# Optimal back-to-front airplane boarding

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**Abstract**

The problem of finding an optimal back-to-front airplane boarding policy is explored, using a mathematical model introduced by Bachmat *et al.* Optimal back-to-front policies with 2 boarding groups are presented. These lead to a nearly 8% improvement in boarding time over random (no policy) boarding. For policies with more groups a recursive procedure for calculating the optimal policies is described. The results indicate that the improvement beyond 2-group policies is negligible.

## 1 Introduction

Airlines and their passengers alike have a mutual interest in minimizing the time spent at the gate while the passengers are boarding the airplane. For the airlines and airport infrastructure, reducing the boarding time means decreased operational costs and increased passenger throughput capacity. The passengers, in their turn, benefit from reducing the boarding time, because waiting, either in the line at the gate or aboard the airplane is a contributing factor to the overall fatigue and dissatisfaction from the trip.

A common airline strategy aimed at decreasing the boarding time is to employ an *announcement policy*. The boarding is performed in stages, by means of announcements such as, "passengers from row number 40 and above are now welcome to board the airplane; all other passengers, please remain seated". Typically, such policies try to board passengers from the back of the airplane first announcing *groups* of rows whose passengers are allowed to board at the same time. We call such policies *back-to-front* policies. We will study such policies using a mathematical model which was introduced in [6]. The model is based on two dimensional Lorentzian (space-time) geometry. The model was validated against detailed computer simulations in

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[7]. The validation shows that the estimated boarding time produced by the model is a very good predictor for ranking boarding policies. We will use the estimate as our measure of boarding efficiency.

A key parameter of the model is the *congestion factor* $k$, the formal definition of which will be given later in Section 2. Intuitively, $1/k$ measures the fraction of passengers that can stand along the aisle. This is best measured when passengers are preparing to exit the plane. This parameter depends on the aircraft design, namely, on the inter-row distance and the number of passengers per seat. modern airplanes have a congestion factor around $k = 4$.

Mathematically, the set of all back-to-front boarding policies is parametrized by the infinite dimensional simplex $\Delta_\infty$ consisting of all partitions $0 < r_1 < r_2 < ... < r_m < 1$ of the unit interval. For a given policy, $m$ is the number of announcements, or passenger groups. If the airplane has $n$ rows, the corresponding announcements first call passengers in rows $m$ through $[r_m]n + 1$, then rows $[r_m]n$ through $[r_{m-1}]n$ and so on. Estimated boarding time is normalized so that for random boarding (no policy), estimated boarding time is 1. In this way, estimated boarding time is a non-negative continuous function on $\Delta_\infty$. Since this space is not compact there is a priori no reason to think that estimated boarding time achieves a minimal value and in fact it does not. Instead we would like to consider the infimum of the function and near optimal policies whose estimated boarding time is close to the infimum. It is not clear a priori that the infimum is positive. Indeed, in the unrealistic case where $k \leq 1$ (all passengers can stand in the aisle at the same time), boarding policies can be arbitrarily good. A detailed analysis of the rate in which boarding time approaches 0, over a larger class of back-to-front boarding policies, when $k = 0$, (cardboard thin passengers) was presented in [5].

The situation changes dramatically in the realistic case where $k > 1$. Bachmat and Elkin, [8], established an upper bound on the performance of any announcement*back-to-front* boarding policy. They showed that the infimum of the estimated boarding time is at least

$$\frac{\sqrt{k-1}}{\sqrt{k} + \frac{1-\ln 2}{\sqrt{k}}}. \tag{1}$$

In particular, with a congestion factor $k \approx 4$, a back-to-front policy can be at most 20% better than random boarding.

Given this bound, two questions remain. First, what is the exact saving achieved by an optimal $m$-group boarding policy over random boarding, and what does a near optimal policy look like? In particular, what is the maximal number of groups that can be justified by non-negligible additional savings over a boarding policy with less groups?

In the course of the present work, which can be viewed as the culmination of this line of study, the following results on back-to-front policies are established:

- A combined analytical and numerical solution, describing the optimal 2-group policy, was found for any $k > 0$. When $k = 4$, the optimal 2-group policy achieves approximately 8% savings over random boarding. The policy first boards roughly 15% of the passengers from the back of the airplane, followed by all other passengers.
- Near optimal policies with at most 4 passenger groups were found for various congestion factors.
- Based on the calculated savings as achieved by these near optimal policies, any further partitioning beyond 2 groups was found to be impractical.

The calculations are conducted inductively on the number of groups $m$ in a policy. The computation is similar to dynamic programming, however, each stage has a continuous family of states and possible transitions. One of the technical difficulties involved in the analysis, is that the stage transitions are subject to various combinatorial constraints, thus the optimization problem has a mixture of continuous and discrete aspects. The key for maintaining control over the amount of computations is to take advantage of the scaling structure of the problem.

The remainder of this work is organized as follows. In Section 2 we describe the mathematical model of airplane boarding which we use. We then proceed to calculations of an optimal 2-group back-to-front boarding policy in Section 3, further generalizing this approach to policies with $m > 2$ groups in Section 4. The $m$-group case involves a recursive computation which comes from a dynamical programming approach. Further on in Section 5 we present the numerical portion of the obtained results, including the actual optimal policies found. We end with some conclusions and future work.

### 1.1 Related work

Researchers have mostly been studying boarding policies through discrete event simulations, see [13, 19, 17, 12, 16]. In addition, mathematical programming formulations appeared in [17, 18, 9]. The target function, to be minimized, in these studies is not an estimate of the boarding time, but rather a related function which counts instances of passengers blocking each other. Computer software is then applied to the mathematical programming problem in order to find efficient boarding policies. The model of [17] emphasizes blocking instances which relate to the internal ordering of passengers within a row leading to policies which board window passengers, followed by middle passengers and finally aisle passengers. Such policies are not within the scope of the present paper, see [7] for further discussion. The model of [9] essentially corresponds to the case where the congestion parameter $k$ is zero, a setting which we consider to be much less realistic. The results in this case markedly differ from those of the various simulation studies.

It is important to note that all these studies consider policies where all groups have roughly equal size. The simulation studies have found that in this case back-to-front policies are not necessarily effective and might, in fact, be detrimental when compared with random boarding. For example, [12] shows that, in some cases, disturbances of the back-to-front scenario caused by passengers boarding outside the time slot assigned for their group will actually *shorten* the boarding time. In contrast we show that with unequal group sizes it is possible to mildly improve upon random boarding using back-to-front policies.

The study, [13] by Marelli *et al.* of Boeing Corp. emphasizes the effect of airplane interior design on boarding time, again using discrete event simulation methods. The paper describes a commercial simulation product, and simulation results are only sketched, making it difficult to analyse or compare to the results of other studies.

We note that the simulation-based models and the mathematical programming models offer a black-box approach. Every new potential strategy has to be evaluated from scratch as a dedicated simulation scenario or program, without any advance performance estimates, nor any clue as to how much a given policy might be further improved. On the other hand, our approach, which is more analytic, treats large families of possible policies in a single computation.

The mathematical model that we use describes the boarding process as a "wave propagation" in a discrete spacetime. As explained in [4], airplane boarding shares this description with discrete processes such as the polynuclear growth process in statistical mechanics, [15], the patience sorting algorithm for finding the longest increasing subsequence of a permutation, [1, 11] and the Andrews-Bender-Zhang algorithm for optimal I/O scheduling in a disk drive [2, 3]. Other connections of this model with combinatorics, physics, random matrix theory, and other applications are further mapped out in [10].

## 2 The mathematical model of airplane boarding

We shall introduce the mathematical model of [6, 7]. We shall only summarize the parts that pertain to back-to-front boarding. Since only back-to-front boarding policies are the subject of this work, we shall omit the words "back-to-front" henceforth.

Let

$n$ be the number of passengers,

$R$ — the number of rows in the airplane,

$m$ — the number of boarding groups, whose queueing times are announced.

$k$ — the congestion factor. The parameter $k$ is the number of passengers per row, times the average aisle length occupied by a passenger when boarding, divided by the distance between successive rows in the airplane.

The passengers will be represented as points $(q, r)$ in the unit square $[0, 1]^2$. The $q$ coordinate will represent the position of a given passenger in the boarding queue, divided by $n$, while the $r$ coordinate will represent the assigned row number, divided by $R$. An $m$-group boarding policy $F$ will be represented by a partition of the unit interval $1 = \rho_0 > \rho_1 > \ldots > \rho_m = 0$. For more compact computations, it will be more convenient to consider the equivalent partition in the queue coordinate. We shall denote this as $F = (x_0, \ldots, x_m)$, where $x_i = 1 - \rho_i$. Passengers in the $i$-th boarding group, $1 \leq i \leq m$, will have their normalised row numbers satisfy $\rho_{i-1} \geq r \geq \rho_i$. To implement such a policy, an airline would first announce the boarding of the passengers with row numbers $R \cdot \rho_1$ and above, then — $R \cdot \rho_2$ to $R \cdot \rho_1$, etc. We say that a policy is uniform if all boarding groups are of equal size, i.e., $\rho_i = \frac{i}{m}$. We denote such a policy with $m$ groups by $F_m$. The boarding policy $F_1$, has just one group consisting of all passengers, and therefore represents random boarding with no airline control policy. This is the yardstick against which we will measure all other policies.

The choice of a policy $F$ leads to a joint probability distribution

$$p_F(q, r)\, dq\, dr$$

on the row and queue location coordinates of passengers. The density function $p(q, r) = p(q, r)_F$ associated with a policy is defined by

$$p(q, r) = \begin{cases} \frac{1}{\rho_{i-1} - \rho_i}, & \rho_{i-1} \geq r \geq \rho_i,\ 1 - \rho_{i-1} \leq q \leq 1 - \rho_i,\ i = 1, \ldots, m; \\ 0, & \text{otherwise.} \end{cases}$$

When restricted to a given boarding group, it induces a uniform distribution since the ordering of passengers within a group is uniformly random. Let

$$\alpha(q, r) = \alpha_F(q, r) = \int_r^1 p(q, z)\, dz.$$

Let $\Psi = \Psi_{F,k}$ be the set of all piecewise differentiable functions $\varphi(q)$ defined on an interval $[q_0, q_1] \subset [0, 1]$ and with values in $[0, 1]$, such that

$$\varphi'(q) + k \cdot \alpha(q, \varphi(q)) \geq 0. \tag{2}$$

Such curves $\varphi$ are *legitimate*.

For a policy $F$ we define

$$T(F,k) = \max_{\varphi \in \Psi} L(\varphi), \tag{3}$$

where

$$L(\varphi) = L_{F,k}(\varphi) = \int_{q_0}^{q_1} \sqrt{p(q,\varphi(q)) \cdot (\varphi'(q) + k \cdot \alpha(q,\varphi(q)))}\, dq.$$

The functional $L(\varphi)$ on the set of legitimate curves coincides with the length (proper time) functional on causal curves in a compact domain of a Lorentzian (space-time) manifold, see [14] for definitions. As such, it has been studied extensively in the context of Lorentzian geometry, among others. In particular, it is known that $L(\varphi)$ attains a finite maximal value on the set of legitimate curves $\psi$, see [11, 14, 7]. We call a legitimate curve which maximizes the functional $L$ over the set of legitimate curves, a *maximal curve*. According to the model which will be used in this paper, the estimated boarding time of $n$ passengers, with policy $F$ and congestion parameter $k$ is

$$T(F,k)\sqrt{n} \tag{4}$$

We therefore consider $T(F,k)$ to be our target function when comparing policies.

the non-negative value $L(\varphi)$, for a legitimate curve $\varphi$ is the *length* of the curve. In the context of Lorentzian geometry, legitimate curves are *causal*. The border case legitimate curves, namely, the ones for which $L(\varphi) = 0$, are called *lightlike*. Curves which locally maximize $T(F,k)$ are called *geodesics*.

For the random policy $F_1$, the length functional takes the form

$$L(\varphi) = \int_{q_0}^{q_1} \sqrt{\varphi' + k(1-\varphi)}\, dq. \tag{5}$$

The general solution for the corresponding Euler-Lagrange equation has the form

$$\varphi(q) = c_1 e^{2kq} + c_2 e^{kq} + 1, \tag{6}$$

and its length is

$$L(\varphi) = \left(e^{kq_1} - e^{kq_0}\right)\sqrt{\frac{c_1}{k}}. \tag{7}$$

Using these formulas and an examination of boundary conditions one gets

$$T(F_1,k) = \begin{cases} \sqrt{\frac{e^k-1}{k}}, & 0 < k \le \ln 2; \\ \sqrt{k} + \frac{1-\ln 2}{\sqrt{k}}, & k \ge \ln 2. \end{cases} \tag{8}$$

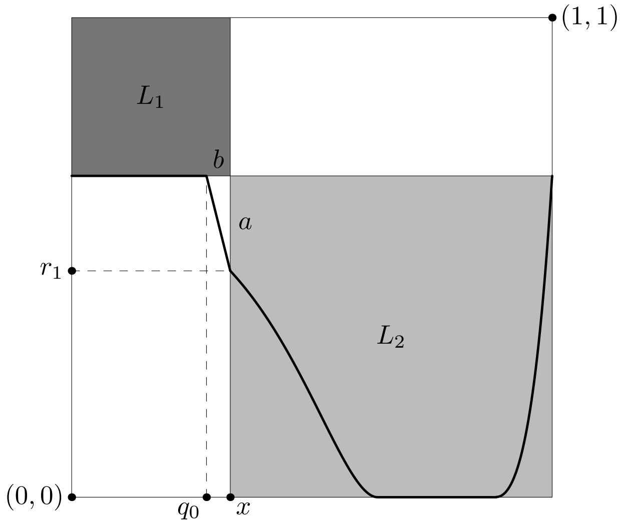


Figure 1: An example of a maximal curve for a 2-group policy. Darker shade indicates higher joint probability density $p(q, r)$.

According to [8], for an airplane model with the congestion factor $k \geq 1$, the savings of any policy $F$ over the random policy $F_1$ is at most

$$1 - \frac{T(F,k)}{T(F_1,k)} \leq 1 - \frac{\sqrt{k-1}}{\sqrt{k} + \frac{1-\ln 2}{\sqrt{k}}}. \tag{9}$$

## 3 An optimal 2-group policy

We are looking for a 2-group boarding policy that would be optimal for a given congestion factor $k$. A 2-group policy is given by a single parameter $\rho_1 = x$. We compute the boarding time as a function of $x$ and minimize. Until further notice, we fix some value of $x$.

Figure 1 shows the situation for a typical value of $x$. The maximal curve $\varphi$ is drawn in black.

We first consider the non-degenerate case when the maximal curve spans both square cells $L_1$ and $L_2$ as shown in the figure.

In this case, it is shown in [7], that the maximum length curve must consist of a horizontal line segment between $(0, 1-x)$ and $(q_0, 1-x)$, for some $0 \leq q_0 \leq 1$, then a straight-line segment sloping down to $(x, r_1)$ with slope $-k$, and then the maximal curve in the lower-right square $L_2$, ending at $(1, 1-x)$. The three possibilities for the portion of the curve which is contained in $L_2$ are shown (in a rescaled version) in figure 2. Each segment of the maximal curve $\varphi$ must either be of the form in equation (6) or be a boundary component of a cell. The three cases shown in figure 2 correspond

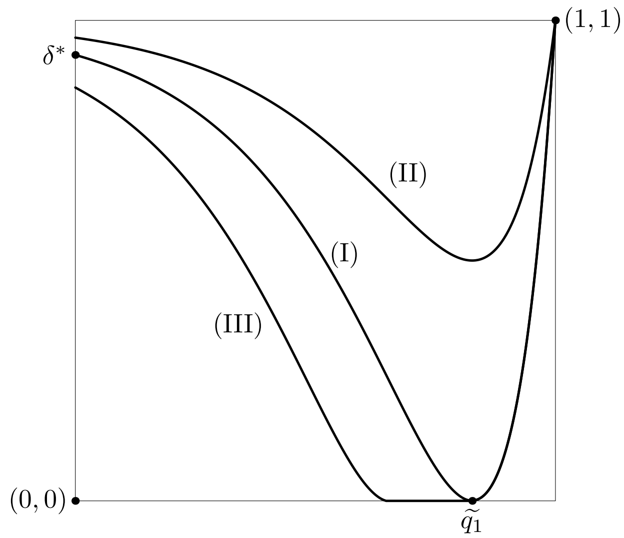


Figure 2: Three possible cases for the maximal curve in $L_2$.

to the case in which $\varphi$ restricted to $L_2$ has no boundary component, is tangent to the boundary or contains a boundary component. Possibility (III), where a boundary component exists, is the one shown in figure 1.

Let $L$ be the length of the maximal curve and let $\tilde{L}_1$ and $\tilde{L}_2$ denote the lengths of the portions of the maximal curve in the corresponding cells $L_1$ and $L_2$. Since the density distribution is uniform on each cell we can apply equations (5-8) after scaling to a unit size square. Let us call $L_1$ and $L_2$ the lengths of the resulting scaled curves. A simple computation reveals that scaling the density distribution and coordinates introduces a square root factor, so $\tilde{L}_1 = \sqrt{x}L_1$ and similarly $\tilde{L}_2 = \sqrt{1-x}L_2$, leading to

$$L = \sqrt{x}L_1 + \sqrt{1-x}L_2 \tag{10}$$

since the maximal curve in $L_1$ is a horizontal line segment we have by (3)

$$L_1 = \sqrt{k} \cdot \frac{q_0}{x} \tag{11}$$

Next, we compute $r_1$ as a function of $(q_0, x)$. As $\frac{a}{b} = k$; $a = 1 - x - r_1$; $b = x - q_0$, we have

$$r_1 = 1 - x - a = 1 - x - kb = 1 - x - k(x - q_0) = 1 - (k+1)x + kq_0. \tag{12}$$

Now let us compute $L_2$. Consider the second cell scaled to unit size, the maximal curve enters the square at

$$\delta = \frac{r_1}{1-x} \tag{13}$$

By equation 12, $\delta$ is constrained by the inequality $\delta \in (\frac{1-x-xk}{1-x}, 1) \cap (0,1)$, i.e.,

$$1 > \delta > \begin{cases} \frac{1-x-xk}{1-x}, & 0 < x < \frac{1}{k+1}; \\ 0, & \frac{1}{k+1} \leq x < 1. \end{cases} \tag{14}$$

We would like to express $L_1$ via $x$ and $\delta$. (See Figure 1.)

$$\delta(1-x) = r_1 = 1-(k+1)x + kq_0, \tag{15}$$

$$kq_0 = \delta(1-x) + (k+1)x - 1. \tag{16}$$

Therefore

$$q_0 = \frac{1}{k}\left(\delta(1-x) + (k+1)x - 1\right). \tag{17}$$

Substituting (17) into (11) we obtain

$$L_1 = \sqrt{k} \cdot \frac{1}{xk}\left(\delta(1-x) + (k+1)x - 1\right) = \frac{\delta(1-x) + (k+1)x - 1}{x\sqrt{k}}. \tag{18}$$

We compute the value of $\delta$ for case (II) of tangency. We do it by applying to the general solution (6) the tangency condition

$$\begin{cases} r(0) = \delta &= c_1 + c_2 + 1, \\ r(\widetilde{q_1}) = 0 &= c_1 e^{2k\widetilde{q_1}} + c_2 e^{k\widetilde{q_1}} + 1, \\ r'(\widetilde{q_1}) = 0 &= 2kc_1 e^{2k\widetilde{q_1}} + kc_2 e^{k\widetilde{q_1}}, \\ r(1) = 1 &= c_1 e^{2k1} + c_2 e^{k1} + 1. \end{cases}$$

We introduce the notation $X \equiv e^{kx}$, $Q = e^{kq}$, $Q_0 = e^{kq_0}$, etc.

Under this notation,

$$r' = \frac{dr}{dq} = 2kc_1 e^{2kq} + kc_2 e^{kq} = 2kc_1 Q^2 + kc_2 Q.$$

The tangency condition $r' = 0$ then implies

$$Q = -\frac{c_2}{2c_1}, \tag{19}$$

i.e., the solution is tangent to the axis $q = 0$ at

$$\widetilde{q_1} \equiv q = \frac{1}{k} \ln\left(-\frac{c_2}{2c_1}\right). \tag{20}$$

Also, from (3),

$$-\frac{c_2}{2c_1} = \frac{e^k}{2},$$

and thus

$$c_1 = \frac{1-\delta}{e^k - 1}, \tag{21}$$

$$c_2 = \frac{-e^k}{e^k - 1}(1-\delta). \tag{22}$$

So

$$\widetilde{q_1} = \frac{1}{k} \ln\left(\frac{e^k}{2}\right) = \frac{1}{k}(\ln e^k - \ln 2) = \frac{1}{k}(k - \ln 2) = 1 - \frac{\ln 2}{k}.$$

Plugging $c_1$, $c_2$, and $\widetilde{q_1}$ into (3), we have

$$\begin{aligned} r(\widetilde{q_1}) = 0 = c_1 \left(\frac{e^k}{2}\right)^2 + c_2 \left(\frac{e^k}{2}\right) + 1 = \\ = \frac{1-\delta}{e^k - 1} \cdot \frac{e^{k2}}{4} - \frac{e^k}{e^k - 1}(1-\delta)\frac{e^k}{2} + 1 = \\ = (1-\delta) \cdot \frac{e^{2k}}{e^k - 1}\left(\frac{1}{4} - \frac{1}{2}\right) + 1 \end{aligned}$$

and

$$(1-\delta) \cdot \frac{e^{2k}}{4(e^k - 1)} = 1,$$

which gives

$$\begin{aligned} \delta^* \equiv \delta = 1 - \frac{4(e^k - 1)}{e^{2k}} = \\ = \frac{1}{e^{2k}}(e^{2k} - 4e^k + 4) = \frac{1}{e^{2k}}(e^k - 2)^2 = \\ = (1 - 2e^{-k})^2. \end{aligned}$$

Thus, for $\delta \geq \delta^*$, by equation (7) and (21)

$$L_2 = \left(e^k - 1\right)\sqrt{\frac{c_1}{k}} = \left(e^k - 1\right)\sqrt{\frac{1-\delta}{e^k - 1} \cdot \frac{1}{k}} = \sqrt{\frac{(1-\delta)(e^k - 1)}{k}}. \qquad (23)$$

The case of equality $\delta = \delta^*$, is depicted by case (I) of Figure 2, while case (II) describes the situation when $\delta > \delta^*$.

In the remaining case (see Figure 2, case (III)) of $\delta < \delta^*$ we have by [7]

$$L_2 = \sqrt{\frac{1}{k}}\left(\sqrt{\delta} + \ln\left(1 - \sqrt{\delta}\right) + k + 1 - \ln 2\right). \qquad (24)$$

We shall denote by $L_2(\delta)$ the maximal length of a legitimate curve in the unit square, constrained to pass through the boundary point $(0, \delta)$. The length $L_2(\delta)$ was computed, in equations (23) and (24), to be

$$L_2(\delta) = \sqrt{\frac{1}{k}} \cdot \begin{cases} \sqrt{(1-\delta)(e^k - 1)}, & \delta \geq \delta^*; \\ \sqrt{\delta} + \ln\left(1 - \sqrt{\delta}\right) + k + 1 - \ln 2, & \delta < \delta^*. \end{cases} \qquad (25)$$

Combining (10), (18), (25) and rescaling, we get a formula for the length of the maximal curve, $\varphi$ conditioned to pass through the point $(x, \delta(1-x))$.

$$\begin{aligned}L(\delta) &= \sqrt{x}\cdot\frac{\delta(1-x)+(k+1)x-1}{x\sqrt{k}} + \sqrt{1-x}\cdot L_2(\delta) = \\ &= \frac{\delta(1-x)+(k+1)x-1}{\sqrt{kx}} + \\ &\qquad + \sqrt{\frac{1-x}{k}}\cdot\begin{cases}\sqrt{(1-\delta)(e^k-1)}, & \delta\ge\delta^*;\\ \sqrt{\delta}+\ln\left(1-\sqrt{\delta}\right)+k+1-\ln 2, & \delta<\delta^*.\end{cases}\end{aligned} \tag{26}$$

Given $x\in(0,1)$, we seek to maximize $L$ as a function of $\delta$. For this, we need to explore the derivative

$$\frac{dL}{d\delta} = \frac{1-x}{\sqrt{kx}} + \sqrt{\frac{1-x}{k}}\cdot\begin{cases}-\frac{\sqrt{e^k-1}}{2\sqrt{1-\delta}}, & \delta\ge\delta^*;\\ \frac{1}{2\sqrt{\delta}}-\frac{1}{2\left(1-\sqrt{\delta}\right)\sqrt{\delta}}, & \delta<\delta^*.\end{cases}$$

As

$$\begin{aligned}\frac{1}{2\sqrt{\delta}}-\frac{1}{2\left(1-\sqrt{\delta}\right)\sqrt{\delta}} &= \frac{1}{2\sqrt{\delta}}\left(1-\frac{1}{1-\sqrt{\delta}}\right) = \\ &= \frac{1}{2\sqrt{\delta}}\cdot\frac{1-\sqrt{\delta}-1}{1-\sqrt{\delta}} = \frac{-1}{2\left(1-\sqrt{\delta}\right)},\end{aligned}$$

thus we obtain

$$\frac{dL}{d\delta} = \frac{1-x}{\sqrt{kx}} - \frac{1}{2}\cdot\sqrt{\frac{1-x}{k}}\cdot\begin{cases}\frac{\sqrt{e^k-1}}{\sqrt{1-\delta}}, & \delta\ge\delta^*;\\ \frac{1}{1-\sqrt{\delta}}, & \delta<\delta^*.\end{cases} \tag{27}$$

For $\delta\ge\delta^*$ we get

$$\begin{aligned}&\frac{dL}{d\delta}=0 \iff \\ &\iff \frac{1}{\sqrt{1-\delta}} = \frac{1-x}{\sqrt{kx}}\cdot 2\cdot\sqrt{\frac{k}{(1-x)(e^k-1)}} = 2\sqrt{\frac{1-x}{x(e^k-1)}} \iff \\ &\iff 1-\delta = \frac{x(e^k-1)}{4(1-x)} \iff \\ &\iff \delta = \frac{4-3x-e^kx}{4(1-x)},\end{aligned} \tag{28}$$

We now check the condition $\delta\ge\delta^*$ for the critical value. Assume

$$\frac{4-3x-e^kx}{4(1-x)} \ge \delta^*$$

as $x < 1$

$$\begin{aligned}
4 - 3x - e^k x &\geq 4(1-x)\delta^* \\
4x\delta^* - 3x - e^k x &\geq 4\delta^* - 4 \\
x(4\delta^* - 3 - e^k) &\geq 4(\delta^* - 1)
\end{aligned}$$

Compare the l.h.s. to 0:

$$\begin{aligned}
4\delta^* - 3 - e^k = 4(1-2e^{-k})^2 - 3 - e^k = 4(1 - 4e^{-k} + 4e^{-2k}) - 3 - e^k = \\
= 1 - 16e^{-k} + 16e^{-2k} - e^k \geq 0 \iff \\
\overset{*e^{2k}}{\iff} -e^{3k} + e^{2k} - 16e^k + 16 \geq 0 \quad (29)
\end{aligned}$$

Letting $t \equiv e^k$ (which is strictly positive), we may transform (29) into an equivalent form

$$-t^3 + t^2 - 16t + 16 = (t-1)(-t^2 - 16) = -(t-1)(t^2 + 16) \geq 0$$

which implies

$$e^k \leq 1$$

and

$$k \leq 0$$

which never holds. we conclude that there is no extremal value of $L$ in the open interval $(\delta^*, 1)$. The second derivative for $\delta \geq \delta^*$ is

$$\begin{aligned}
\frac{d^2 L}{d\delta^2} = -\frac{1}{2} \cdot \sqrt{\frac{(1-x)(e^k - 1)}{k}} \cdot \left(\frac{1}{\sqrt{1-\delta}}\right)' = \\
= -\frac{1}{2} \cdot \sqrt{\frac{(1-x)(e^k - 1)}{k}} \cdot \frac{-1}{2\,(1-\delta)^{3/2}} \cdot (-1) < 0
\end{aligned}$$

so the function is decreasing and hence its maximal value is at $\delta^*$. We conclude that the maximal value of $L$ is obtained when $\delta \leq \delta^*$.

when $\delta < \delta^*$, let $\delta_{crit}$ denote the value for which $\frac{dL}{d\delta} = 0$.

$$\frac{dL}{d\delta} = 0 \iff$$

$$\iff \frac{1-x}{\sqrt{x}} = \frac{1}{2\left(1-\sqrt{\delta}\right)} \cdot \sqrt{1-x} \iff$$

$$\iff 2\left(1-\sqrt{\delta}\right) = \sqrt{\frac{x}{1-x}} \iff$$

$$\iff \sqrt{\delta} = 1 - \frac{1}{2} \cdot \sqrt{\frac{x}{1-x}} \iff$$

$$\iff \delta = 1 - \sqrt{\frac{x}{1-x}} + \frac{x}{4(1-x)} \iff$$

$$\iff \delta = \frac{4(1-x) - 4\sqrt{x(1-x)} + x}{4(1-x)} \iff$$

$$\iff \delta_{crit} = \frac{4 - 3x - 4\sqrt{x - x^2}}{4(1-x)}.$$

also

$$\frac{d^2L}{d\delta^2} = -\frac{1}{2} \cdot \sqrt{\frac{1-x}{k}} \cdot \left(\frac{1}{1-\sqrt{\delta}}\right)' = -\frac{1}{2} \cdot \sqrt{\frac{1-x}{k}} \cdot \frac{-1}{\left(1-\sqrt{\delta}\right)^2} \cdot \frac{-1}{2\sqrt{\delta}} < 0.$$

We need to check the admisiblity conditions for the critical point, namely $\delta_{crit} < \delta^*$ and $\delta_{crit} > \max\left\{0, \frac{1-x-xk}{1-x}\right\}$. We consider the first constraint

$$\begin{aligned}
\frac{4 - 3x - 4\sqrt{x - x^2}}{4(1-x)} &\quad < \quad \delta^*, \\
4 - 3x - 4\sqrt{x - x^2} &\overset{(0<x<1)}{<} 4(1-x)\delta^*, \\
\underbrace{4 - 3x + 4(x-1)\delta^*}_{a} &\quad < \quad \underbrace{4\sqrt{x - x^2}}_{b}.
\end{aligned}$$

Since for any $0 < x, \delta^* < 1$, both $a, b > 0$, the last inequality holds iff $a^2 < b^2 \iff x \in (x_1, x_2) \cap (0,1)$, where $x_1 \leq x_2$ are the roots of $a^2 = b^2$. It turns out that both the roots

$$x_{1,2} = 1 - \frac{1}{5 \mp 8\sqrt{\delta^*} + 4\delta^*}$$

fall within $(0,1)$ for any $\delta^* \in (0,1)$, so $\delta_{crit}$ is an admissible root of the derivative whenever, $\delta^* = \delta^*(k)$ and $x$ satisfy

$$1 - \frac{1}{5 - 8\sqrt{\delta^*} + 4\delta^*} < x < 1 - \frac{1}{5 + 8\sqrt{\delta^*} + 4\delta^*}. \tag{30}$$

Regarding the other constraint, $\delta_{crit} > \max\left\{0, \frac{1-x-xk}{1-x}\right\}$, it further remains to check (14). For $x \geq \frac{1}{k+1}$ it reduces to the trivial $\delta_{crit} > 0$.

Now assume $x < \frac{1}{k+1}$. One needs to check that

$$\delta_{crit} > \frac{1-x-xk}{1-x},$$

and in case it holds, $\delta_{crit}$ is admissible; otherwise, the boundary value at $L(\frac{1-x-xk}{1-x})$ should be considered instead of $L(\delta_{crit})$.

The above inequality holds if (and only if)

$$\frac{16}{17+8k+16k^2} < x. \tag{31}$$

Summarizing, we have the following cases. We let $\delta^* = (1-2e^k)^2$, $\delta_{crit} = \frac{4-3x-4\sqrt{x-x^2}}{4(1-x)}$ and $\delta_{min} = \frac{1-(k+1)x}{1-x}$ If $\delta^* > \delta_{crit} > \delta_{min}$ then the maximal value is $L(\delta_{crit})$. If $\delta^* < \delta_{min}$ then the maximal value is $L(\delta_{min})$. Finally, if $\delta_{crit} > \delta^* > \delta_{min}$ then the maximal value is $L(\delta^*)$. Equations (30) and (31) provide criteria for the different cases. In addition to the maximal value of $L(\delta)$ which we have just analyzed, we need to consider two other posibilities, that the maximal curve is wholly contained in $L_1$ or that it is wholly contained in $L_2$. The first possibility is relevant when $x > 1/2$ and the other when $x \leq 1/2$. In the first case the maximal curve within the first square will be the maximal curve for $F_1$, with its length determined by (8) and scaling

$$L_{1'} = \sqrt{x} \cdot T(F_1, k).$$

We note that this is simply the estimated boarding time for random boarding with $xn$ passengers (passengers of the first group) instead of $n$ passengers. Likewise, if the maximal curve only spans the second group, its length will be

$$L_{2'} = \sqrt{1-x} \cdot T(F_1, k)$$

The maximal length curve $L(x)$ will be given by the maximum of $L(\delta)$, $L_{1'}$ and $L_{2'}$. The quantity $L(x)$ is our estimated boarding time for the 2 group policy whose first group has $xn$ passengers from the back rows. Our goal is to minimize $L(x)$ as a function of $x$.

### 3.1 The case of $k = 4$

We show how the calculations of the previous section can be applied specifically to the case $k = 4$ which we consider to be realistic. For this case, using the upper bound (9), we conclude that the possible boarding time savings given by any policy cannot exceed 20%.

By (8), the random boarding time is $T(F_1, 4) = 2.5 - \frac{\ln 2}{2} \approx 2.153426409$

To calculate $L(x)$ we need to consider $\max\{L(\delta_{crit}), L_{1'}, L_{2'})\}$ as long as (30) and (31) simultaneously hold, i.e., when $x$ is within

$$(\underbrace{\sim 0.005338746}_{x_1}, \underbrace{\sim 0.939095941}_{x_2}) \cap (\frac{16}{305}, 1) = (\frac{16}{305}, x_2). \tag{32}$$

Within this interval we wish to minimize $Max(L(\delta_{crit}), L_{1'}, L_{2'}$ with respect to $x$.

Numerical search by $x$ for the meeting point of $L(\delta_{crit})$ and $L_{2'}$ in the interval (32) shows that the minimum of $\max\{L(\delta_2), L_{1'}, L_{2'}\}$ is achieved at $x \approx 0.148531234$ where $L(\delta_{crit}) = L_{2'}$. The mutual value is $L = \sqrt{1-x} \cdot T(F_1, 4) \approx 1.987075623$. (Note that this already beats $F_1$).

The interval $(0, \frac{16}{305})$ can be excluded immediately from the consideration since $\sqrt{1-x}$ and consequently $L_{2'}$ is a decreasing function.

Numerical search for the other meeting point where the maximal value of $L(\delta)$ equals $L_{1'}$ in the regime, of $x \geq x_2$ discovers another, somewhat larger, local minimum of $\max\{L(\delta^*), L_{1'}, L_{2'}\}$ at $x \approx 0.975885874$, where $L(\delta^*) = \sqrt{x} \cdot T(F_1, 4) \approx 2.127303971$.

Thus, when $k = 4$, the optimal 2-group policy is achieved at

$$x \approx 0.148531234,$$

which saves

$$1 - \frac{L_{2'}}{T(F_1, 4)} \approx 0.077249348 \approx 8\%$$

of the estimated boarding time in comparison with the random (no) boarding policy.

While these explicit calculations were done for the case $k = 4$, they can also be implemented for any other value of $k$.

# 4 An optimal $m$-group policy

We would like to compute the optimal $m$-group policy and to measure its efficiency relatively to $T(F_1, k)$. The approach is similar to dynamic programming. We will follow a similar procedure to what we have done with 2 groups. The computation is done inductively on $m$. We will need an auxillary quantity which we denote by $L_1^{(m)}(z)$ the length of the maximal curve in an $m$-cell partition of the unit square constrained to end at the point $(z, 0)$. We will also compute $L_1^{(m)}(z)$ inductively. To begin the induction we know from (11) that

$$L_1^{(1)}(z) = \sqrt{k} \cdot z. \tag{33}$$

In order to proceed with the induction we need to compute the maximal length of a curve in the unit square which begins at the point $(0, r)$ and ends at $(q, 0)$. We call this the *corner length* and denote it by $L_C(r, q)$. The length is computed in [7]. We need to distinguish two cases. Let $q^* = \frac{1}{k} ln(\frac{1}{1-\sqrt{r}})$. The maximal curve between $(0, \delta_0)$ and $(q^*, 0)$ is tangent to the bottom edge $r = 0$ at the endpoint $(q^*, 0)$. If $q \geq q^*$, the maximal curve is obtained by concatenating the maximal curve between $(0, r)$ and $(q^*, 0)$ with the bottom edge segment between $(q^*, 0)$ and $q, 0)$. The length of this curve is $\sqrt{\frac{r}{k}} + (q - q^*)\sqrt{k}$. If $q < q^*$ the maximal curve will be given by a solution to the Euler-Lagrange equation between the endpoints.

Given $r$ and $q$ and letting $Q = e^{kq}$ we find the corresponding $c_1$, $c_2$:

$$\begin{aligned} c_1 &= r - 1 - c_2; && (34) \\ 0 &= (r - 1 - c_2)Q^2 + c_2 Q + 1 = (r - 1)Q^2 + c_2 Q(1 - Q) + 1, && (35) \end{aligned}$$

thus

$$c_2 = \frac{(1 - r)Q^2 - 1}{Q(1 - Q)} \tag{36}$$

$$c_1 = \frac{(r - 1)Q(1 - Q) + (r - 1)Q^2 + 1}{Q(1 - Q)} = \frac{(r - 1)Q + 1}{Q(1 - Q)} \tag{37}$$

Given $r$ and $q$ and thus $c_1$ and $c_2$ from (37) and (36), we need to check when the corresponding curve (6) is legitimate. Given $r$ we find the minimal value of $q$ for which the solution is a legitimate curve. For the minimal $q$ the solution $\varphi(q)$ will be light-like and will have zero length. The functional $L(\varphi)$, is in this case expressed by (5). Thus, we are looking for $\varphi$ s.t.

$$\varphi' + k(1 - \varphi) = 0,$$

which has the general solution

$$\varphi(q) = 1 + e^{kq} c_1. \tag{38}$$

Given $r$ we get the initial condition $\varphi(0) = r$, so we get the solution

$$\varphi(q) = 1 + e^{kq}(r - 1).$$

which meets the axis $r = 0$ at $q$ which satisfies

$$1 + e^{kq}(r - 1) = 0 \iff e^{kq}(r - 1) = -1.$$

$Q = \frac{1}{1-r}$, so the minimal value is

$$q_* = q_*(r) = \frac{1}{k} \ln\left(\frac{1}{1 - r}\right).$$

We can also invert the relation. For a given $q$, the minimal $r$ for which there is a legitimate curve from $(0, r)$ to $(q, 0)$ is given by

$$r_* = r_*(q) = 1 - e^{-kq}. \tag{39}$$

Putting the two cases together and using equation (7) we get

$$L_C(r,q) = \begin{cases} \sqrt{\frac{(Q-1)((1-r)Q-1)}{kQ}}, & q_* \le q < q^*; \\ \frac{(Q^*-1)(1-\sqrt{r})}{\sqrt{k}} + (q-q^*)\sqrt{k}, & q > q^*; \end{cases}$$

$$= \begin{cases} \sqrt{\frac{(Q-1)((1-r)Q-1)}{kQ}}, & q_* \le q < q^*; \\ \frac{kq+\sqrt{r}+\ln\left(1-\sqrt{r}\right)}{\sqrt{k}}, & q > q^*. \end{cases} \quad (40)$$

We can now use the corner length to compute inductively $L_1^{(m)}$. We shall mostly reuse the construction from section 3. However, instead of $L_1$ (see Figure 1), computed over a non-partitioned group of the back rows, we shall use $L_1^{(m-1)}$. For $z \ge \frac{1}{k+1}$, we are led to the inductive formula

$$L_1^{(m)}(z) = \min_{0<x<z} \max\left\{ \sqrt{1-x} \cdot L_1^{(1)}\left(\frac{z-x}{1-x}\right), \right.$$
$$\max_\delta \left\{ \sqrt{x} \cdot L_1^{(m-1)}\left(\frac{\delta(1-x)+(k+1)x-1}{kx}\right) + \right.$$
$$\left.\left. + \sqrt{1-x} \cdot L_C\left(\delta, \frac{z-x}{1-x}\right)\right\}\right\}$$

where $\max_\delta$ is, by (14), taken over $\max\left\{0, \frac{1-x-xk}{1-x}\right\} < \delta$; the upper limit is now further restricted by $\delta < 1 - e^{-k\cdot\frac{z-x}{1-x}}$ according to (39).

For $z < \frac{1}{k+1}$ we also need to consider the possibility that $x > z$ and that only the upper-left $m-1$ cells, will contribute, thus, if $z < \frac{1}{k+1}$, then $L_1^{(m)}(z)$ is defined as the minimum of the above and

$$\min_{z<x<1-kz} \left\{ \sqrt{x} \cdot L_1^{(m-1)}\left(\left(z - \frac{1-x}{k}\right)/x\right)\right\}.$$

Given $L_1^(m)(z)$ we can finally compute $T_m$, the optimal boarding time for an $m$ group back-to-front policy. We compute $T_m$ recursively using $L_1^{(m-1)}$ and $L_2(\delta)$. We consider the first $m-1$ groups and the last group. The analysis of the contribution of the last cell is the same as the analysis of the contribution of the second (last) cell in the two group case. The first cell in the 2-group case is replaced by $L_1^{(m-1)}(z)$ for the appropriate $z$ which matches $\delta$ and $x$. specifically, following the analysis of the 2-group case we have the recursive formula

$$L^{(m)}(\delta) = \sqrt{x} \cdot L_1^{(m-1)}\left(\frac{\delta(1-x)+(k+1)x-1}{kx}\right) + \sqrt{1-x} \cdot L_2(\delta) \quad (41)$$

for a maximal curve spanning both the first $m-1$ cells and the last cell.

| $k$ | $T_2/T_1$ | $x$ |
|---|---|---|
| 4/3 | 0.770 | 0.40700 |
| 2.00 | 0.840 | 0.29436 |
| 3.00 | 0.894 | 0.20152 |
| 4.00 | 0.923 | 0.14853 |

Table 1: The boarding times under the optimal two-group policy, and the optimal partitioning point $x$ for various congestion factors $k$.

Combining with the cases where not all cells are used we see that the optimal time is given by

$$T_m = \min_{0<x<1} \max \left\{ \sqrt{x} \cdot T_{m-1}, \sqrt{1-x} \cdot T_1, \max_{\delta} L^{(m)} \right\}, \tag{42}$$

where the $\max_\delta$ is, by (14), taken over the interval

$$\max \left\{ 0, \frac{1-x-xk}{1-x} \right\} < \delta < 1.$$

**Conjecture 1.** *For $k \geq 4$, the curve $\max_\delta L^{(m)}$ will be a piecewise concave function of $x$; and therefore, computing the intersections of $L^{(m)}$ with $\sqrt{x}T_{m-1}$ and with $\sqrt{1-x}T_1$ and taking the smaller value will produce the optimum.*

# 5 Results

In Table 1 we give a comparison of the boarding times between the optimal 2-group policy $F_2^*$, and random boarding $F_1$, for various congestion factors. The times for $F_2^*$ were computed using the method described in Section 3. They were normalised according to $F_1^*$ as given by (8).

One can see that the optimal policy achieves visible savings over the random boarding with congestion as high as $k = 4$. The last column in the table shows the optimal partitioning point $x$ defining the optimal policy $F_2^* = (0, x, 1)$. Based on the table results we would recommend, that, roughly, 15–20 percent of the passengers from the back rows board first, followed by the other passengers. Such a strategy seems robust in the face of possible changes in the value of $k$.

Table 2 summarizes the results of looking for the optimal $m$-class policy. Again, the last columns list the inner points $\rho_{m-1}, \ldots, \rho_1$ of the partition comprising $F_m^*$. This time, we relied on the recursion described in Section 4 to obtain $T_m$.

From this table we conclude that, for $k = 4$, further partitioning beyond $m = 2$ is impractical, while for more spacious airplanes with $k = 2$, a 3-class policy might still be relevant.

| $k$ | $m$ | $T_m/T_1$ | $F_m^*$ | | |
|---|---|---|---|---|---|
| 2 | 2 | 0.840 | 0.29436 | | |
| 2 | 3 | 0.792 | 0.11030 | 0.37226 | |
| 2 | 4 | 0.774 | 0.04477 | 0.15009 | 0.40041 |
| 4 | 2 | 0.923 | 0.14853 | | |
| 4 | 3 | 0.911 | 0.02526 | 0.17004 | |
| 4 | 4 | 0.909 | 0.00439 | 0.02953 | 0.17368 |

Table 2: The boarding times under the optimal $m$-group policies, and the inner partitioning points $\rho_{m-1}, \dots, \rho_1$ of the optimal policy for $m = 2, 3, 4$; the congestion factors are $k = 2, 4$.

# 6 Conclusions and future work

In this work we have analyzed in detail back-to-front airplane boarding policies. We have found near optimal boarding policies within this class, depending upon the congestion parameter $k$. We have shown that 2 passenger groups suffice to achieve near optimal results and that the first group of passengers chould consist of between 15 and 20 percent of all passengers. For congestion values close to $k = 4$ one can expect a nearly 8 percent improvement upon random boarding, with increasing gains for lower values of $k$. Our results complement the upper bounds derived in [8].

It is natural to try and extend this work to the setting of boarding policies which are not back-to-front. For non-back-to-front policies the iterative formulae will be significantly more complex and will require some bound on $m$ the number of groups in the policy.

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