1)$-dimensional annulus,
$\mathcal{A}' \subseteq \mathcal{H}$, centered at $\frac{h}{||\delta||^2} \cdot \delta$,
containing vectors b such that

$$(T - \frac{h^2}{||\delta||^2}) - g \leq ||b - \frac{h}{||\delta||^2} \cdot \delta||^2 \leq (T - \frac{h^2}{||\delta||^2}) \ .$$

$T' = T - \frac{h^2}{||\delta||^2} \leq T$
(the new squared radius).

We need to estimate
$\hat{W} = \hat{\mathcal{S}} \cap \mathcal{H} = \mathcal{A} \cap C \cap \mathcal{H} = \mathcal{A}' \cap C$.

Let $\tilde{W} = \mathcal{A}' \cap \mathcal{C}$ be the continuous analogue of $\hat{W}$, i.e., $\hat{W}$ is the set of integer points of $\tilde{W}$.

We provide an upper bound on $Vol(\tilde{W})$.

The Strategy

- To use estimates on the discrepancy between volume and number of integer points to obtain an upper bound for $\hat{W}$ using an upper bound for $Vol(\tilde{W})$.

- *But* the body $\tilde{W} = \mathcal{A}' \cap \mathcal{C} = \mathcal{A} \cap \mathcal{H} \cap \mathcal{C}$ is complex; it is hard to compute its volume or the discrepancy between volume and number of integer points.

- We build a nicer body $\tilde{Q}$ s.t. $\tilde{W} \subseteq \tilde{Q}$, estimate its volume and discrepancy, and get an upper bound for the number of integer points in $\tilde{W}$.

The Body $\tilde{Q}$

- $\tilde{W} = \mathcal{A}' \cap \mathcal{C}$.

- $\tilde{Q}$ is $\mathcal{A}'$ intersected with a (relatively) small number of orthants, after the axes are appropriately rotated.

- The number of orthants will be $2^{\epsilon \cdot k}$.

  Recall: $g = \epsilon \cdot k$ is the squared width of $\mathcal{A}$; i.e., ϵ is a parameter that we control.

Crude Attempts

Taking $\tilde{Q} = \mathcal{A}'$ is too crude.
Intuitively, we get an extra factor of roughly $2^k \gg y = 2^{k/2}$.

2^k is the number of orthants.

$\mathcal{A}'$ contains points in all orthants in $\mathbb{R}^{k-1}$;
($\mathcal{A}' \subseteq \mathcal{H}$; if we put the origin in the center of $\mathcal{A}'$, and use any orthonormal basis,
then $\mathcal{A}'$ intersects all orthants.)

$\tilde{W} = \mathcal{A}' \cap \mathcal{C}$ is contained in the
positive orthant in $\mathbb{R}^k$ (because $\mathcal{C}$ is).

Using the k-volume of one orthant of $\mathcal{A}$
(rather than $(k-1)$-volume of $\mathcal{A}'$)
is too crude either, as we pay
an extra factor of y.

Rotating the Space

To define the "right" superset $\tilde{Q}$ of $\tilde{W}$ (and subset of $\mathcal{A}'$) we rotate the space.

- $\mathcal{H}' = \{\alpha \in \mathbb{R}^k \mid \langle \alpha, \delta \rangle = 0\}$ is the parallel hyperplane to $\mathcal{H} = \{\alpha \in \mathbb{R}^k \mid \langle \alpha, \delta \rangle = h\}$ that passes through the origin.

- We'll build an orthonormal basis $\Upsilon = \{\gamma_1, \gamma_2, \ldots, \gamma_{k-1}\}$ for $\mathcal{H}'$.

- Recall: $0 < ||\delta||^2 \leq g = \epsilon \cdot k$, $\delta \in \mathbb{R}^k$, δ is an integer vector.

$$\Downarrow$$

δ may have $\leq \epsilon \cdot k$ non-zero entries.

$I = \{i \in [k] : \delta_i \neq 0\}$.
Let $m = |I| \leq \epsilon \cdot k$.

Defining the Basis

- For each $\alpha = (a_1, a_2, \ldots, a_k) \in \mathcal{H}'$,

$$\sum_{i \in I} a_i \cdot \delta_i = \langle \delta, \alpha \rangle = 0 \ .$$

- $\{\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(m-1)}\}$ -
  an arbitrary orthonormal basis for the solution space of this equation.

  $\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(m-1)} \in \mathrm{I\!R}^m$,

- Define $\widehat{\gamma}^{(1)}, \widehat{\gamma}^{(2)}, \ldots, \widehat{\gamma}^{(m-1)} \in \mathrm{I\!R}^k$ by:
  $\widehat{\gamma}^{(j)}$ agrees with $\gamma^{(j)}$ in all coordinates $i \in I$, and has 0 elsewhere.

- Also, $\forall j \in [k] \setminus I$,
  we insert into the basis Υ the vectors
  $e_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0) \in \mathrm{I\!R}^k$,
  with 1 in the jth entry.

Properties of the New Basis

- $\#$vectors $= m - 1 + k - m = k - 1 = dim(\mathcal{H}')$.
  To complete it to the basis of $\mathrm{I\!R}^k$,
  add the vector $\frac{\delta}{||\delta||}$.

- $\forall j \in [k] \setminus I,\ e_j \in \mathcal{H}'$,
  because δ has 0 in jth entry.

- All vectors have unit norm.

- $\langle \widehat{\gamma}^{(i)}, e_j \rangle = 0$,
  because $\widehat{\gamma}^{(i)}$ has 0 in each entry $j \in [k] \setminus I$.

- Hence $\Upsilon = (\widehat{\gamma}^{(1)}, \ldots, \widehat{\gamma}^{(m-1)}) \circ (e_j \mid j \in [k] \setminus I)$
  is an orthonormal basis for $\mathcal{H}'$.

- We do not know much about the
  $m - 1 \leq \epsilon \cdot k - 1$ vectors $\widehat{\gamma}^{(1)}, \widehat{\gamma}^{(2)}, \ldots, \widehat{\gamma}^{(m-1)}$.
  But the other $k - m$ vectors of this basis
  are e_j's, $j \in [k] \setminus I$.

Changing the Axes

Notation: $\Upsilon = \{\gamma_1, \gamma_2, \ldots, \gamma_{k-1}\}$.

- Move the origin to the point $\frac{h}{||\delta||^2} \cdot \delta$ (the center of the annulus $\mathcal{A}'$).

- Rotate the space so that the axes become colinear with vectors of Υ and with $\frac{\delta}{||\delta||}$.

- This rotation is volume-preserving.

- For a vector $\tau \in \mathcal{H}'$,
  $\tau_1[\Upsilon], \tau_2[\Upsilon], \ldots, \tau_{k-1}[\Upsilon]$ -
  the coordinates of τ in the new basis.

$$\tau_i[\Upsilon] = \langle \tau - \frac{h}{||\delta||^2} \cdot \delta, \gamma_i \rangle$$

Properties of Υ

Lemma: $\forall \tau \in \tilde{W} = \mathcal{A}' \cap \mathcal{C}$, and
$\forall i \in \{m, m+1, \ldots, k-1\}$,
$\tau_i[\Upsilon] \geq 0$.

Hence in this basis τ has $\geq (1-\epsilon) \cdot k$ non-negative coordinates.

Proof: $\langle \tau - \frac{h}{||\delta||^2} \cdot \delta, \gamma_i \rangle = \langle \tau, \gamma_i \rangle$
(because $\langle \delta, \gamma_i \rangle = 0$, because $\gamma_i \in \mathcal{H}'$, and $\mathcal{H}' \perp \delta$)

Observe that $\tau \in \mathcal{C}$.

Thus, its coordinates are non-negative.

For $i \in \{m, m+1, \ldots, k-1\}$,
$\gamma_i = e_{j_i}$, for some $j_i \in [k] \setminus I$.

Hence $\langle \tau, \gamma_i \rangle = \langle \tau, e_{j_i} \rangle \geq 0$. QED

Defining $\tilde{Q}$

Recall: $\tilde{Q}$ is a superset of $\tilde{W}$
(but a subset of $\mathcal{A}'$),
which is nicer to work with.

$(\tilde{W} = \mathcal{A}' \cap \mathcal{C} = \mathcal{A} \cap \mathcal{H} \cap \mathcal{C})$

$$\begin{aligned}\tilde{Q} \;=\; &\{\alpha \in \mathcal{A}'(= \mathcal{A} \cap \mathcal{H}) \mid \\ &\forall i \in \{m, m+1, \ldots, k-1\}, \alpha_i[\Upsilon] \geq 0\}\end{aligned}$$

Instead of intersecting $\mathcal{A}'$ with the cube,
we intersect it with the 2^{m-1}
orthants of the rotated space.
The first $m-1$ coordinates may be
either positive or negative;
the rest are non-negative.

By the last lemma,
$\tilde{W} = \mathcal{A}' \cap \mathcal{C} \subseteq \tilde{Q}$.

Let $\mathcal{A}'' = (\mathrm{I\!R}^+)^{k-1} \cap \mathcal{A}'$,
(in the rotated basis).

$\mathcal{A}''$ is one single orthant
of the annulus $\mathcal{A}'$.

$\tilde{Q}$ is 2^{m-1} orthants of $\mathcal{A}'$.

Thus:

$$Vol(\tilde{Q}) = 2^{m-1} \cdot Vol(\mathcal{A}'') \leq 2^{\epsilon k-1} \cdot Vol(\mathcal{A}'')$$

So $Vol(\tilde{W}) \leq Vol(\tilde{Q}) \leq 2^{\epsilon k-1}\, Vol(\mathcal{A}'')$.

Wrapping up

Lm: $Vol(\mathcal{A}'') \leq g \cdot \left(\frac{\pi e}{6}\right)^{k/2} \cdot y^{k-3} \cdot 2^{O(\sqrt{k})}$.

(The proof is by a direct calculation.)

Hence $Vol(\tilde{W}) \leq 2^{\epsilon k} \left(\frac{\pi e}{6}\right)^{k/2} \cdot y^{k-3}$.

It can be shown that
$\hat{W} \leq 2^{\epsilon k} \left(\frac{\pi e}{6}\right)^{k/2} \cdot y^{k-3}$ too,
where $\hat{W}$ is the set of integer points in $\tilde{W}$.

(We argue that the #integer points in $\tilde{Q}$ is at most this. This is easy because $\tilde{Q}$ is "nice", and so the discrepancy between volume and #integer points can be easily estimated. Then we conclude it for $\tilde{W}$, because $\tilde{W} \subseteq \tilde{Q}$.)

Recall that $|\hat{\mathcal{S}}| = \Omega(\epsilon \cdot \sqrt{k} \cdot y^{k-2})$,
where $\hat{\mathcal{S}} = \mathcal{A} \cap C$ is the original point set
(from which $\mathcal{H}$ wipes off the points of $\hat{W}$).

We need to show that $|\hat{W}|$ is a small fraction (ideally, $\frac{1}{y}$-fraction) of $|\hat{\mathcal{S}}|$.

Wrapping Up (Cont.)

Roughly, we compare $\left(\frac{\pi e}{6}\right)^{k/2}$ with $y \approx 2^{k/2}$.

$\left(\frac{\pi e}{6}\right)^{k/2} \approx y^{0.51}$.

So each hyperplane $\mathcal{H}$ wipes off roughly $\leq \frac{1}{\sqrt{y}}$-fraction of all integer points of $\hat{\mathcal{S}}$.

All the $2^{\eta(\epsilon)\cdot k} = y^{2\eta(\epsilon)}$ hyperplanes wipe off $\leq \frac{1}{y^{1/2-2\eta}}$-fraction of all integer points of $\hat{\mathcal{S}}$.

The set $\check{\mathcal{S}}$ of the remaining points contains $\geq (1 - \frac{1}{y^{1/2-2\eta}})$-fraction $\geq$ half of all integer points of $\hat{\mathcal{S}}$, and it is a CIS.

$\check{\mathcal{S}}$ translates into progression-free $\check{S}$ of the same cardinality, via Freiman isomorphism.

QED

Summary and Open Problems

- **Progression-free sets:**
  Our lower bound is

$$\nu(n) = \Omega\left(\frac{n}{2^{2\sqrt{2}\sqrt{\log_2 n}}} \cdot \log^{1/4} n\right).$$

  Bourgain's upper bound is

$$\nu(n) = O\left(\frac{n}{\log^{2/3} n}(\log\log n)^2\right).$$

- O'Bryant showed that these ideas can be extended to k-term progressions.

  Does it extend to improve *Ruzsa's construction*?
  (Dense sets with no solutions to certain Diophantine equations.)