A Near-Optimal Distributed Fully Dynamic Algorithm for Maintaining Sparse Spanners

Michael Elkin

Ben-Gurion University

The Message-Passing Model

- n processors reside in vertices of an unweighted undirected graph $G = (V, E)$.
  Each processor v has a unique Id $I(v)$.

- Interconnected via links of E.

- *Short* messages ($O(\log n)$ bits).

- Unlimited computational power.
  Local computation requires zero time.

The Message-Passing Model (Cont.)

Synchronous setting (for this talk).

- Communication in *discrete* rounds.
- Messages sent in the beginning of a round R, arrive before the round $R+1$ starts.

Running Time = #rounds.

Message Complexity = # messages.

Dynamic Model

Edges and vertices may
appear or crash at will.

The weakest studied model.
(Weaker than controlled and
partially controlled dynamic models.)

- Endpoints of a crashing edge are notified by a link-level protocol.

- A message is lost only if its edge crashes.

Motivation for the dynamic model:
real-life networks,
modern ad-hoc, sensor, wireless networks.

Primitive devices require *simple* algorithms!

Quiescence Complexity

Topology updates cease occuring at time α. β is the time when all vertices stop processing updates. At this point the algorithm maintains a correct structure.

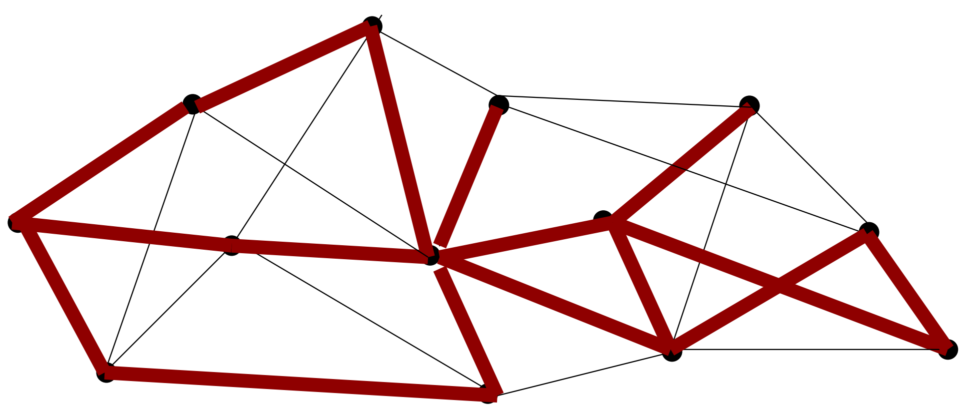
Quiescence time $= \max\{\beta - \alpha\}$.
Quiescence message = # messages sent within $[\alpha, \beta]$.

Spanners

Spanners = skeletons that approximate metric properties.

For $t \geq 1$,
$G' = (V, H)$ is a t-*spanner* of $G = (V, E)$, $H \subseteq E$,
if $\forall\ u, w \in V$,

$$dist_{G'}(u, w) \ \leq \ t \cdot dist_G(u, w) \ .$$



The Basic Tradeoff

[Peleg,Schaffer,89]
$\forall$ graph $\forall t$ $\exists$ $O(t)$-spanner
with $O(n^{1+1/t})$ edges.

The best-known result
[Althofer,Das,Dobkin,Joseph,Soares,90] -
$(2t-1)$-spanner of size $O(n^{1+1/t})$.

An inherent tradeoff between the
stretch parameter and the number of edges.

Optimal under Erdos girth conjecture.

Applications of Spanners

An underlying construct for many distributed algorithms.

- Synchronization.
  [Peleg,Ullman,89],
  [Awerbuch,Peleg,90]

- Routing.
  [Hassin,Peleg,99]

- Approximate Distances and Shortest Paths Computation.
  [Awerbuch,Berger,Cowen,Peleg,93],
  [Elkin,01]

- Broadcast.
  [Awerbuch,Goldreich,Peleg,Vainish,89],
  [Awerbuch,Baratz,Peleg,92]

Distributed Spanners

State-of-the-art distributed *static* algorithm.

[Baswana,Sen,03],
[Baswana,Kavitha,Mehlhorn,Pettie,05]

For $t = 1, 2, \ldots$, and n-vertex G, constructs $(2t-1)$-spanner with expected $O(t \cdot n^{1+1/t})$ edges.

Time: $O(t)$.
Message: $O(|E| \cdot t)$.
Space: $O(deg(v) \cdot \log n)$.

Near-optimal tradeoff.

Dynamic State-of-the-Art

[Baswana,Sen,03] composed with
the simulation technique of
[Awerbuch,Patt-Shamir,Peleg,Saks,92]:
$(2t-1)$-spanner of expected size $O(t \cdot n^{1+1/t})$,
Quiescence time: $O(t \cdot \log^3 n)$.
Quiescence message: $O(t \cdot |E| \cdot \log^3 n)$.
Space: $O(deg(v) \cdot \log^4 n)$.

Drawbacks of APSPS simulation technique:

Extremely *complex* (a reset procedure,
neighborhood covers, a bootstrap technique,
a local rollback).

Heavy local computations -
unsuitable for *simple* devices.

Our Result

$(2t-1)$-spanner of expected size $O(t \cdot n^{1+1/t})$.

Quiescence time: $3t$ instead of $O(t \cdot \log^3 n)$.
Note: $t \leq \log n$.

Quiescence message: worst-case $O(|E| \cdot t)$, expected $O(|E|)$.
Space: $O(deg(v) \cdot \log n)$.
Expected local processing per edge: $O(1)$.

Lower bound: $2t/3$.
$t-1$ under Erdos girth conjecture.

Better performance in purely incremental and purely decremental settings.

In both algorithms: non-adaptive adversary, oblivious to coin tosses.

Additional Features of our Algorithm

- **Treatment time:** If edges stop crashing at time α, but are still allowed to appear, then at time $\alpha + 3t$ the spanner takes care of all edges present at time α.

  Stronger than a bound on quiescence time!

- **Incremental setting:** bound of $2t$.

  If update set F is a matching, quiescence time is 1!

- **Decremental setting:**

  If update set is of size $o(n^{1/t})$, the expected quiescence time is $1 + o(1)$.

Historyless Dynamic Algorithm

Standard approach: maintain *history* of communication, undo operations based on the history.

Very expensive in terms of *local computation*. *Unfeasible* in wireless, sensor, ad-hoc networks.

Our approach: No history is stored!

Look for a “replacement” for crashing edges.

Undo operations, but the list-to-undo is deduced from the current state of affairs.

Reminiscent of *memoryless online* algorithms.

The Incremental Variant: Initialization

Focus on incremental algorithm.

Set a parameter $p \approx n^{-1/t}$.

Each v picks a radius $r = r(v)$ from the truncated geometric distribution

$\mathbb{P}(r = k) = p^k \cdot (1 - p)$, for $k \in [0, \ldots, t-2]$, and $\mathbb{P}(r = t-1) = p^{t-1}$.

Memoryless distribution

$\mathbb{P}(r \geq k+1 \mid r \geq k) = p$ for $k \in [0, 1, \ldots, t-2]$.

[Linial,Saks,92],[Bartal,96]

Labels

Each v has a unique id $I(v)$,
and a label $P(v) = (B(P(v)), L(P(v)))$.

Initially, $P(v) \leftarrow (I(v), 0)$.

$P = (B(P), L(P))$.

Implicitly, the algorithm maintains a *tree cover*.
(A set of not necessarily disjoint trees
that cover all vs.)

$B(P)$ - the id of a tree τ to which
the vertex v labeled by P currently belongs.

$L(P)$ - the distance between v and
the root of τ.

The vertex $w = w_P$ s.t. $I(w) = B(P)$ is
the *base* vertex of P.
w_P is the root of the tree $B(P)$.

$r(w_P)$ - maximum distance to which
$B(P) = I(w_P)$ is allowed to propagate.
The tree $B(P)$ cannot be deeper than $r(w_P)$.

$\Rightarrow$ For each label P, $L(P) \le r(w_P)$.

A label P is *selected* if $L(P) \ < \ r(w_P)$.
In this case v may be
an internal vertex of the tree $B(P)$.

For a label P,
$\mathbb{P}(P \ \text{ is } \ \text{selected}) \ =$
$= \ P(r(w_P) \ge L(P) + 1 \mid r(w_P) \ge L(P)) \ \le$
$\le \ p \ \approx \ 1/n^{1/t}$.

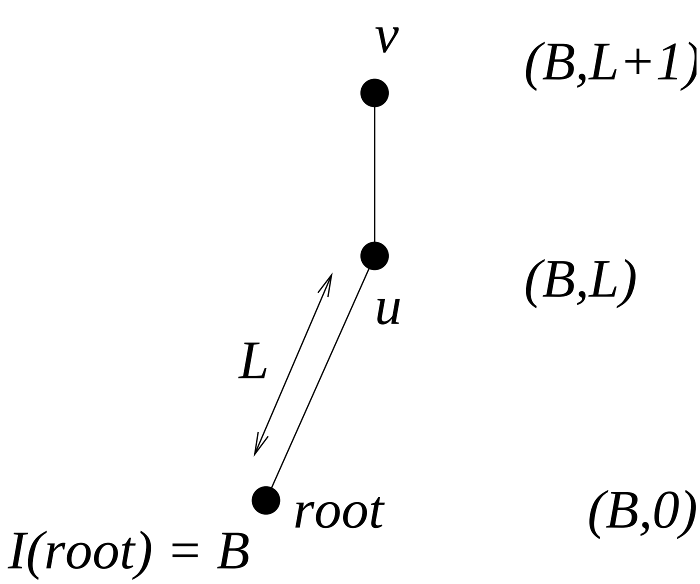
Probability of a label to reach level $t-1$ is
$\mathbb{P}(r = t-1) \ = \ \mathbb{P}(r = t-1 \mid r \ge t-2) \cdot$
$\mathbb{P}(r \ge t-2 \mid r \ge t-3) \cdot \ldots \cdot \mathbb{P}(r \ge 1 \mid r \ge 0) \ =$
$= \ p^{t-1} \ \approx \ \left(\frac{1}{n^{1/t}}\right)^{t-1}.$

Hence, whp, the #labels of level $t-1$
$\approx \ n^{1/t}$.

Comparing labels:

$P(v) \succ P(v')$ iff *either*
$(L(v), B(v)) > (L(v'), B(v'))$ *or*
$(((L(v), B(v)) = (L(v'), B(v'))) \wedge (I(v) > I(v')))$.

Vertices *adopt* labels from their neighbors.
When v adopts a label from u, it becomes its child in the tree $B(P)$, $P = P(u)$.
When a label P is adopted, $L(P)$ is incremented, but $B(P)$ stays unchanged.



Data Structures

Every v maintains an edge set $Sp(v)$.

Initially, $Sp(v) = \emptyset$.

$Sp(v)$ grows monotonely.

$Sp(v) = T(v) \cup X(v)$.

$T(v)$ - the *tree edges* of v.

$X(v)$ - the *cross edges* of v.

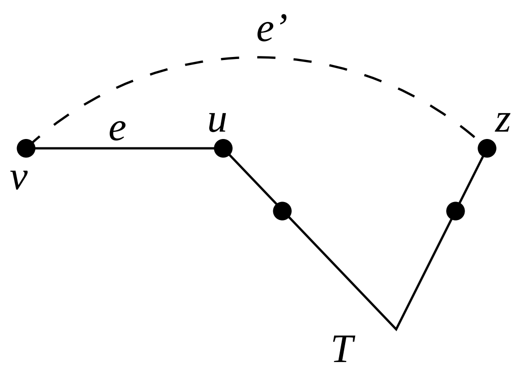
An implicit construction of a *tree cover*. Edges of the tree cover are stored in $T(v)$'s.

The spanner also has edges connecting different trees. Those are edges of $X(v)$'s.

Data Structures (Cont.)

For each vertex v, the algorithm maintains a table $M(v)$.
Initially, $M(v) = \emptyset$.

$M(v)$ is the set of trees to which v is already connected in the spanner.



e'= (v,z) in X(v) ==> B(P(z)) in M(v)
B(P(z)) = B(P(u))

$\Downarrow$

e can be dropped!

The Algorithm (for a vx v)

For $2t$ rounds from the beginning *or* after detecting a new edge do

Go over all received messages and do
while $\exists$ message $P(u)$ with $P(u) \succ P(v)$
// adopt the label of u
  if $P(u)$ is selected
    $B(P(v)) \leftarrow B(P(u))$;
    $L(P(v)) \leftarrow L(P(u)) + 1$;
    add (v, u) to $Sp(v)$; // to $T(v)$
  else if $B(P(u)) \notin M(v)$
    add $B(P(u))$ to $M(v)$;
    add (v, u) to $Sp(v)$; // to $X(v)$
  end-if
end-while
Send to all neighbors the message $P(v)$.

Remark: Testing whether $P(u)$ is selected is done by standard techniques.

The Algorithm: Discussion

Very simple:

1. One type of messages.
2. The same behavior on each round.
3. A handful of local variables.
4. Basic data structures.

Definition - Scanning an Edge

v *scans* $e = (v, u)$ if $P(u)$ passes
the while-loop condition of the vertex v.

It may happen that on a given round,
neither v nor u scan the edge (v, u).
(Due to different *order* in which
v and u process edges.)

Example

At the beginning of a round,
$P(v) \succ P(u)$.

The vertex v considers the message $P(u)$
before all other messages,
and discovers that $P(v) \succ P(u)$.
Thus v does not scan e.

The vertex u considers first
another message $P(z)$.
As a result u increases its label
to $P'(u) \succ P(v) \succ P(u)$,
and then considers the message $P(v)$.

However, since $P'(u) \succ P(v)$,
it does not scan e either.

So, neither v nor u
scan e on this round!

Analysis - Scanning Edges

Lm: Every edge $e = (v, u)$ is eventually scanned.

Pf: v increases its label $\leq t - 1$ times.

The same applies for u.

Hence among $2t$ rounds
$\exists$ round (other than the first)
on which neither v nor u increase their labels.

On this round either v or u scan e.

QED

Analysis - Stretch

Lm: Suppose v was labeled by P at some point.
Then $\exists$ path between w_P and v of length $\leq L(P)$ in $\bigcup_{z\in V} T(z)$.

Pf: Induction on $L(P)$.

For v to get label P,
it must have inserted an edge (v,u) into $T(v)$,
s.t. $L(P(u)) = L(P) - 1, \quad B(P(u)) = B(P)$.

The induction hypothesis is applicable to u.
QED

Cor: If v used to be labeled by P,
and v' by P', and $B(P) = B(P')$, then
$\exists$ path of length $\leq \; L(P) + L(P') \leq 2t - 2$
between v and v' in $\bigcup_{z\in V} T(z)$.

Analysis - Stretch (Cont.)

Lm: If v scans $e = (v, u)$,
from that point on $\exists$ path of length $\leq 2t - 1$
between v and u in the spanner.

Pf:
If $((P(u)$ is selected$)$ or $(B(P(u)) \notin M(v)))$,
$e \in Sp(v)$, and we are done.

If $((P(u)$ is not selected$)$ and $(B(P(u)) \in M(v)))$,
$\exists u'$ s.t. u' used to have label P'
with $B(P') = B(P(u))$, and
$e' = (v, u') \in X(v) \subseteq Sp(v)$.

$$\Downarrow$$

$\exists$ uu'-path of length $\leq 2t - 2$
in the spanner, and
$\exists$ uv-path of length $\leq 2t - 1$. QED

Local Processing

$\tilde{O}(t \cdot n^{1/t})$ space.

Maintain a data structure of
$\tilde{O}(t \cdot n^{1/t})$ base values.

Support existence and insertion queries.

Naively:
$\log \tilde{O}(t \cdot n^{1/t}) = O\left(\frac{\log n}{t} + \log\log n\right)$
time-per-query whp.
(a balanced search tree (BST))

More sophisticatedly:
$o(\log\log n)$ time-per-query whp.
(hash + BST in each entry).

Open question: $O(1)$ whp?

Summary

- Optimal solution for
  the dynamic distributed spanner problem.
- Historyless paradigm for devising
  dynamic distributed algorithms.
- Lower bound of $\Omega(t)$.
- Streaming algorithm.
- Centralized dynamic algorithm.

Open Questions

- Applications for the dynamic distributed spanners: Synchronization (?), Routing (?), Online load balancing (?).

- Applications for the historyless paradigm.

- Achieve $O(1)$ processing time-per-edge whp.

- Achieve spanner size of $O(n^{1+1/t})$ instead of $O(t \cdot n^{1+1/t})$ (even for *static* model).

- Derandomize.
  Less challenging - devise algorithm for an adaptive adversary, or for a non-oblivious one.