Letter to Eitan Bachmat on positive definite L-functions

April 2011

Dear Eitan,

This is a revised version of my letter to you from a few weeks ago. Bjorn Poonen pointed out some results in [D-P-Z] on zeros of random polynomials which have led me to revise some of the heuristic reasoning below. I have also corrected some small errors and added new comments about local positive definiteness.

Your interest is in elliptic curves $E/\mathbb{Q}$ for which the modular parametrization $f_E(z)$ is positive on the y axis, or put another way $f_E(z)$ has no zeros on the geodesic iy, $y>0$. You report that among such curves of conductor at most 1000, 319 have rank zero (an obvious necessary condition in view of the BSD conjecture) and of those 283 have this positivity property.

To begin with I formulate this property in the more general context of automorphic L-functions. Let π be a self-dual automorphic form on $GL_m/\mathbb{Q}$ for some $m \geq 1$. Denote by $\Lambda(s,\pi)$ its completed standard L-function. In order to include Dedekind Zeta functions as well as Rankin-Selberg L-functions we allow $L(s,\pi)$ to have a pole of order k at $s = 1$ and if it does we include a factor of $\left(\frac{s(1-s)}{2}\right)^k$ in $\Lambda(s,\pi)$. We normalize π so that it has a unitary central character and so that it is expected to satisfy the Ramanujan Conjectures (see below). Thus $\Lambda(s,\pi)$ is entire and satisfies a functional equation

$$\Lambda(1-s,\pi) = \varepsilon_\pi N_\pi^{s-1/2} \Lambda(s,\pi). \tag{1}$$

Here $\varepsilon_\pi = \pm 1$ is the root number and $N_\pi \geq 1$ is the conductor. For positive definiteness we must assume that $\varepsilon_\pi = 1$. Under these assumptions $\Lambda(s,\pi)$ is real for s real and following Riemann we set

$$\xi(t,\pi) := \Lambda\left(\tfrac{1}{2}+it,\pi\right) \qquad (2).$$

As a function of t, ξ is entire and even. We are interested in whether $\xi(t,\pi)$ for $t\in\mathbb{R}$ viewed as a group under addition, is positive definite. That is for any $t_1,\dots,t_n$ in $\mathbb{R}$, $\left(\xi(t_j-t_k)\right)_{\substack{j=1,2,\dots n\\k=1,2,\dots n}}$ is a positive definite hermitian matrix.

Equivalently the function

$$h_\pi(y) := \frac{1}{2\pi}\int_{-\infty}^{\infty}\xi(t,\pi)\,y^{-it}\,dt, \qquad (3)$$

on $(0,\infty)$ is positive (we will take this to mean strictly positive).

If $L(s,\pi_\infty)$ is the archimedian factor of the L-function then setting

$$W_{\pi_\infty}(y) := \frac{1}{2\pi i}\int_{Re(s)=2} L(s,\pi_\infty)\,y^{-s+\frac{1}{2}}\,ds \qquad (4)$$

yields

$$h_\pi(y) = \sum_{n=1}^{\infty}\lambda_\pi(n)\,W_{\pi_\infty}(ny). \qquad (5)$$

Here $\lambda_\pi(n)$ are the coefficients in the series expansion

$$L(s,\pi) = \sum_{n=1}^{\infty} \lambda_\pi(n) n^{-s}. \qquad (6)$$

The expected Ramanujan property is

$$\lambda_\pi(n) = O_{\varepsilon,\pi}(n^\varepsilon) \text{ for any } \varepsilon > 0. \qquad (7)$$

The functional equation is equivalent ~~to~~

$$h_\pi(y) = h_\pi(1/N_\pi y). \qquad (8)$$

Thus the positivity of $h_\pi(y)$ depends on the coefficients $\lambda_\pi(n)$ and on the shape of the local "Whittaker" function $W_{\pi_\infty}(y)$.

Whether $W_{\pi_\infty}(y)$ is positive or put another way $\xi(t,\pi_\infty) = L(\frac{1}{2}+it, W_{\pi_\infty})$ is positive definite on $\mathbb{R}$, is clearly relevant and is an interesting local question.

<u>Definition</u> We say that π is positive definite if $\xi(t,\pi)$ is positive definite.

This property is of interest for a number of reasons:

(1) If π is positive definite then $\Lambda(s,\pi)$ has no real zeros (and hence no Siegel zeros according to any definition of the latter).

(2) If π on $GL_2/\mathbb{Q}$ corresponds to a classical holomorphic (hecke) eigenform f_π on $\mathbb{H}$, then π is positive definite iff $f_\pi(z)$ has no zeros on the y axis. The location of the zeros of such forms is a classical problem in modular forms.

(3) If π corresponds to a classical Maass (hecke) eigenform ψ_π on $\mathbb{H}$, then π being positive definite is equivalent to the nodal line of ψ_π not meeting the y-axis. This is especially interesting for ψ a Maass cusp form ~~with~~ with eigenvalue $1/4$ (which most people

expect correspond exactly to even 2-dimensional finite Galois representations). According to Selberg's eigenvalue conjecture, these φ's will have the smallest nonzero eigenvalue on the corresponding modular surface, and hence by Courant's nodal domain theorem these will have very few nodal domains. So the location of the nodal lines for these is very interesting.

(4). The fundamental example of a positive definite π, is $\pi = \mathbb{1}$ the trivial representation on GL_1, that is $L(s, \pi) = \zeta(s)$. In this case Riemann gave an explicit expression for $\xi(t, \mathbb{1})$ and for the corresponding $h(y)$. In [Bo] Bombieri gives a nice symmetric form for this;

$$h(y) = -\mathcal{D}\left(\theta(y^2)\right) \qquad \text{——— (9)}$$

where $\mathcal{D} = D(1-D)$, $D = -y\frac{d}{dy}$ and θ is the theta function

$$\theta(y) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 y} \qquad \text{——— (10)}$$

Explicitly

$$h(y) = 4y^{5/2}\sum_{n=1}^{\infty}(2\pi^2 n^2 y^2 - 3\pi n^2)e^{-\pi n^2 y^2}. \qquad \text{——— (11)}$$

From (11) it is immediate that $h(y) > 0$ for $y \geq 1$ and hence for all y from (8) (here $N_1 = 1$). Thus $\mathbb{1}$ is positive definite. The probability density $h(y)\, y^{-1/2} dy$ on $(0, \infty)$ arises in large deviation theory of the simple random walk on $\mathbb{Z}$ and it is related to the Kolmogorov-Smirnov statistic and to Brownian bridges. This is discussed in detail in [B-P-Y]. Perhaps this probabilistic connection is related to your search for positive definite π's corresponding to elliptic curves. After all one of the points in your monograph [B] is that such self dual probability densities on $(0, \infty)$ arise naturally in queuing theory.

In [Bo] the positive function $h(y)$ in (4) is a ground state in his variational problem associated to the explicit formula. As Bombieri suggests, as such it should play a central role in the study of the much sought after "Weil positivity" in that context.

(5). The question of the existence of positive definite π's when $\pi = \chi_d$ is a quadratic Dirichlet character of conductor d (on GL_1) has an amusing history. Apparently Fekete proposed that all χ_d's are positive definite and Polya [P] showed that this is false for infinitely many d's. Later Chowla [C] repeated this false conjecture and Heilbronn [H] disproved it similarly. As with your elliptic curves, most χ_d's with d small are positive definite and so it is interesting that Baker and Montgomery [B-M] show that almost all χ_d's in the sense of density are not positive definite.

Discussion:

We begin with a couple of observations. The first is that if π is positive definite then $\xi(t, \pi)$ has a strict global maximum for $t \in \mathbb{R}$ at $t=0$. In particular if $L(\frac{1}{2}, \pi) = 0$

(either for the trivial reason that $\varepsilon_\pi = -1$ or for deeper arithmetic reasons when $\varepsilon_\pi = 1$) then π is not positive definite. The second is that if π_1 and π_2 are positive definite then so is the isobaric sum $\pi_1 \boxplus \pi_2$ since $\Lambda(s, \pi_1 \boxplus \pi_2) = \Lambda(s, \pi_1)\Lambda(s, \pi_2)$ and hence

$$h_{\pi_1 \boxplus \pi_2}(y) = \int_0^\infty h_{\pi_1}(\xi)\, h_{\pi}(y\xi^{-1}) \frac{d\xi}{\xi}.$$

In this way the (many) positive definite π's that have been found numerically may be used to generate new ones using $\boxplus$. The interesting question is whether the (infinite) set of positive definite π's is finitely generated in this way. This leads to the basic question

<u>Question:</u> Is the set of cuspidal π's which are positive definite, finite or infinite?

One way of investigating this question is to examine it for π's in families $\mathcal{F}$ as defined in [Sa 1].

<u>$\mathcal{F}_1$</u>: For π's which are self dual on $GL_1/\mathbb{Q}$ the question amounts to the finiteness (or not) of the set of positive definite χ_d's in (5) above. Now if χ_d is say even ($\chi_d(-1)=1$) then we have

$$L_\infty(s,\chi_d) = \Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2) \qquad (12)$$

so that

$$W_{\pi_d}(y) = y^{1/2} e^{-\pi y^2}. \qquad (13)$$

Hence $L_\infty(\frac{1}{2}+it, \chi_d)$ is positive definite and

$$h_\chi(y) = \sum_{n=1}^{\infty} \chi_d(n)\, y^{1/2} e^{-\pi n^2 y^2}. \qquad (14)$$

It is a common belief that for $n \le d^\alpha$ with $\alpha < 1/2$ (that is n less than the square-root of the conductor), the $\chi_d(n)$'s behaves like a random multiplicative function. For a random function (not multiplicative but for the purpose of the following heuristic this should not matter much) the expected density of zeros of the function in (14) can be computed using the "Kac-Rice" formula (see for example [E-K]) and it yields the density;

$$c_1 \frac{dy}{y}, \quad \text{for} \quad d^{-\alpha} \leq y \leq 1. \tag{15}$$

Hence the expected number of zeros of $h_\chi(y)$ is $c_2 \log d$, with $c_2 > 0$ and fixed. This is consistent with most of the χ_d's not being positive definite. However this expected number grows slowly and probing further into the probability of the random function in (14) having <u>no</u> zeros in $[d^{-1/2}, \infty)$, one finds according to the theorem of Dembo-Poonen and Zeitouni

[D-P-Z], that this probability is of order ⓶
$d^{-\gamma}$, with $0<\gamma<1$! Hence by a Borel-Cantelli argument the probability that there are infinitely many d's for which $h_{random}(y)$ has no zeros, is equal to one!
If we are to believe such heuristic arguments (and having little else to go by) we should expect that there ~~are~~ are infinitely many χ_d's which are positive definite.

$\underline{\mathcal{F}_2}$: We consider self dual π's on $GL_2/\mathbb{Q}$. There are many more of these than for $GL_1/\mathbb{Q}$ and they correspond to classical holomorphic ~~and~~ to Maass cusp forms according to the local representation π_∞ of $GL_2(\mathbb{R})$. The local question for these, that is whether $L(\frac{1}{2}+it, \pi_\infty)$ is positive definite its relevant and interesting. Of course π_∞ must be tempered (according to Selberg's conjecture) and a quick inspection [since $\Gamma_{\mathbb{R}}(s-i\nu)\Gamma_{\mathbb{R}}(s+i\nu)$ is not positive definite if $\nu>0$ since the corresponding Whitaker function $W(y)$ oscillates through 0 as $y\downarrow 0$] shows that $L(\frac{1}{2}+it,\pi_\infty)$ is positive definite iff

$$L(s,\pi_\infty)=\begin{cases}\Gamma_{\mathbb{R}}(s+\varepsilon_1)\Gamma_{\mathbb{R}}(s+\varepsilon_2)\,, & \varepsilon_1=\varepsilon_2=0 \text{ or } \varepsilon_1=\varepsilon_2=1\\ \Gamma_{\mathbb{C}}\left(s-\frac{k-1}{2}\right)\,, & k\geq 1 \text{ an integer.}\end{cases} \qquad (16)$$

That is, π_∞ is positive definite and tempered iff π_∞ corresponds to a Maass form with eigenvalue $1/4$ in the first case and a holomorphic form of weight k in the second. These are exactly the π's on GL_2 that are called algebraic in [Cl] and in [B-G].

Returning to the positive definiteness question for the global L-functions in this family $\mathcal{F}_2$, it is natural to consider various subfamilies. For example one can fix the level

N (ie the conductor N_π) and let π run over all Maass forms ordered by their analytic conductors $C(\pi)$ (see [I-S] for a definition). For these $C(\pi) \sim t_\pi^2$ where $\lambda_\pi(\infty) = \frac{1}{4} + t_\pi^2$. We denote this family by $\mathcal{F}_2^{\text{Maass}}(N)$. Fixing N and letting π run over all holomorphic cusp forms of weight k and level N (here $C(\pi) \sim k^2$) we get a family which we denote by $\mathcal{F}_2^{\text{hol}}(N)$. Finally we can fix π_∞ and vary the level N.

For $\mathcal{F}_2^{\text{Maass}}(N)$ with $t_\pi \to \infty$, not only is $L(s, \pi_\infty)$ not positive definite but $W_{\pi_\infty}(y)$ oscillates with a frequency increasing with t_π for $y \ll t_\pi$. From this it is easy to see ([Sa 2]) that $h_\pi(y)$ must have zeros (or equivalently the nodal line of the corresponding high energy Maass form must meet the y axis) for $y \sim k$. It follows that $\mathcal{F}_2^{\text{Maass}}(N)$ contains only finitely many positive definite members.

The family $\mathcal{F}_2^{hol}(N)$ is much more difficult to analyze as far as positive definiteness. For these $W_{\pi_\infty}(y)$ is positive and the question is whether the corresponding holomorphic form $f_\pi(z)$ must develop zeros on the y axis as $k \to \infty$. The heuristic argument gotten by considering random co-efficients predicts $\sqrt{k} \log k$ such zeros. This is relatively large and indicates that that there might only be finitely many π's in $\mathcal{F}_2^{hol}(N)$ which are positive definite. One of the main results in the recent paper [G-S] is a proof of this finiteness:

- Fix N, then there are only finitely many holomorphic cusp forms of level N (and of any weight) which are positive definite.

For $N=1$ (i.e. full level) one could in principle list all such forms. Note that for $k=12$ and $f(z)=\Delta(z)$ the unique such cusp form, $\Delta(z)\neq 0$ for $z\in\mathbb{H}$ so that Δ is positive definite.

In the level aspect and Π_∞ fixed with Π_∞ positive definite as in (16), the finiteness question is again very subtle. For example fix $k=2$ and let f vary over holomorphic cusp forms of weight 2 and varying level N. The random model predicts that $f(z)$ has $\log N$ zeros on the y axis. The story is apparently very similar to that of the family $\mathcal{F}_1$ and if we push the heuristic reasoning to its limit we should expect that there are infinitely many positive definite such f's. The same reasoning applies to your family of f_E's.

$\mathcal{F}_m$, $m\geq 3$:

For families of self dual automorphic forms π on GL_m, $m\geq 3$ the picture is similar. There are many more self dual forms and the ones for which the local function $L(s,\pi_\infty)$ is positive definite are special. If one wants to match this local positive definiteness with the condition of π_∞ being algebraic in the sense of [C] and [B-G] then it should probably be in terms of the positivity of the full Whittaker function $W_{\pi_\infty}\left(\begin{pmatrix} y_1 & & & \\ & y_2 & & \\ & & \ddots & \\ & & & y_m \end{pmatrix}\right)$ rather than the restriction to $\begin{pmatrix} y & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$ in (4). I have not examined in any detail if this works.

What can be approached in quite some generality is an extension of the Baker-Montgomery result to families $\mathcal{F}$ as in [Sa 1]. That is given such a family $\mathcal{F}$ of self dual π's, then when ordered by conductor almost all of them are not positive definite. In particular this applies to your family of elliptic curves. The formulation and proof of this general sparsity in families of positive L-functions is due to J. Jung in his Princeton Ph.D. thesis 2011/12.

Best regards

Peter Sarnak.

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