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OPTIMAL BOARDING POLICIES FOR THIN PASSENGERS

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Abstract

We deal with the problem of seating an airplane's passengers optimally, namely in the fastest way. Under several simplifying assumptions, whereby the passengers are infinitely thin and react within a constant time to boarding announcements, we are able to rewrite the asymptotic problem as a calculus of variations problem with constraints. This problem is solved in turn using elementary methods. While the optimal policy is not unique, we identify a rigid discrete structure which is common to all solutions. We also compare the (non-trivial) optimal solutions we find with some simple boarding policies, one of which is shown to be near optimal.

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1. Introduction

The process of airplane boarding is experienced daily by millions of passengers worldwide. Airlines have adopted a variety of boarding strategies in the hope of reducing the gate turnaround time for airplanes. Significant reductions in gate delays would improve on the quality of life for long-suffering air travelers, and yield significant economic benefits from more efficient use of aircraft and airport infrastructure; see van den Briel et al. [6], van Landeghem and Beuselinck [10], Marelli et al. [11].

Despite the fact that the boarding process is an important part of the customer's flying experience, there has been little effort in the way of airplane boarding analysis; see van Landeghem and Beuselinck [10]. The efforts (van den Briel et al. [6, 7], van Landeghem and Beuselinck [10], Marelli et al. [11] and Ferrari and Nagel [9]) thus far have been simulation and heuristics based. To the best of our knowledge, no previous attempt at a rigorous mathematical analysis has been made.

The most pervasive strategy currently employed links boarding time to seat assignment. The airline specifies which rows may start boarding at any given time. The policy is implemented by announcements of the form "Passengers from rows 30 and above are now welcome to board the plane". Such strategies are *back-to-front* boarding strategies since they board passengers from the back of the airplane first. Moreover, current policies may be considered to be *jump policies* in the sense that the announcements allow passengers in a certain group of contiguous rows to join the queue simultaneously. It is possible to consider other back-to-front policies which gradually allow passengers to join the boarding queue, one row at a time. Such policies may be implemented using a display at the terminal which shows which row (and those beyond it) may now join the queue. In jump policies the row numbers on display will decrease by jumps, while in gradual policies they will steadily decrease by at most one at any given time. Among the gradual policies we can single out the constant pace policies in which the displayed row number decreases by one after a constant time interval. When assessing boarding policies we also need to model the way in which passengers react to the policies. In this paper we assume that passengers join the boarding queue uniformly within a constant time interval of being allowed to do so by the policy. The size of the interval, which we denote by α, is a reciprocal measure of the attentiveness of passengers. The rate at which passengers from new rows are allowed to join the boarding queue in the constant rate policy, and the size of jumps in jump policies, may be adjustable parameters, depending on how attentive the passengers are to the boarding policy. The optimal boarding policies and boarding times will also depend on α.

In this paper we find a family, parameterized by α, of asymptotically optimal back-to-front boarding policies under the assumption that passengers are infinitely thin or, equivalently, that the distance between successive rows in the airplane is very large. While this assumption may seem unrealistic, it is still instructive to consider this case since the obtained results are rather precise and the methods introduced to solve it can be applied more generally. The problem is solved by reducing the problem of airplane boarding to the problem of finding the longest increasing subsequence among uniformly sampled points in a planar domain. The problem is then reduced to an isoperimetric type problem of finding an optimal domain. In a forthcoming series of papers, [4, 5], we will show that airplane boarding for non infinitely thin passengers can be reduced to a connect-the-dots type problem [2], which is obtained by sampling points from a domain in a 2-dimensional Lorentzian manifold with respect to the volume form. For a popular science description of this work, see [3]. The case considered in this paper corresponds to certain domains in flat Minkowski space (corresponding to special relativity theory rather than general relativity). It is thus easier and can be solved completely. The optimal policies we present are rather complex; see Section 4 for complete details. However, we can show that constant rate policies are near optimal and are better than group policies, after the policy parameters are adjusted to the attentiveness of the passengers. The optimal policies are also not unique, but we identify an α-dependent discrete substructure, common to all optimal solutions.

The paper is organized as follows. In Section 2 we present carefully our model of the boarding process. We also show that increasing subsequences in a permutation, attached to the boarding process, correspond to sequences of passengers, each blocking his successor's way to his assigned seat. Thus, boarding time will coincide with the length of the maximal increasing subsequence. In Section 3 we digress to a discussion of the asymptotics of maximal increasing subsequences of permutations. The main results are stated in Section 4. Section 5 is devoted to the proofs of our results. In

Section 6 we consider the examples of jump and constant rate policies and compare them with the optimal policies.

2. The model of airplane boarding

2.1. The boarding process

Let us describe our model of the airplane boarding process. We assume that passengers are assigned seats in the airplane in advance of the boarding process. For simplicity we also assume that each row in the airplane is designed for a single passenger. The airplane has S seats. We denote the total amount of time allotted for queue joining by Q. We represent passengers by points (x, y) in the plane, where x is the row assigned to the passenger and y the time at which he joins the boarding queue. For convenience, we shall normalize both variables; x is replaced by x/S, and y by y/Q. Via these normalizations, the point (x, y) will lie in the unit square. Passengers are assumed to be infinitely thin. Once passengers arrive at their designated seat, it takes them a fixed amount of time to get organized and clear the aisle by sitting down. This assumption leads to a synchronous boarding process, which we describe in terms of rounds. Initially, all passengers line up in a queue in front of the airplane gate. In the first round, all passengers who can walk unobstructed all the way to their assigned row do so. Those who cannot reach their seat, due to another passenger with a smaller row number obstructing their way, proceed as far as possible and queue behind an obstructing passenger. At the end of the round, all passengers who have reached their assigned row sit down simultaneously. Once the first round of passengers is seated, the remaining passengers advance forward again, beginning a second round of movement. The process repeats until everyone is seated. The number of rounds needed will be taken as our measure of total boarding time.

We define a natural partial order on passengers. A passenger A *blocks* another passenger B, and we denote $A \prec B$, if the latter may sit only after the former has done so. Formally, let $A = (x_A, y_A)$, $B = (x_B, y_B)$. Then $A \prec B$ if $x_A < x_B$ and $y_A < y_B$. Intuitively, this condition means that B arrives after A and sits behind him. A chain in this partial order is an *increasing sequence* or an *upright sequence*.

In terms of the partial order, the boarding process is a well-known "peeling" process, which can be traced to the work of Cantor on ordinal arithmetic. The process peels the partially ordered set by successively eliminating (in rounds) the minimal elements in the partial order. In our case, the minimal elements are precisely the passengers which are unobstructed, and can thus proceed directly to their assigned seat. This shows that the two processes indeed coincide. We recall that a set C in a partial order is a *chain* if every two elements $x, y \in C$ are comparable, i.e., either $x \prec y$ or $y \prec x$. A set I is *independent* if no two elements of I are comparable. The peeling process provides simultaneously a minimal decomposition of the poset into independent sets and a maximal chain in the poset. To see this, consider R_i, the set of passengers arriving at their seat during round i. By the definition of the peeling process, each R_i is independent. To show that the R_i's form a minimal decomposition into independent sets, we construct a chain, whose length equals the number of rounds, as follows. Each passenger (not seated in the first round) is assigned a pointer, which points to the last passenger who was responsible for blocking his/her way to the assigned row. This passenger obviously sat down in the previous round. Following the trail of pointers starting from a passenger who arrived at his seat in the last round, we identify a longest chain in the partial order. In particular, the number of rounds needed is the size of the longest chain in the partial order, or the longest increasing subsequence. The boarding process in fact coincides with patience sorting, a procedure which computes the longest increasing subsequence in a permutation. The corresponding permutation is obtained by indexing the passengers according to their x-coordinate, while the value of the permutation is the y-coordinate index. We refer the reader to Aldous and Diaconis [1] for a description of patience sorting and its importance in studying increasing subsequences.

2.2. Airline policies

An airline policy is represented by a function $L(x)$, which provides the earliest time at which a passenger from row x may join the queue. Stated otherwise, the policy represented by L allows passengers in row xS to join the queue at time yQ. Back-to-front policies will obviously correspond to non-increasing functions.

In addition to a boarding policy, we need a probabilistic model for the passenger's reaction to the boarding policy.

We consider the following reaction model, which we call the *attentive reaction model.*

We assume that passengers follow the airline policy in the sense that a passenger in row x does not join the queue before time $L(x)$. We also assume that there exists an attentiveness parameter α, such that a passenger at row x joins the queue uniformly within α time units of the earliest allowed boarding time $L(x)$. α is a reciprocal measure of the attentiveness of the passengers, with a small value of α corresponding to very attentive passengers. When considering policies L in conjunction with this reaction model, we shall always assume that passengers are allowed at least α time units to board, or, equivalently, that $L(x) \leq 1 - \alpha$. The policy L coupled with the attentive reaction model with parameter α determine a density $p_{L'\alpha}$ which describes the probability of a passenger from row x to join the queue at time y. More precisely, our assumption that passengers from row x do not join the queue before time $L(x)$, but do join within α time units, means that $p_{L,\alpha}(x, y) = 0$ for $y \leq L(x)$ and $y \geq L(x) + \alpha$. The assumption of uniform boarding within the time interval $L(x) \leq y \leq L(x) + \alpha$ means that $p_{L,\alpha}(x, y) = 1/\alpha$ for $L(x) \leq y \leq L(x) + \alpha$. Given some value of α, it will sometimes be more convenient for us to represent a policy by the function $U = L + \alpha$ rather than by L itself. The representation of a policy by U is the *upper representation.* We will use both representations interchangeably, and thus $p_{L,\alpha}$ is the same as $p_{U,\alpha}$.

Following the discussion in Subsection 2.1, we may define the random variable $T_{L,\alpha,n}$ which represents the boarding time of n passengers, given the policy L and attentiveness parameter α, as follows. We choose n i.i.d. points from the unit square using the density $p_{L,\alpha}$. $T_{L,\alpha,n}$ is the length of the longest increasing subsequence among the n points.

3. The asymptotics of increasing subsequences

The following result of Deuschel and Zeitouni [8] describes the asymptotic behavior of the maximal increasing subsequence of n i.i.d. points in the unit square chosen with respect to a density p. Following the previous section, we will apply this result to densities of the form $p_{L,\alpha}$. In the following theorem, the notation w.h.p (with high probability) refers to a property which holds with probability approaching 1 as n approaches infinity.

Theorem 1. *Let $p(x, y)$ be a differentiable density distribution on a domain U in the unit square, and let S be a set of n i.i.d. points in the unit square, chosen with respect* *to the density p. Let A and B be points in U with $A \prec B$. Denote by $K = K(A,B)$ the largest increasing subset of S, whose points lie between A and B. Then:*

1) *For all $\varepsilon > 0$, w.h.p, $|K - C(A,B)\sqrt{n}| < \varepsilon\sqrt{n}$. Here $C = C(A,B)$ is given by*

$$C_{p,A,B} = \max_\phi 2\int_{x_A}^{x_B} \sqrt{\phi'(x)p(x,\phi(x))}dx,$$

where ϕ runs through all differentiable, nondecreasing functions on the unit interval with boundary conditions $\phi(x_A) = y_A$ and $\phi(x_B) = y_B$.

2) *For any $\varepsilon, \delta > 0$, w.h.p, an increasing subset of size $K - \varepsilon\sqrt{n}$ whose elements lie between A and B can be found in a δ-neighborhood of ϕ if ϕ maximizes the above functional. Here a δ-neighborhood is the set of all points with a vertical distance of less than δ from some point of the form $(x, \phi(x))$.*

4. The main result

Let $0 < \alpha \leq 1$, and let $\mathcal{D}_\alpha$ be the family of all non-increasing functions from $[0,1]$ to $[\alpha,1]$. We view the family $\mathcal{D}_\alpha$ as representing all back-to-front policies via the upper representation $U = L + \alpha$. Let I be the family of all differentiable non-decreasing functions from $[0,1]$ to itself. (In the sequel, if the upper function is denoted by U^* or $\tilde{U}$, say, instead of U, then the corresponding lower function is L^* or $\tilde{L}$, respectively, etc.) Since boarding time $T_{U,\alpha,n}$ is given by the size of the longest increasing subsequence, Theorem 1 shows that the boarding time T of n passengers, given a back-to-front airplane policy U and attentiveness parameter α, satisfies w.h.p

$$-\varepsilon\sqrt{n} \leq T - \mathcal{O}_{U,\alpha}\sqrt{n} \leq \varepsilon\sqrt{n} \tag{1}$$

for all $\varepsilon > 0$, where

$$\mathcal{O}_{U,\alpha} = \sup_{\psi \in I} 2\int_0^1 \sqrt{p_{U,\alpha}(z,\psi(z))\psi'(z)}dz, \tag{2}$$

The problem of finding an optimal policy is thus reduced to

Problem 1. Find a function U^*_α such that

$$\mathcal{O}_{U^*_\alpha,\alpha} = \min_{U \in \mathcal{D}_\alpha} \mathcal{O}_{U,\alpha} \tag{3}$$

Our main result is the following:

Theorem 2. *For given* $0 < \alpha \leq 1$, *denote* $N(\alpha) = \lceil \frac{2/\alpha - 3 + \sqrt{(2/\alpha - 2)^2 + 1}}{2} \rceil$ *and* $M(\alpha) = \frac{(N(\alpha)+1)\alpha - 1}{N(\alpha)^2}$, *and consider the partition* $[0,1] = \bigcup_{i=0}^{N(\alpha)} I_i$, *where:*

$$I_0 = [0, \tfrac{M(\alpha)}{\alpha}],$$

$$I_i = ((2i-1)\tfrac{M(\alpha)}{\alpha}, (2i+1)\tfrac{M(\alpha)}{\alpha}], \qquad 1 \leq i \leq N(\alpha) - 1, \tag{4}$$

$$I_{N(\alpha)} = ((2N(\alpha)-1)\tfrac{M(\alpha)}{\alpha}, 1].$$

Then the function U^*_α, *defined by*

$$U^*_\alpha(x) = \frac{i^2 M(\alpha)}{x} + 1 - i\alpha, \qquad x \in I_i, \ \ 0 \leq i \leq N(\alpha), \tag{5}$$

forms a solution of Problem 1. The optimal value of the objective function in (2) is

$$\mathcal{O}_\alpha = \frac{2}{N(\alpha)} \sqrt{N(\alpha) + 1 - 1/\alpha}.$$

We remark that the function U^*_α is composed of a concatenation of hyperbola segments. This fact hints at the Lorentzian interpretation of the problem since hyperbolas are the "circles" of Minkowski space.

It will turn out in the course of the proof that the function U^*_α in Theorem 2 is not the only solution of Problem 1. In fact, Problem 1 admits uncountably many solutions for any fixed α. The proof of Theorem 2 implies that the solution $U^*_\alpha(x)$ is special, though, in the sense that it bounds from above all the solutions. In other words, denoting by $\mathcal{O}pt_\alpha$ the set of all solutions of Problem 1, we have

$$U^*_\alpha(x) = \sup_{U \in \mathcal{O}pt_\alpha} U(x), \qquad x \in [0,1].$$

However, there are several points at which all solutions of the problem coincide, as the following proposition shows.

Proposition 1. *Let* $0 < \alpha \leq 1$, *and put* $N = \lceil \frac{2/\alpha - 3 + \sqrt{(2/\alpha - 2)^2 + 1}}{2} \rceil$. *For every solution* $U \in \mathcal{O}pt_\alpha$ *we have:*

$$U\left(\frac{i}{N(\alpha)}\right) = 1 - \frac{i}{N(\alpha)}(1 - \alpha), \qquad 0 \leq i \leq N(\alpha).$$

5. Proofs

In this section we fix some $0 < \alpha < 1$. To prove the main result, we need a few lemmas first.

Lemma 1. *Let $U \in \mathcal{D}_\alpha$ be an arbitrary fixed function and $p = p_{U,\alpha}$. Then:*

$$\sup_{\psi \in I} \int_0^1 \sqrt{p(z,\psi(z))\psi'(z)}dz = \sqrt{\frac{1}{\alpha}} \cdot \sqrt{\sup_{0 \le a < b \le 1} (b-a) \cdot (U(b-) - L(a+))}.$$

Proof. Denote

$$S = \sup_{\psi \in I} \int_0^1 \sqrt{p(z,\psi(z))\psi'(z)}dz$$

and

$$C = \sup_{0 \le a < b \le 1} (b-a) \cdot (U(b-) - L(a+)).$$

First we will show that

$$S \le \sqrt{\frac{1}{\alpha}} \cdot \sqrt{C}. \tag{6}$$

Let $\psi_0 \in I$ be arbitrary and fixed. Put:

$$a_0 = \sup\{x \in [0,1] : \ \psi_0(x) < L(x)\}, \qquad b_0 = \inf\{x : \ \psi_0(x) > U(x)\},$$

(where we agree that $a_0 = 0$ if $\psi_0(0) \ge L(0)$ and $b_0 = 1$ if $\psi_0(1) \le U(1)$).

If $a_0 = b_0$ then $\int_0^1 \sqrt{p(z,\psi_0(z))\psi_0'(z)}dz = 0 \le \sqrt{\frac{1}{\alpha}} \cdot \sqrt{C}$. Otherwise we have $a_0 < b_0$. Since ψ_0 is non-decreasing and U is non-increasing, we have $L(x) \le \psi_0(x) \le U(x)$ throughout the interval (a_0, b_0), and therefore

$$\int_0^1 \sqrt{p(z,\psi_0(z))\psi_0'(z)}dz = \int_{a_0}^{b_0} \sqrt{\frac{\psi_0'(z)}{\alpha}}dz. \tag{7}$$

By Cauchy-Schwarz's inequality

$$\begin{aligned} \int_{a_0}^{b_0} \sqrt{\tfrac{\psi_0'(z)}{\alpha}}dz &\le \sqrt{\int_{a_0}^{b_0} \psi_0'(z)dz \cdot \int_{a_0}^{b_0} \tfrac{1}{\alpha}dz} \\ &= \sqrt{\tfrac{b_0 - a_0}{\alpha}} \cdot \sqrt{\psi_0(b_0) - \psi_0(a_0)}. \end{aligned} \tag{8}$$

Since

$$\sqrt{\psi_0(b_0) - \psi_0(a_0)} \le \sqrt{U(b_0-) - L(a_0+)},$$

by (7) and (8) we obtain

$$\int_0^1 \sqrt{p(z,\psi_0(z))\psi_0'(z)}dz \leq \sqrt{\frac{1}{\alpha}} \cdot \sqrt{(b_0 - a_0)(U(b_0-) - L(a_0+))} \leq C,$$

which implies (6).

Now we will show that

$$S \geq \sqrt{\frac{1}{\alpha}} \cdot \sqrt{C}. \tag{9}$$

Denote:

$$\tilde{B} = \{(a,b) : \ 0 \leq a < b \leq 1, \ U(b-) > L(a+)\}.$$

If a is a continuity point of U, then $(a,b) \in \tilde{B}$ for all b in a sufficiently small right neighborhood of a. Since U has at most countably many discontinuities, this implies $\tilde{B} \neq \emptyset$. Hence:

$$C = \sup_{(a,b)\in\tilde{B}} (b-a) \cdot (U(b-) - L(a+)) > 0. \tag{10}$$

Take an arbitrarily small $\varepsilon > 0$. By (10) there exists a pair $(a^*, b^*) \in \tilde{B}$ such that

$$C - \varepsilon \leq (b^* - a^*) \cdot (U(b^*-) - L(a^*+)), \tag{11}$$

or, equivalently,

$$\sqrt{\frac{C-\varepsilon}{\alpha}} \leq \sqrt{\frac{b^* - a^*}{\alpha} \cdot (U(b^*-) - L(a^*+))}. \tag{12}$$

Let $\tilde{\psi}$ be a linear function on $[a^*, b^*]$, with $\tilde{\psi}(a^*) = L(a^*+)$ and $\tilde{\psi}(b^*) = U(b^*-)$. Obviously:

$$\begin{aligned} \textstyle\int_{a^*}^{b^*} \sqrt{p(z,\tilde{\psi}(z))\tilde{\psi}'(z)}dz &= \textstyle\int_{a^*}^{b^*} \sqrt{\frac{1}{\alpha} \cdot \frac{\tilde{\psi}(b^*) - \tilde{\psi}(a^*)}{b^* - a^*}}dz \\ &= \sqrt{\frac{b^* - a^*}{\alpha} \cdot (U(b^*-) - L(a^*+))}. \end{aligned} \tag{13}$$

If $L(a^*+) \neq 0$ and $U(b^*-) \neq 1$, define $\tilde{\psi}$ outside $[a^*, b^*]$, in such a way that $\tilde{\psi} \in I$. By (13) and (12):

$$S \geq \int_0^1 \sqrt{p(z,\psi(\tilde{z}))\tilde{\psi}'(z)}dz = \int_{a^*}^{b^*} \sqrt{p(z,\psi(\tilde{z}))\tilde{\psi}'(z)}dz \geq \sqrt{\frac{C-\varepsilon}{\alpha}}. \tag{14}$$

If $L(a^*+) = 0$, then, by changing $\tilde{\psi}$ slightly on the interval $[a^*, a^* + \delta)$ for an arbitrarily small $\delta > 0$, we obtain similarly

$$S \geq \sqrt{\frac{C - \varepsilon - 2\delta}{\alpha}}. \tag{15}$$

The case $U(b^*-) = 1$ leads to (15) in a similar way. Letting ε and δ approach 0, we obtain (9).

The combination of (6) and (9) completes the proof of the lemma.

In view of the lemma, Problem 1 is equivalent to

Problem 2. For a given $0 < \alpha \leq 1$, find

$$M = \min_{U \in \mathcal{D}_\alpha} \sup_{0 \leq a < b \leq 1} (b - a) \cdot (U(b-) - L(a+)). \tag{16}$$

Lemma 2. *If Problem 1 has a solution U, then it has a continuous solution U^*. Moreover, U^* may be chosen so that $U^*(x) \geq U(x)$ for every $x \in [0, 1]$.*

Proof. Denote:

$$C(U) = \sup_{0 \leq a < b \leq 1} (b - a) \cdot (U(b-) - L(a+)), \qquad U \in \mathcal{D}_\alpha. \tag{17}$$

Consider an arbitrary function $U_0 \in \mathcal{D}_\alpha$. If U_0 is discontinuous, we will construct a continuous function $U_0^* \in \mathcal{D}_\alpha$, such that $C(U_0^*) \leq C(U_0)$ and $U_0^*(x) \geq U_0(x)$ for $x \in [0, 1]$.

Define a function $\tilde{U_0}$ by:

$$\tilde{U_0}(x) = \begin{cases} U_0(0), & x = 0, \\ \\ U_0(x-), & x > 0. \end{cases} \tag{18}$$

Clearly, $\tilde{U_0}$ belongs to $\mathcal{D}_\alpha$, is continuous from the left, bounds U_0 from above, and $C(\tilde{U_0}) = C(U_0)$. Hence we may assume to begin with that U_0 is continuous from the left.

Let J_0 be the (countable) set of all discontinuity points of U_0. For any $w \in J_0$, denote by $j(w) = U_0(w) - U_0(w+)$ the jump of U_0 at w. Suppose that the maximal jump of U_0 is at w_1.

Now we construct a new function $U_1 \in \mathcal{D}_\alpha$, which will be identical with U_0, except (perhaps) on a small interval $I_1 = [w_1, w_1 + \delta_1^{'}]$, such that U_1 will be continuous on I_1 and $C(U_1) \leq C(U_0)$. Take some

$$0 < \delta_1 \leq \min \left\{ \frac{C(U_0) \cdot j(w_1)}{1 + j(w_1)}, 1 - w_1 \right\},$$

and denote

$$K_1 = \tfrac{U_0(w_1) - U_0(w_1 + \delta_1)}{\delta_1},$$
$$x_0 = \inf\{x \in [w_1, w_1 + \delta_1] \ : \ U_0(w_1) - K_1(x - w_1) \leq U_0(x)\}. \tag{19}$$

Put:

$$\delta_1^{'} = x_0 - w_1. \tag{20}$$

Clearly, $0 < \delta_1^{'} \leq \delta_1$. Define:

$$U_1(x) = \begin{cases} U_0(w_1) - K_1(x - w_1), & w_1 \leq x \leq w_1 + \delta_1^{'}, \\ \\ U_0(x), & \text{otherwise.} \end{cases} \tag{21}$$

The function U_1 belongs to $\mathcal{D}_\alpha$, is continuous from the left, and $U_1(x) \geq U_0(x)$ throughout $[0, 1]$. We want to prove that:

$$C(U_1) \leq C(U_0). \tag{22}$$

To this end, we have to show that, for any points $x_1,\ x$ with $0 \leq x_1 < x \leq 1$, we have

$$(x - x_1) \cdot (U_1(x) - L_1(x_1+)) \leq C(U_0). \tag{23}$$

Denote:

$$\Delta x = x - x_1, \tag{24}$$

$$\Delta y = U_1(x) - L_1(x_1+). \tag{25}$$

If both x_1 and x lie outside the interval $[w_1, w_1 + \delta_1^{'}]$, then the left-hand side of (23) is left unchanged if U_1 and L_1 are replaced by U_0 and L_0, respectively, so that (23) holds. If both of them are within the interval, then

$$\Delta x \cdot \Delta y \leq \Delta x \leq \delta_1^{'} \leq \delta_1 \leq \frac{C(U_0) \cdot j(w_1)}{1 + j(w_1)} \leq C(U_0).$$

If $w_1 \le x_1 \le w_1 + \delta_1^{'} < x$, then

$$\Delta x \cdot \Delta y \le \Delta x \cdot (U_0(x) - L_0(x_1+)) \le C(U_0).$$

It remains to deal with the case $x_1 < w_1 \le x \le w_1 + \delta_1^{'}$. Rewrite Δx in the following form

$$\Delta x = x - w_1 + w_1 - x_1 = x - w_1 + \Delta w_1, \tag{26}$$

where $\Delta w_1 = w_1 - x_1$ and Δy in the form

$$\Delta y = -K_1(x - w_1) + \Delta y_1, \tag{27}$$

where $\Delta y_1 = U_1(w_1) - L_1(x_1+)$.

Now, if $\Delta w_1 \le C(U_0) - \delta_1^{'}$, then

$$\Delta x \cdot \Delta y \le \Delta x \le \Delta w_1 + \delta_1^{'} \le C(U_0).$$

If $\Delta w_1 > C(U_0) - \delta_1^{'}$, then by (26) and (27) we have

$$\Delta x \cdot \Delta y = \Delta w_1 \cdot \Delta y_1 + (x - w_1)(\Delta y_1 - K_1(x - w_1) - K_1 \Delta w_1). \tag{28}$$

Since $0 < \delta_1' \le \delta_1 \le \frac{C(U_0)\cdot j(w_1)}{1+j(w_1)}$ we obtain:

$$\frac{\delta_1^{'}}{j(w_1)} \le C(U^*) - \delta_1^{'}. \tag{29}$$

Note that $K_1 \ge \frac{j(w_1)}{\delta_1^{'}}$, and (29) implies:

$$K_1 \ge \frac{1}{C(U_0) - \delta_0^{'}}. \tag{30}$$

Since $\Delta y_1 \le 1$, by (30) this yields:

$$\begin{aligned}
\Delta y_1 - K_1((x - w_1) + \Delta w_1) &\le 1 - \frac{(x - w_1) + \Delta w_1}{C(U_0) - \delta_1^{'}} \\
&= \frac{C(U_0) - \delta_1^{'} - \Delta w_1 - (x - w_1)}{C(U_0) - \delta_1^{'}} \\
&= \frac{C(U_0) - \delta_1^{'} - \Delta w_1}{C(U_0) - \delta_1^{'}} < 0.
\end{aligned}$$

The last inequality, combined with (28), provides

$$\Delta x \cdot \Delta y \leq \Delta w_1 \cdot \Delta y_1 \leq C(U_0).$$

Thus we have shown that in all cases

$$C(U_1) \leq C(U_0). \tag{31}$$

Employing the same process we construct a sequence of functions $(U_i)_{i=0}^{\infty}$ in $\mathcal{D}_\alpha$ such that

$$U_0(x) \leq U_1(x) \leq \ldots \leq U_i(x) \leq U_{i+1}(x) \leq \ldots, \qquad 0 \leq x \leq 1,$$

and

$$C(U_0) \geq C(U_1) \geq \ldots \geq C(U_i) \geq C(U_{i+1}) \geq \ldots .$$

(Actually, if at some finite stage we obtain a continuous U_i, we stop the process. In what follows we shall refer only to the case where each U_i is still discontinuous.)

Each function U_i is continuous from the left and agrees (for $i \geq 1$) with U_{i-1} except (perhaps) on the interval $I_i = [w_i, w_i + \delta_i']$, where w_i is the point of maximal jump of U_{i-1}. Moreover, U_i is linear, and in particular continuous, on I_i. We may assume that

$$I_i \bigcap I_j = \emptyset, \qquad i \neq j. \tag{32}$$

By (32), for each $x \in [0,1]$ the sequence $(U_i)_{i=0}^{\infty}$ is eventually constant, and hence the sequence $(U_i)_{i=0}^{\infty}$ converges pointwise to some function U_0^*. Clearly, $U_0^* \in \mathcal{D}_\alpha$ and $U_0^*(x) \geq U_0(x)$ for each $x \in [0,1]$. Let J_i, $i \geq 0$, and J_0^* be the sets of discontinuity points of U_i and $U_0^*(x)$, respectively. Clearly, $J_0^* \subset J_i \subseteq \{w \in J_0 : \ j(w) \leq j(w_{i+1})\}$ for each i. Since $j(w_i) \underset{i \to \infty}{\longrightarrow} 0$, the function $U_0^*(x)$ is continuous. Take any two points $x_1, \ x$ with $0 \leq x_1 < x \leq 1$. For sufficiently large i we have $U_0^*(x_1) = U_i(x_1)$ and $U_0^*(x) = U_i(x)$, and therefore

$$(x - x_1) \cdot (U_0^*(x) - L_0^*(x_1)) \leq C(U_i) \leq C(U_0), \tag{33}$$

which implies $C(U_0^*) \leq C(U_0)$, and provides the lemma.

For $0 < \alpha \leq 1, \ M > 0$, put:

$$\mathcal{D}_{\alpha,M} = \left\{ U \in \mathcal{D}_\alpha : \sup_{0 \leq a < b \leq 1} (U(b-) - L(a+))(b-a) \leq M \right\}. \tag{34}$$

Similarly to Lemma 2, one can prove

Lemma 3. *If $\mathcal{D}_{\alpha,M} \neq \emptyset$ for given $0 < \alpha \leq 1$ and $M > 0$, then for any function $U \in \mathcal{D}_{\alpha,M}$ there exists a continuous function $U^* \in \mathcal{D}_{\alpha,M}$, such that $U^*(x) \geq U(x)$ for $x \in [0,1]$.*

Let $\mathcal{C}_M$ be the family of all non-increasing continuous functions $U : [0,1] \longrightarrow (-\infty, 1]$, satisfying

$$\sup_{0 \leq a < b \leq 1} (U(b) - L(a))(b-a) = M, \qquad M > 0.$$

Lemma 4. *Suppose $\mathcal{C}_M \neq \emptyset$ for a given $M > 0$. Let:*

$$U_0(x) = \sup_{U \in \mathcal{C}_M} U(x), \qquad x \in [0,1].$$

Then U_0 is non-increasing and satisfies:

$$(U_0(b-) - L_0(a+))(b-a) \leq M, \qquad 0 \leq a < b \leq 1. \tag{35}$$

Proof. Clearly, U_0 is non-increasing. For arbitrary fixed $\delta > 0$ we have:

$$\begin{aligned} &(U_0(b-) - L_0(a+))(b-a) \\ &\leq \frac{b-a}{b-a-2\delta}((b-\delta) - (a+\delta))(U_0(b-\delta) - L_0(a+\delta)). \end{aligned} \tag{36}$$

For arbitrary fixed $\varepsilon > 0$, select a function $U^* \in \mathcal{C}_M$ such that

$$U_0(b-\delta) \leq U^*(b-\delta) + \varepsilon.$$

Since $L^*(a+\delta) \leq L_0(a+\delta)$, we obtain:

$$\begin{aligned} &((b-\delta) - (a+\delta))(U_0(b-\delta) - L_0(a+\delta)) \\ &\leq ((b-\delta) - (a+\delta))(U^*(b-\delta) + \varepsilon - L^*(a+\delta)) \leq M + \varepsilon. \end{aligned}$$

Substituting in (36), we find that:

$$(U_0(b-) - L_0(a+))(b-a) \leq \frac{b-a}{b-a-2\delta} \cdot (M + \varepsilon).$$

Letting $\delta \to 0$ and $\varepsilon \to 0$ we obtain (35), which completes the proof.

For given $0 < M \le \alpha \le 1$, denote $N = \lceil \frac{\alpha/M-1}{2} \rceil$, and consider the partition $[0,1] = \bigcup_{i=0}^{N} J_i$, where:

$$J_0 = [0, \tfrac{M}{\alpha}],$$

$$J_i = ((2i-1)\tfrac{M}{\alpha}, (2i+1)\tfrac{M}{\alpha}], \qquad 1 \le i \le N-1, \tag{37}$$

$$J_N = ((2N-1)\tfrac{M}{\alpha}, 1].$$

Proposition 2. *Put:*

$$U^*(x) = \frac{i^2 M}{x} + 1 - i\alpha, \qquad x \in J_i, \;\; 0 \le i \le N. \tag{38}$$

Then:

(i) $U^* \in \mathcal{C}_M$,

(ii) $U^*(x) = \sup_{U \in \mathcal{C}_M} U(x), \;\; x \in [0,1]$.

Proof. (i) It is easy to verify that U^* is non-increasing, continuous and bounded above by 1. It remains to show that

$$\sup_{0 \le a < b \le 1} (U^*(b) - L^*(a))(b-a) = M.$$

Let $0 \le i \le N$ and $b \in J_i$ be an arbitrary fixed point of J_i. We shall split the proof into four cases.

If $0 \le i \le 2$ and $a \in I_0$ then:

$$\begin{aligned} (U^*(b) - L^*(a))(b-a) &= \left(\tfrac{i^2 M}{b} - (i-1)\alpha\right)(b-a) \\ &\le i^2 M - (i-1)\alpha b \;\le\; M. \end{aligned} \tag{39}$$

For any $0 \le i \le N$, if $b - \frac{M}{\alpha} < a < b$, then:

$$(U^*(b) - L^*(a))(b-a) = (U^*(b) - U^*(a) + \alpha)(b-a) < \alpha \cdot \frac{M}{\alpha} = M. \tag{40}$$

If $i \ge 3$ and $a \in J_k$ for some $0 \le k \le i-3$, then $0 \le a \le (2i-5)\frac{M}{\alpha}$, and therefore:

$$U^*(b) - L^*(a) \le \frac{i^2 M}{(2i-1)\frac{M}{\alpha}} - \frac{(i-3)^2 M}{(2i-5)\frac{M}{\alpha}} - 2\alpha = -\frac{\alpha}{(2i-1)(2i-5)} < 0. \tag{41}$$

Finally, let $i \geq 1$ and $a \in J_k$ for some $\max(1, i-2) \leq k \leq i$. Denote

$$A = Mi^2 - \alpha b(i - k - 1)$$

and

$$d = \frac{b - a}{M/\alpha}.$$

Define a quadratic function h by:

$$h(z) = -MAz^2 + \alpha b(A - M(k^2 - 1))z - \alpha^2 b^2.$$

A routine calculation gives:

$$(U^*(b) - L^*(a))(b - a) - M = \frac{M}{\alpha^2 ab} \cdot h(d).$$

It is easy to verify that $A > 0$ and that the discriminant of h satisfies:

$$\Delta = \alpha^2 b^2 (A - M(k+1)^2)(A - M(k-1)^2) \leq 0.$$

It follows that $h(d) \leq 0$, which implies

$$(U^*(b) - L^*(a))(b - a) \leq M. \tag{42}$$

Altogether, by (40), (41), (39), and (42):

$$(U^*(b) - L^*(a))(b - a) \leq M, \qquad 0 \leq a \leq b \leq 1. \tag{43}$$

On the other hand, let $b \in J_i$ for any $1 \leq i \leq N$, and set $a_0 = \frac{i-1}{i} b$. It is easy to verify that

$$(U^*(b) - L^*(a_0))(b - a_0) = M, \qquad b \in J_i,\ 1 \leq i \leq N, \tag{44}$$

which, combined with (43), completes the first part of the proof.

(ii) Put:

$$U_0(x) = \sup_{U \in \mathcal{C}_M} U(x), \quad x \in [0, 1].$$

Suppose U_0 is not identical with U^*. Let j be the smallest index i for which the restrictions of U_0 and U^* to J_i are non-identical. Clearly, $1 \leq j \leq N$ and $U_0(x) = U^*(x)$ for each point $x \in J_{j-1}$. Let $b_0 \in J_j$ be a point with $U_0(b_0) > U^*(b_0)$. Put $a_0 = \frac{j-1}{j} b_0$. Then $a_0 \in J_{j-1}$ and $U_0(a_0) = U^*(a_0)$. As in (44),

$$(U_0(b_0) - L_0(a_0))(b_0 - a_0) > (U^*(b_0) - L^*(a_0))(b_0 - a_0) = M,$$

which contradicts Lemma 4, and thus completes the second part of the proof.

Proof of Theorem 2. If $N = \lceil \frac{2/\alpha - 3 + \sqrt{(2/\alpha-2)^2+1}}{2} \rceil$ and $M = \frac{(N+1)\alpha - 1}{N^2}$, then $U^*(1) = \alpha$, where U^* is defined as in (38). By part (i) of Proposition 2 we have $U^* \in \mathcal{C}_M$, and thus $U^* \in \mathcal{D}_\alpha$. Hence $U^* \in \mathcal{D}_{\alpha,M}$.

Let $0 < M < \frac{(N+1)\alpha-1}{N^2}$ be arbitrary and fixed. Suppose there exists some $\tilde{U} \in \mathcal{D}_{\alpha,M}$. By Lemma 3, there exists a continuous function $\tilde{U_0} \in \mathcal{D}_{\alpha,M}$. Since, by part (ii) of Proposition 2, U^* is the supremum of all continuous functions $U \in \mathcal{C}_M$, and it is easy to verify that

$$U^*(1) = N^2 M + 1 - N\alpha < (N+1)\alpha - N\alpha = \alpha,$$

it follows that for any $U \in \mathcal{C}_M$ we have $U(1) < \alpha$. Thus, for any $U \in \mathcal{C}_M$, we obtain $U \notin \mathcal{D}_{\alpha,M}$, which implies that $\mathcal{D}_{\alpha,M} = \emptyset$.

Hence U^*, with $M = \frac{(N+1)\alpha-1}{N^2}$, forms a solution of Problem 1. In view of Lemma 1, the optimal value is $\mathcal{O}_\alpha = \frac{2}{N}\sqrt{N+1-1/\alpha}$.

Proof of Proposition 1. Let $U \in \mathcal{O}pt$. First note that $\frac{i}{N} \in I_i$ for $i = 0, 1, \ldots, N$. We start by examining the value of U at the last of the points i/N, namely $1 \in I_N$. On the one hand, $U(1) = L(1) + \alpha \geq \alpha$, and on the other hand, by Proposition 2.ii and Lemma 2, we have $U(1) \leq U^*(1) = \alpha$. Thus, every $U \in \mathcal{O}pt$ satisfies $U(1) = U^*(1) = \alpha$. Next consider the point $\frac{N-1}{N} \in I_{N-1}$. Since U is a solution of Problem 2 with optimal value $M = \frac{(N+1)\alpha-1}{N^2}$, we have

$$\left(U(1) - L\left(\frac{N-1}{N}\right)\right) \cdot \left(1 - \frac{N-1}{N}\right) \leq \left(U(1-) - L\left(\frac{N-1}{N}+\right)\right) \cdot \frac{1}{N} \leq M.$$

On the other hand, U^* also provides the same optimal value M of Problem 2, and by Proposition 2.ii and Lemma 2 imply $L\left(\frac{N-1}{N}\right) \leq L^*\left(\frac{N-1}{N}\right)$. Thus

$$\left(U(1) - L\left(\frac{N-1}{N}\right)\right) \cdot \frac{1}{N} \geq \left(U^*(1) - L^*\left(\frac{N-1}{N}\right)\right) \cdot \frac{1}{N} = M,$$

which implies

$$U\left(\frac{N-1}{N}\right) = U^*\left(\frac{N-1}{N}\right) = 1 - \frac{N-1}{N}(1-\alpha).$$

Now suppose that for some $1 \leq i \leq N$ we have proved that $U\left(\frac{i}{N}\right) = 1 - \frac{i}{N}(1-\alpha)$. One can easily check that

$$\left(U^*\left(\frac{i}{N}\right) - L^*\left(\frac{i-1}{N}\right)\right)\frac{1}{N} = M, \qquad 1 \leq i \leq N,$$

which, in the same way as above, implies that $U\left(\frac{i-1}{N}\right) = 1 - \frac{i-1}{N}(1-\alpha)$. This proves the proposition.

6. Examples

6.1. Jump policies

A jump policy is a back-to-front policy, where the function $L(x)$ is of the form $L(x) = t_i$ for $r_i \leq x \leq r_{i+1}$, for some constants $1 = t_0 > t_1 > t_2 > \ldots > t_k = 0$ and $0 = r_1 < r_1 < \ldots < r_{k+1} = 1$, for $i = 1, \ldots, k$. The parameter k is the number of jumps in the policy, and the i-th jump allows the group of passengers from rows r_i to r_{i+1} to join the queue. The idea is to allow the passengers from the k-th (last) group board first, followed by the passengers from the next to last group, and so on. This will occur if $\alpha \leq \min_i \ t_{i-1} - t_i$, an inequality which we assume. Under such circumstances, the boarding time will in fact be independent of the differences $t_{i-1} - t_i$, and we may assume that $t_i = 1 - \frac{i}{k}$ and that $\alpha = 1/k$. Applying Theorem 1, we see that the boarding time (after division by $\sqrt{n}$) is asymptotically $\max\limits_i \ 2\sqrt{r_{i+1} - r_i}$. The boarding time is minimized by choosing all jumps (groups) to be of equal size, in which case we obtain an asymptotic boarding time of $2\sqrt{\frac{1}{k}} = 2\sqrt{\alpha}$.

6.2. Constant rate policies

The constant rate policy with parameter α is given by the affine function $L_\alpha(x) = (1-\alpha)(1-x)$. As noted in the introduction, such policies allow passengers from new rows to join the boarding queue at a constant rate, which is adjusted to the attentiveness of the passengers. We know by Lemma 1 that the boarding time $\mathcal{O}_{L_\alpha}$ is given by $\max\limits_{0 \leq a < b \leq 1} \ 2\sqrt{(b-a)(U_\alpha(b) - L_\alpha(a))}$. By elementary computations we have

$$\mathcal{O}_{L_\alpha} = 2\sqrt{\frac{\alpha}{4(1-\alpha)}} = \sqrt{\frac{1}{1-\alpha}}\sqrt{\alpha} \tag{45}$$

for $\alpha \leq 2/3$, while

$$\mathcal{O}_{L_\alpha} = 2\sqrt{2 - \frac{1}{\alpha}} \tag{46}$$

for $\alpha \geq 2/3$.

A comparison with jump policies shows that constant rate policies are preferable to jump policies. We now compare constant rate policies to the optimal policies. When

$\alpha \geq 2/3$, the boarding time coincides with that of the optimal policy, and hence the constant rate policy is optimal for $\alpha \geq 2/3$. For $\alpha < 2/3$ the constant rate policy is rarely optimal, but we show that it is worse by at most a factor of $\frac{3}{2\sqrt{2}} \approx 1.06$ than the optimal policy for all values of α. Let $R(\alpha)$ denote the ratio between the performance of the optimal policy and that of the constant rate policy. We are interested in the maximal value of $1/R$. We have $R(\alpha) = 1$ for $\alpha \geq 2/3$. For $\alpha \leq 2/3$ we have

$$R(\alpha) = \frac{2}{N(\alpha)} \sqrt{\frac{((N(\alpha)+1)\alpha - 1)(1-\alpha)}{\alpha^2}}. \tag{47}$$

$N(\alpha)$ is an integer-valued function, which assumes the value $N \geq 1$ in the range $\alpha_{N-1} \leq \alpha < \alpha_N$, where $\alpha_N = \frac{2N+1}{N^2+3N+1}$ for $N > 0$ and $\alpha_0 = 1$. Fixing some N, it is easy to see that the function $R_N(\alpha) = \frac{2}{N}\sqrt{\frac{((N+1)\alpha-1)(1-\alpha)}{\alpha^2}}$ has a single maximal value in the range $\alpha > 0$, which is at the point $\beta_N = \frac{2}{N+2}$. A simple computation shows that β_N satisfies $\alpha_{N-1} < \beta_N < \alpha_N$ and that $R(\beta_N) = 1$. Hence the constant rate policy is optimal for each $\alpha = \beta_N$, and only at these values. We also conclude that the minimal value of R in the range $\alpha_{N-1} \leq \alpha < \alpha_N$ must lie either at α_{N-1} or at α_N. Plugging the formula for α_N into R, we obtain a sequence of values $\gamma_n = R(\alpha_N) = \frac{\sqrt{(N+1/2)^2-1/4}}{2(N+1/2)}$. It is clear that γ_N is an increasing sequence, and hence the minimal value is $\gamma_1 = \frac{2\sqrt{2}}{3}$ as required.

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