Approximating Graph Metric by Spanning Trees (2005; Elkin, Emek, Spielman, Teng)

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SYNONYMS: Lower-Stretch Spanning Trees.

1 PROBLEM DEFINITION

Consider a weighted connected graph $G = (V, E, \omega)$, where ω is a function from the edge set E of G into the set of positive reals. Given a spanning tree T of V, we define the distance in T between a pair of vertices $u, v \in V$, $dist_T(u, v)$, to be the sum of the lengths of the edges on the unique path in T between u and v. We can then define the stretch of an edge $(u, v) \in E$ to be

$$stretch_T(u,v) = \frac{dist_T(u,v)}{\omega(u,v)} \ ,$$

and the average stretch over all edges of E to be

$$avestr_T(E) = \frac{1}{|E|} \sum_{(u,v)\in E} stretch_T(u,v) \ .$$

The average stretch of a graph $G = (V, E, \omega)$ is defined as the smallest average stretch of a spanning tree T of G, $avestr_T(E)$. The average stretch of a positive integer n, $avestr(n)$, is the maximum average stretch of an n-vertex graph G. The problem is to analyze the asymptotic behavior of the function $avestr(n)$.

2 KEY RESULTS

The problem was introduced by Alon et al. [1], who showed that

$$\Omega(\log n) \ = \ avestr(n) \ = \ exp(O(\sqrt{\log n \cdot \log\log n})) \ .$$

Elkin et al. [5] improved the upper bound and showed that

$$avestr(n) \ = \ O(\log^2 n \cdot \log\log n) \ .$$

Dhamdhere et al. [4] further improved the upper bound of [5] to $O(\log^2 n)$.

3 APPLICATIONS

The main currently known application of low stretch spanning trees is for solving symmetric diagonally dominant linear systems of equations. Boman and Hendrickson [2] were the first to discover the surprising relationship between these two seemingly unrelated problems. They applied the spanning trees of [1] to design solvers that run in time $m^{3/2} 2^{O(\sqrt{\log n \log\log n})} \log(1/\epsilon)$. Spielman and

Teng [7] improved their results by showing how to use the spanning trees of [1] to solve diagonally-dominant linear systems in time

$$m2^{O(\sqrt{\log n \log\log n})} \log(1/\epsilon).$$

By applying the low-stretch spanning trees developed in [5], the time for solving these linear systems reduces to

$$m \log^{O(1)} n \log(1/\epsilon),$$

and to $O(n(\log n \log\log n)^2 \log(1/\epsilon))$ when the systems are planar. Applying a recent reduction of Boman, Hendrickson and Vavasis [3], one obtains a $O(n(\log n \log\log n)^2 \log(1/\epsilon))$ time algorithm for solving the linear systems that arise when applying the finite element method to solve two-dimensional elliptic partial differential equations.

4 OPEN PROBLEMS

The most evident open problem is to close the gap between the upper bound of $O(\log^2 n)$ and the lower bound of $\Omega(\log n)$ on $avestr(n)$. Another intriguing subject is the study of low-stretch spanning trees for various restricted families of graphs. Progress in this direction was recently achieved by Emek and Peleg [6].

5 CROSS REFERENCES

None is reported. Entry editors please feel free to add some.

6 RECOMMENDED READING

[1] Noga Alon, Richard M. Karp, David Peleg, and Douglas West. A graph-theoretic game and its application to the k-server problem. *SIAM Journal on Computing*, 24(1):78–100, February 1995. Also available Technical Report TR-91-066, ICSI, Berkeley 1991.

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[4] Kedar Dhamdhere, Anupam Gupta and Harald Racke. Improved embeddings of graph metrics into random trees. In *Proceedings of Symp. on Discr. Algorithms, SODA'06*, pp. 61-69, 2006.

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[7] Daniel A. Spielman and Shang-Hua Teng. Nearly-linear time algorithms for graph partitioning, graph sparsification, and solving linear systems. In *Proceedings of the thirty-sixth annual ACM Symp. on Theory of Computing, STOC'04*, pages 81–90, 2004.