Average case analysis for batched disk scheduling and increasing subsequences

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ABSTRACT

We consider the problem of estimating the tour length and finding approximation algorithms for the asymmetric traveling salesman problem arising from the disk scheduling problem. Given N requests, we show that if the seek function has positive derivative at 0 the tour length is concentrated in probability around the value $C_{f,p}N^{1/2}$ for an explicit constant $C_{f,p}$ dependent on the seek function and the distribution of requests. For linear seek function we provide even tighter bounds and provide an $O(Nlog(N))$ time algorithm for finding the optimal tour. The proof uses several results on the size and location of maximal increasing subsequences. To handle more general seek functions we introduce a more general concept of increasing subsequences. we provide order of magnitude estimates on the tour length for a wide class of seek functions with vanishing derivative at 0. For general seek functions we use some geometric information on the location of maximal generalized increasing subsequences obtained via Talagrand's isoperimetric inequalities to produce a probabilistic $1+\varepsilon$ approximation algorithm. These results complement the results on guaranteed approximation algorithms for this problem presented in [**?**].

1. INTRODUCTION

Modern disk drives have the ability to queue incoming read and write requests and to service them in an out of order fashion. In the *Batched disk scheduling* problem we are given a batch of N queued requests and we wish to service them in an order which minimizes the total service time, or equivalently, the number of disk rotations required to service all N requests.

In order to be more specific we have to characterize the geometric properties of mechanical disk motion. Mechanical motion in the disk consists of two components the first is the inward/outward radial motion. The time spent by the disk head performing the first component of motion between successive requests is known as *seek time*. The second component of motion is the rotational motion of the disk around its axis, the time it takes between the arrival of the disk head to the proper radial location and the time the wanted data passes under the disk head is known as (rotational) *latency time* . Given radial and angular distance units r and θ the function $f(\theta)$ is defined to be the radial distance which the disk head can travel starting and finishing at rest (in the radial direction), while the disk rotates through an angular distance θ. The mechanics of disk head motion dictate that the function f is convex. The choice of units and f characterize our model of the physical disk. Given this model we can define using f the distance from a request P to a request Q as the time required to move the disk head from location P to location Q, with no radial velocity at the beginning and end of the movement.

In reality there are some other time components involved in servicing a request such as head switch time and data transfer time. the contribution of these elements to the total service time is often small and can be treated separately.

The main goal of this paper is to present average case analysis of the number of rotations required to optimally service N batched requests.

We assume that we are given a continuous density function $p(r)$ on the space of disk locations, which depends only on r, a restriction which will later be justified. We assume that the N requests are drawn in accordance with this distribution density. If the seek function satisfies $f'(0) > 0$ We show that the number of disk rotations needed to service all N requests is with high probability asymptotic to $C\sqrt{N}$, as N approaches infinity. The constant C is given explicitly in terms of the physical characteristics of the disk and the density $p(r)$.

If $f'(0) = 0$ then the asymptotics are different but the distribution of the number of rotations still concentrates around a certain value for which we provide order of magnitude estimates.

As a byproduct of the analysis techniques we will also obtain a probabilistic polynomial time $1+\varepsilon$ approximation algorithm.

The estimate on the number of rotations in the $f'(0) > 0$ case is analogous to the BHH theorem for the Euclidean traveling salesman problem, see [**?**], that states that the length of an optimal tour between N randomly chosen points in the unit square, equipped with the standard Euclidean metric, is with high probability asymptotic to $C_1\sqrt{N}$. How-

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ever the constant C_1 is analytically unknown. The $1+\varepsilon$ approximation algorithm is analogous to Karp's famous probabilistic approximation algorithm for the Euclidean TSP which is derived from the BHH theorem, see [?]. thus our results provide yet another example of the powerful link between knowledge of the distribution of the size of a geometric functional and approximation algorithms for the functional.

We note however that the maximal length of an optimal tour in the Euclidean case has order of magnitude $\Theta(\sqrt{N})$ while in our case it is easy to construct examples in which the optimal tour has length $\Theta(N)$ (pick all requests to be on the same ray emanating from the center of the disk). This simple observation implies that our problem does not satisfy the *subadditivity property* which is common to the Euclidean TSP, Euclidean minimal spanning tree and the Euclidean minimal matching functionals, see [?] for a thorough account of the research on these geometric functionals. As a result the methods which are required in our case are somewhat different from those which are used in analyzing the other functionals. It also explains how we get asymptotics which are not of the form $C\sqrt{N}$ in the case where $f'(0) = 0$.

We establish the tour length estimate for the case of a linear seek function, $f(\theta) = c\theta$ (seek distance is proportional to seek time), by showing that the length of the optimal tour in this case is very close to the length of the largest increasing subsequence in a particular geometric setting which was studied in [?].

When the seek function satisfies $f'(0) = 0$, the length of the optimal tour is closely related to the length of a maximal generalized increasing subsequence, a notion which we introduce in the paper.

Previous work Disk scheduling was studied extensively during the seventies and eighties, see [?], [?] and [?] among others. At the time, after the head seeked to a certain track it spent there an entire revolution of the disk, hence optimization involved only seek minimization and latency was not considered. The first papers which discuss seek+latency optimization are [?] and [?]. In particular it is noted in [?] that the problem of minimizing the total service time of a batch of requests is a particular instance of the asymmetric shortest Hamiltonian path problem. Further experimental studies which utilized certain heuristic methods for constructing tours are contained in [?]. The scheduling problem was then studied analytically for the first time in the beautiful paper [?]. The authors show that for general convex seek functions f the problem is NP-complete. The main result of [?] is a 3/2-approximation algorithm for this problem. For the case of a linear seek function they actually present a polynomial time algorithm which services the requests optimally.

We note that in the recent preprint [?], the length of the optimal tour is also studied, for batches of requests which are confined to a very small radial annulus of the disk. This assumption completely eliminates The seek component from the problem and the computation involves latency estimates only. In completely practical terms one might say that we study the estimated optimal tour length when the activity to the disk is spread over a data region of size 500MB or more, while [?] explores activity which is confined to an address space of 100MB or less. In that sense the two studies are complementary and together provide an almost complete analysis over almost all ranges.

Finally we note that in an amusing twist of faith the next generation of magnetic storage technology known as MEMS based storage devices will most likely resemble the seek only devices studied in the seventies, see [?] thus completing the cycle of research on this subject.

organization of the paper: In section 2.1 we introduce the problem and many related definitions. In particular we introduce some relaxations which are easier to analyze. Much of this material is based on [?] section 2. In section 2.2 we introduce the probabilistic setting and state all the results on increasing subsequences which will be used.

In section 3 we estimate the size of a relaxed version of our problem which is equivalent to a problem on increasing subsequences.

In section 4 we relate the relaxed version to the disk scheduling problem again using knowledge on the location of maximal increasing subsequences.

Section 5 states our main results which are for the most part easy consequences of the discussion in sections 3 and 4.

Finally section 6 briefly discusses a problem related to the online disk scheduling problem.

2. STATEMENT OF THE PROBLEM, DEFINITIONS AND PRELIMINARIES

In this section we describe the batched disk scheduling problem and briefly survey some results on increasing sequences which are used later on. The batched disk scheduling problem and the associated disk graph were formally introduced in [?] section 2 which we follow closely in the next subsection.

2.1 The disk graph

A computer disk has the shape of an annulus. For convenience we let the radial distance between the inner and outer circles be 1. Each point on the disk is then represented by polar coordinates, $R = (r, \theta)$ where $0 \le r \le 1$ is the radial distance from the inner circle and $0 \le \theta \le 1$ is the angle relative to an arbitrary but fixed ray. A complete circular angle is chosen to be of 1 unit instead of 2π as in [?] for the convenience of future computations. We denote the coordinates of a point R by $r(R)$ and $\theta(R)$ respectively.

Associated with the disk is a *reachability function* $f(\theta)$. The function $f(\theta)$ represents the maximum radial distance the disk head can travel starting and ending with no radial motion, while the disk rotates through an angle θ. The acceleration and deceleration involved in disk head motion dictates that f is a convex function. We will assume throughout the paper that the seek function f is convex and furthermore that $f(\theta) > 0$ for $\theta > 0$ unless otherwise stated.

We denote the number of rotations needed for a full radial stroke by t_s which by definition equals $f^{-1}(1)$, were f^{-1} denotes the inverse function of f.

given two requests for disk data at locations $R_i = (r_i, \theta_i)$, $i = 1, 2$, we may define the distance between the two requests R_1 and R_2 to be

$$d(R_1, R_2) = Min\,\{Integers\ \ k:\ f(\theta_2 - \theta_1 + k) \ge |r_1 - r_2|\}$$

The distance is the number of times the head must cross the line $\theta = 0$ when traveling from R_1 to R_2. Note that the distance function is asymmetrical.

Following section 5 in [?] we define a partial order on the requests. We say that $R_i \le R_j$ iff $d(R_i, R_j) = 0$. It is easy to verify that $\leq$ is indeed a partial ordering of the requests.

We denote by $mc(G)$ the size of a minimal decomposition of the vertices in G into chains with respect to $\leq$. By Dilworth's theorem, see [?], $mc(G)$ is also the size of the maximal independent subset of G with respect to $\leq$.

Given N requests, $R_1, ..., R_N$, consider their associated *disk graph*, $G = G_f$, which is a weighted complete directed graph whose vertices are the requests and the weight of the directed edge (R_i, R_j) is $d(R_i, R_j)$.

We consider the operator T which is defined by $T(r, \theta) = (r, \theta + 1)$. Consider the infinite stripe given by $0 \leq r \leq 1$ We say that R' is an *extended request* if there exists an integer l and a request R such that $R' = T^l(R)$. We say that such R and R' are equivalent It is convenient to think of extended requests in terms of space-time coordinates. equivalent extended requests correspond to the space (disk) location but in space-time R' is the same location l disk rotations later (or earlier). We can extend the relation $\leq$ to extended requests. We say that $R'_1 \geq r'_2$ iff $\theta(R'_1) \geq \theta(R'_2)$ and $f(\theta(R'_1) - \theta(R'_2)) \geq |r(R'_1) - r(R'_2)|$. It is easy to see that this definition agrees with the previous one on requests. Intuitively $R'_1 \geq R'_2$ if there is a feasible disk head motion which starts at the space-time point R'_2 (with no radial motion) and reaches the space-time point R'_1 (again with no radial motion). We define the *extended disk graph* $\hat{G}$ to be the directed graph whose vertices are the extended requests and with an edge (R'_1, R'_2) iff $R'_2 \geq R'_1$.

A *cycle* in G is an ordered set of distinct requests $R_0 = (r_0, \theta_0), ..., R_{k-1} = (r_{k-1}, \theta_{k-1})$. The *predecessor* of a request R_i in a cycle is R_{i-1} with R_{k-1} being the predecessor of R_0. We denote the predecessor of R_i by $p(R_i)$. Let L be a cycle. $l(L)$, the *length* of L, is defined as $\sum_{i=0}^{k-1} d(R_i, R_{i+1})$, where $i+1$ is understood to be Mod k, if $k > 1$. If the cycle is a singleton then we define its length to be 1. By abuse of notation we let L also denote the concatenated cycle of paths $P(R_0, R_1), P(R_1, R_2), ..., P(R_{k-1}, R_0)$.

Denote by $st(G)$ the length of the shortest Hamiltonian cycle in G. $st(G)$ is the minimal number of rotations required to service all N requests starting from one of the requests and returning to it at the end of the service, hence the notation $st(G)$ for service time.

Our goal is to provide average case analysis for $st(G)$.

If L is a cycle a path $P = P_L$ is a function $P : [0, l(L)] \longrightarrow [0, 1]$ such that for all $0 \leq i \leq k-1$ we have$P(\sum_{j=0}^{i-1} d(R_j, R_{j+1}) + \theta_i) = r_i$, $P(0) = P(l(c))$ and which describes a physically feasible disk head motion which visits request R_0 in it's first rotation, stops at request R_i, $\sum_{j=0}^{i-1} d(R_j, R_{j+1})$ rotations later, returning to R_0 after $l(C)$ rotations. Such a path exists by the definition of d.

Let $L_1, ..., L_m$ be a cover of the vertices of G by cycles. The length of the cycle cover is $\sum_{i=1}^{m} l(L_i)$. Let $cc(G)$ denote the length of a minimal cycle cover of G. Obviously $cc(G) \leq st(G)$ it is also easy to verify that $mc(G) \leq cc(G)$.

2.2 The probabilistic setting and increasing subsequences

To discuss average case analysis, we need a probabilistic model for the requests. Our probabilistic model assumes that the N requests are chosen independently in the (r, θ) unit square, $0 \leq r \leq 1$ and $0 \leq \theta \leq 1$ according to some continuous density distribution $p(r, \theta)$. The density $p(r, \theta)$ is a measure of the activity of the data residing at location (r, θ). When laying out data on a disk logically linked data is usually placed at radially adjacent locations although in some systems there is no such connection. Such a relation will imply density differences along the radial axis. It is unlikely on the other hand that such differences will be displayed in the angular direction. Therefore we assume in this paper that p the density does not depend on θ, hence $p = p(r)$.

In this context the phrase "with high probability", w.h.p for short refers to an event that occurs with probability approaching 1 on the probability space of all sets of N requests as N approaches infinity.

In the next section we will show that the disk scheduling problem is intimately connected to problems on increasing subsequences. In the course of the paper we will repeatedly use the following result of Deuschel and Zeitouni, see [?], on increasing subsequences.

We say that a set of points $z_1 = (x_1, y_1), ..., z_k = (x_k, y_k)$ in the plane is *increasing* if $x_i \geq x_j$ iff $y_i \geq y_j$ for all $1 \leq i, j \leq k$.

THEOREM 1. *Let $Q(x, y)$ be a continuous density distribution on the unit square and let S be a set of N points in the unit square chosen with respect to Q. Denote by K the largest increasing subset of S. Then*

1) *For all $\varepsilon > 0$, w.h.p, $|K - C\sqrt{N}| < \varepsilon\sqrt{N}$. Here C is given by the formula*

$$C = Max_\phi 2 \int_0^1 \sqrt{\phi'(x) Q(x, \phi(x))} dx$$

Where ϕ runs through all nondecreasing functions on the unit interval with boundary conditions $\phi(0) = 0$ and $\phi(1) = 1$.

2) *For any $\varepsilon, \delta > 0$, w.h.p, an increasing subset of size $K - \varepsilon\sqrt{N}$ can be found in a δ neighborhood of ϕ if ϕ maximizes the above functional. here a δ neighborhood refers to all points which are at a distance less then δ from a point of the form $(x, \phi(x))$.*

Theorem 1 relies on a more basic result which was shown independently by Vershik and Kerov and by Logan and Shepp, see [?] and [?], which solves the problem for the uniform distribution.

THEOREM 2. *If, in the notation above $Q(x, y) = 1$, then w.h.p*

$$K - 2\sqrt{N} < \varepsilon\sqrt{N}$$

for all $\varepsilon > 0$.

The appearance of the constant 2 in theorem 2 is a deep result. As we shall see later on in theorem 3 the asymptotics of $K - 2\sqrt{N}$ have recently been the focus of intense study, we refer the reader to [?] for an excellent survey.

We will need theorem 1 in a slightly more general context in which the requests are locally distributed with respect to the density Q in some fundamental domain of a group action and then translated by the action. We briefly explain how theorem 1 is derived from theorem 2. One can verify that only local calculations are needed in the process. Moreover the group action that we use is horizontal consisting of translations in the θ coordinate hence as will be seen below the required calculations are never correlated.

The interval [0,1] is subdivided into many intervals of a small fixed length dx. Given a nondecreasing function ϕ, and $1 \leq i \leq \frac{1}{dx}$ an integer, we consider the rectangles $(idx, \phi(idx)), (idx, \phi((i+1)dx)),$
$((i+1)dx, \phi(idx)), ((i+1)dx, \phi((i+1)dx))$. This rectangle has approximate area $\phi'(idx)(dx)^2$ and hence contains approximately $\phi'(idx)(dx)^2 Q(idx, \phi(idx))N$ points. It is shown in [?], that the small variance in the density $Q(x,y)$ over the rectangle does not affect the asymptotics of theorem 2 and hence the size of the maximal increasing subset contained in the rectangle is according to theorem 2 approximately $2\sqrt{\phi'(idx)Q(x,\phi(x))}dx\sqrt{N}$. Taking the union of all increasing subsets from all the rectangles, which is trivially seen to form an increasing subset, we obtain in an ε neighborhood of ϕ an increasing subset of the desired size.
Conversely, any increasing subset can be ordered according to its x coordinate. By definition this also orders the y coordinate. By adjoining (0,0) and (1,1) to the subset and linearly extrapolating between the points of the subset we obtain a nondecreasing ϕ for which we can apply the preceding calculation.

In [?] Baik, Deift and Johansson proved the following very deep refinement of theorem 2

THEOREM 3. *Let $q(x)$ be the Painleve II function solving the differential equation $q''(x) = xq(x) + 2q^3(x)$ which is asymptotic to the Airy function as $x \longrightarrow \infty$ and let*

$$F(x) = Exp(-\int_x^\infty (x-t)q^2(t)dt)$$

Where Exp refers to the natural exponential function. Let L_N denote the random variable describing the length of the longest increasing subsequence among N uniformly chosen points in the unit square, then

$$P(\frac{L_N - 2\sqrt{N}}{N^{1/6}} \leq x) \longrightarrow F(x)$$

as $N \longrightarrow \infty$ for all x.

The following definition and theorem are taken from [?] Consider a probability space Ω. In our case Ω will be the unit square equipped with a probability density function p.

A *Configuration function* L_N is a function from Ω^N to the non negative integers with the following property. Given any $z = (z_1, ..., z_N) \in \Omega^N$ there exists a subset J_z of the set $\{1, ..., N\}$ of size $L_N(z)$ such that for each $y \in \Omega^N$ $L_N(y) \geq Card\{i \in J_z \mid y_i = z_i\}$. The following is a deep theorem of M.Talagrand.

THEOREM 4. *Let L_N be a configuration function and let M be it's median then*

$$P(|L_N - M| \geq u) \leq 2e^{-\frac{u^2}{4M}}$$

Theorem 4 states that the distribution of values of a configuration function concentrates rather strongly around it's mean.

We note that $log()$ always refers to the natural logarithm function

Finally we also note that in several of the proofs some technical details have been omited, they will be detailed in the full version of the paper.

2.3 Strategy

We want to estimate $st(G)$, but we first estimate $mc(G)$ using part 1 of theorem 1. Then we will show that w.h.p $mc(G)$ is equal to $cc(G)$ and that $cc(G)$ is approximately equal to $st(G)$. in the later case the approximation will be up to a bounded additive factor while in the more general cases the approximation will only be in the asymptotic sense. As a result we can use the estimate for $mc(G)$ for $st(G)$ as well. We also establish stronger combinatorial relations between $mc(G), cc(G)$ and $st(G)$ under various assumptions.

3. ESTIMATING MC(G)

We estimate $mc(G)$. In the case of a linear seek function the estimates are fairly tight. For seek functions of the form $f(\theta) = c\theta^a$ we present order of magnitude estimates.

3.1 The linear case

THEOREM 5. *With high probability*

$$|mc(G) - (\sqrt{\frac{2}{c}} \int_0^1 \sqrt{p(r)}dr)\sqrt{N}| < \varepsilon\sqrt{N}$$

PROOF. We first apply to the (r,θ) unit square the linear transformation $V : (r,\theta) \longrightarrow (r/c, \theta)$. On the new rectangle we use the induced measure preserving density and we define the induced partial order $\leq_V$ by $V(R_i) \leq_V V(R_j)$ iff $R_i \leq R_j$. V transforms f into $V(f)(\theta) = \theta$, hence denoting the coordinates of $V(R_i)$ by $(r_{V,i}, \theta_{V,i})$ we have $V(R_i) \leq_V V(R_j)$ iff $|r_{V,j} - r_{V,i}| \leq \theta_{V,j} - \theta_{V,i}$.
We now compose V with an affine transformation W which shrinks (expands) our rectangle by a factor of $\sqrt{2}\frac{1}{c+1}$, then rotates it clockwise by a 45 degree angle and finally shifts it upwards by $\frac{1}{c+1}$.
It is easy to verify that the image of our rectangle is a rectangle, which we call I_c, whose vertices are

$$(\frac{1}{c+1}, 0), (0, \frac{1}{c+1}), (\frac{c}{c+1}, 1), (1, \frac{c}{c+1})$$

We also note that further inducing the relation $\leq$ now with respect to the composition WV produces the relation $WV(R_i) \leq_{WV} WV(R_j)$ which holds iff $r_{WV,j} \leq r_{WV,i}$ and $\theta_{WV,j} \geq \theta_{WV,i}$. Stated otherwise, a set of points form an independent set with respect to $\leq$ iff their images under WV form an increasing subsequence

We are now in a position to use theorem 1. Let us first assume that $p(r)$ is the constant function 1. Since the area of I_c is $\frac{2c}{(c+1)^2}$ the induced density after applying WV is $Q(x,y) = \frac{2c}{(c+1)^2}\chi_{I_c}$. Here χ simply denotes the characteristic function of the set I_c.

The convexity of the square root function and Jansen's inequality easily imply that for Q of the form $a\chi_B$ where a is a constant and B is a convex set in the unit square, the functional of theorem 1 achieves it's maximum on a curve ϕ which when restricted to B forms a straight line (this is essentially stated already in [?] section 4).

Among the lines passing through I_c it is easy to verify that we only need to consider those whose one boundary point (x_1, y_1) lies on the segment $A = (\frac{1}{c+1}, 0), (0, \frac{1}{c+1})$ and whose second boundary point (x_2, y_2) lies on the segment $B = (\frac{c}{c+1}, 1), (1, \frac{c}{c+1})$. Considering only nondecreasing lines we define $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ and then $\phi'(x) =$

$\frac{\Delta y}{\Delta x}$ and hence we must maximize $\sqrt{\Delta x}\sqrt{\Delta y}$. Since both segments A and B are inclined by 45 degrees sliding the boundary point will not change $\Delta x + \Delta y$ and hence the maximum is obtained when they are equal. This means that $\phi(x) = x$ maximizes the functional for all c.

Consider now the case of a general density $p(r)$. Consider the densities of the following form. Let $0 = r_0 < r_1 < r_2 < ... < r_{n-1} < r_n = 1$ and let c_i $i = 1, ..., n$ be arbitrary non negative coefficients with $\sum_{i=1}^{n} c_i(r_i - r_{i-1}) = 1$. Let P have the form $\sum_{i=1}^{n} c_i \chi_{I_{r_{i-1},r_i}}$ where $\chi_{I_{r_{i-1},r_i}}$ is the characteristic function of the interval $[r_{i-1}, r_i]$.

Since it is known that finite linear combinations of characteristic functions of intervals are dense in the space of Lebesgue measurable functions it is enough to prove the theorem for such densities.

Consider now the rectangular stripe J_i in the unit square consisting of points (r, θ) with $r_{i_1} \le r \le r_i$. The image of J_i under WV is up to expansion and a shift by a point of the form (x, x) isometric to $I_{c'}$ for an appropriate c'. In addition P is constant on J_i and hence $\phi(x) = x$ will maximize the functional on $WV(J_i)$ with the induced density. Obviously if a function ϕ optimizes the functional on all the J_i separately then it is optimal (the problem could have been that the functions that optimize each piece separately would not have matched at the boundary conditions), hence $\phi(x) = x$ is optimal for all densities and the formula in the statement of the theorem is obtained by plugging ϕ into the formula in theorem 1. □

Let $\varepsilon > 0$ be some positive number. We say that an independent set I is *nearly maximal* (with respect to ε) if $Card(I) \ge mc(G)(1 - \varepsilon)$.

We say that a seek function f has the *weak vertical property* if for all $\varepsilon, \delta > 0$ and every line ϕ_b of the form $\theta = b$ one can find w.h.p a nearly maximal independent set for $I_{\varepsilon,\delta}$ of G_f in a δ neighborhood of ϕ_b. We say that f has the *strong vertical property* if in addition any continuous curve satisfying the above condition is of the form ϕ_b.

It follows from the proofs of theorem 1 and theorem 5 that linear seek functions are strongly vertical.

Using (in a rather crude manner) theorem 3 we can strengthen theorem 5 in the case of the uniform distribution as follows

THEOREM 6. *If $f(\theta) = c\theta$ and p is the uniform distribution then w.h.p*

$$Blog(N)^{2/3} < mc(G) - \sqrt{\frac{2}{c}}\sqrt{N} < Alog(N)^{2/3}$$

with $A = \frac{1}{4}(2c)^{1/6} - \varepsilon$ and $B = 5^{2/3}A + \varepsilon$.

Sketch of proof: The first order approximation to $mc(G)$ is given by theorem 5. The main difference between the unit square for which theorem 3 holds and the rectangle I_c constructed in theorem 5 is that the optimal curve in the unit square case is unique (the diagonal) while there are many optimal solutions (the images of ϕ_b) in the case of I_c. Therefore in the I_c case the longest increasing subsequence is obtained as the maximum over dependent trials X_b one for each ε neighborhood of ϕ_b of the random variable described in theorem 3. The problem then becomes one of estimating the effective number of "independent" trials.

We recall that the Airy function is asymptotically equal to $\frac{Exp(2x^{3/2}/3)}{2\sqrt{\pi}x^{1/4}}$. Denote by $G(x)$ the exponent in the expression for $F(x)$ in theorem 3, then it follows that

$$Exp(-(4/3 + \varepsilon)x^{3/2} < G(x) < Exp(-(4/3 - \varepsilon)x^{3/2})$$

Consider the pair of points $S = (0, b), T = (1, b)$ and add them as "virtual requests" to the graph G, we denote the resulting graph G_b. Let $I(G_b)$ be the size of the maximal independent set in G_b containing S and T. We denote the set itself Z_b. Using the notation of theorem 3 and the techniques of proof of theorem 5 it can be verified that $I(G_B)$ is equal as a distribution to $L_{N/2c}$ Let

$$H(x) = P(\frac{I(G_b) - \sqrt{\frac{2}{c}}}{N^{1/6}} < x)$$

then by theorem 3

$$H(x) = P(\frac{L_N - 2\sqrt{N}}{N^{1/6}} < x(2c)^{1/6}) = F(x(2c)^{1/6})$$

we further claim that Z_b is contained in a $N^{-\rho}$ neighborhood of ϕ_b for all $\rho < 1/6$. The method of proof is similar to the argument in theorem 1. Let $U = (r, \theta)$ be a point of Z_b not in the neighborhood and assume for simplicity that $c = 1$. Let $Y = (r, b)$, consider the two rectangles A and B whose pairs of diagonal vertices are S,U and T,U respectively. Let C and D be the two rectangles whose pairs of diagonal vertices are S,Y and Y,T respectively. Let Z'_b be the maximal independent set in $C \cup D$ which is the union of maximal independent subsets of C and D whose sizes we denote by I_C and I_D. Similarly Z_b is the union of maximal independent subsets of A and B whose sizes are denoted by I_A and I_B. Since $\mu(B) \le \mu(D)$ we have by theorem 3 $I_B - I_D \le N^{1/6+\delta}$ for all $\delta > 0$. On the other hand $\frac{\mu(A)}{\mu(C)} < 1 - \beta N^{-2\rho}$ for some constant β hence w.h.p the number of points in A is smaller then the number of points in C by at least $N^{1-2\rho-\delta'}$ for all $\delta' > 0$, applying theorem 3 to A and B now yields w.h.p $I_C - I_A > \alpha N^{1/2-2\rho-\delta''}$ for all $\delta'' > 0$. Comparing the estimates we find that Z_b is not maximal.

For $i = 1, ..., N^{\rho}$ we consider the random variables $X_i = I(G_{iN^{-\rho}})$. After shifting to a Poissonized model, see [?] for example, and by the above considerations the X_i are independent samples of the random variable $L_{N/2c}$. If x is large then $1 - H(x) = 1 - F(x(2c)^{1/6})$ approximately equals $G(x(2c)^{1/6})$. by our estimates for G we have $G(x(2c)^{1/6}) > N^{-\rho+\delta}$ if $x \le [(\rho - \delta)\frac{4}{3}(2c)^{1/4}logN]^{2/3}$ letting ρ approach 1/6 yields the desired lower bound. The upper bound is proved similarly by examining the overlapping (dependent) variables $I(G_b)$, for where $b = iN^{-5/6}$, $i = 1, ..., N^{5/6}$ and noting that w.h.p $st(G) = Max_i I(G_b)$ up to an additive factor of the order of magnitude $N^{1/6}$.

Remark 1: It is tempting to conjecture that $mc(G)$ will behave very much like the extreme value distribution associated with N^w independent trials of the distribution $F(x)$, where w is such that the expected width of a maximal increasing subsequence is $\Theta(N^{-w})$. We hope to test this experimentally in the near future.

3.2 More general seek functions

Let f be an increasing function on the non negative real numbers with $f(0) = 0$. a set of points $z_1 = (r_1, \theta_1), ..., z_n = (r_n, \theta_n)$ is said to be a *generalized increasing sequence* w.r.t f, or *an f-sequence*, if for all $1 \le i, j \le n$ $|r_i - r_j| \ge f(|\theta_i -$

$\theta_j|$). we will always assume that an f-sequence is numbered so that $r_1 \leq r_2 \leq ... \leq r_n$.

The relation to increasing sequences is obtained by choosing $f(\theta) = \theta$ and rotating the points by 45 degrees clockwise.

It is easy to verify that given a disk graph G_f, the notion of an f-sequence coincides with the notion of an independent set (antichain) and the notion of an f^{-1}-sequence coincides with the notion of a chain.

We also note that from this interpretation in terms of f-sequences it is easy to verify that for a given f, $mc(G)$ is a configuration function in the sense of Talagrand, see section 2.2. This is seen simply by choosing J_z to be the longest f-subsequence of z.

THEOREM 7. *Let f be a seek function of the form $f(\theta) = c\theta^a$. If $a \geq 1$ There exist constants A and B, such that w.h.p $Ac^{\frac{-1}{a+1}}N^{\frac{a}{a+1}} < mc(G_f) < Bc^{\frac{-1}{a+1}}N^{\frac{a}{a+1}}$. The same conclusion holds for all $a > 0$ if we allow B to depend on a.*

PROOF. Given some $0 < b < 1$ consider the rectangles $E_{i,b}$ defined by

$$ic^{\frac{1}{a+1}}N^{\frac{-a}{a+1}} < r < (i+1)c^{\frac{1}{a+1}}N^{\frac{-a}{a+1}}$$

$$b < \theta < b + (cN)^{\frac{-1}{a+1}}$$

for $0 < i < c^{\frac{-1}{a+1}}N^{\frac{a}{a+1}}$. It is easy to verify that the area of $E_{i,b}$ is $1/N$, and that any point in $E_{i,b}$ is independent with respect to G_f from any point in $E_{j,b}$ for all $i \neq j$ both even. The probability that $E_{i,b}$ contains a point is $1 - e^{-1}$, hence we obtain w.h.p an independent set of size

$$\frac{1-e^{-1}}{2}c^{\frac{-1}{a+1}}N^{\frac{a}{a+1}}$$

We note that actually A is independent of a. To prove the upper bound, we estimate the probability that a set of size $k+1$ is independent w.r.t f. Given a set of $k+1$ points, we may sort them by their r coordinate. Let $P_i = (r_i, \theta_i)$ be the i'th point after the sort. P_{i+1} is by definition independent of P_i if $|\theta_{i+1} - \theta_i| < c^{-1/a}(r_{i+1} - r_i)^{\frac{1}{a}}$, hence the probability of that event is at most $2c^{-1/a}(r_{i+1} - r_i)^{\frac{1}{a}}$. These events for different i are independent of each other. Let $\Delta_{m,t}$ denote the m-1 dimensional simplex of points $(x_0, ..., x_{m-1})$ which satisfy $\sum_{i=0}^{m-1} x_i = t$. The volume of the simplex $\Delta_{m,1}$ is $\frac{1}{m!}$. Denote by $P_{a,c,k}$ the probability that $k+1$ points are independent with respect to G_f. By the discussion above we have

$$P_{a,c,k} \leq 2^k k! c^{-k/a} G(k,1)$$

where

$$G(k,t) = \int_{\Delta_{k,t}} (x_0 x_1 ... x_{k-1})^{\frac{1}{a}} dx_0 ... dx_{k-1}$$

This is seen by setting $x_i = r_{i+1} - r_i$. By the change of variables $x_i \to tx_i$, we have $G(k,t) = t^k G(k,1)$.

This Dirichlet integral is known and can be computed inductively, indeed

$G(k,1) = \int_{\Delta_{k,1}} (x_0 ... x_{k-1})^{1/a} dx_0 ... dx_{k-1}$

$= \int_0^1 x_{k-1}^{1/a} \int_{\Delta_{k-1,1-x_{k-1}}} (x_0 ... x_{k-2})^{1/a} dx_0 ... dx_{k-2}$

Hence

$G(k,1) = \int_0^1 x_{k-1}^{1/a} G(k-1, 1-x_{k-1}) dx_{k-1}$

$= \int_0^1 x_{k-1}^{1/a}(1-x_{k-1})^{k-1} dx_{k-1} G(k-1,1)$

$= \beta(\frac{a+1}{a}, k) G(k-1,1)$

$= \frac{\Gamma(\frac{a+1}{a})\Gamma(k)}{\Gamma(\frac{a+1}{a}+k)} G(k-1,1)$

where we have used the definition of the β function and it's relation to the Γ function. Multiplying out the induction we obtain

$$G(k,1) = (\Gamma(\frac{a+1}{a}))^k \Pi_{i=1}^k \frac{\Gamma(k)}{\Gamma(\frac{a+1}{a}+k)}$$

It is well known, see [?] that $(k-1)^s\Gamma(x) \leq \Gamma(k+s) \leq \Gamma(k)n^s$ for all s. As a result we obtain the asymptotic value of $(\Gamma(\frac{a+1}{a}))^k (k!)^{\frac{-(a+1)}{a}}$ for $G(k,1)$. Returning to the probability we obtain an upper bound of

$$P_{a,c,k} \leq 2^k c^{-k/a} \Gamma(\frac{(a+1)}{a})^k (k!)^{-1/a}$$

Denote the left hand side by $Q_{a,c,k}$. The expected number of independent sequences of size $k+1$ in a set of size N is given by $E_{a,c,k,N} = \binom{N}{k} P_{a,c,k} \leq \binom{N}{k} Q_{a,c,k}$. Using Stirling's formula to estimate the right hand side one shows that given a, c, N, the right hand side approximately equals 1 when k is approximately

$$e 2^{a/(a+1)} \Gamma(\frac{a+1}{a})^{a/(a+1)} c^{-1/(a+1)} N^{a/(a+1)}$$

and becomes exponentially small once we multiply this expression by $1 + \varepsilon$. since the expected number of sequences of length k is obviously greater then the probability for such a sequence we are done.

We see that choosing $B = 2e$ provides an upper bound for all $a \geq 1$ since $\frac{a+1}{a}$ then ranges between 1 and 2 and $\Gamma(x) \leq 1$ in that range. □

4. FROM MC(G) TO ST(G)

In this section we show that the quantities $mc(G)$ and $st(G)$ are approximately equal. The procedure will be effective in the sense that we will provide a polynomial time procedure that converts a minimal chain decomposition of G into a Hamiltonian cycle whose length is approximately the same.

4.1 From mc(G) to cc(G)

We will show that w.h.p $mc(G)$ and $cc(G)$ are approximately equal by comparing both to the intermediate quantity $mc(\hat{G})$. For linear seek functions we will obtain the much finer estimate $cc(G) - mc(G) < \frac{1}{c}$.

We first must eliminate the arbitrary choice of the angle $\theta = 0$ involved in the construction of the disk graph. To that end we consider the extended disk graph $\hat{G}$.

LEMMA 1.

1) *If f has the strong vertical property then w.h.p*

$$mc(G) = mc(\hat{G}).$$

2) *If f has the weak vertical property then w.h.p*

$$\frac{mc(\hat{G}) - mc(G)}{mc(G)} < \varepsilon$$

PROOF. We first treat the first statement. Even though $\hat{G}$ is an infinite graph it is easy to verify from the periodicity of the graph construction that the maximal independent set is always finite and can be computed by considering the subgraph induced by the extended requests in a rectangle given by $0 \leq r \leq 1$ and $0 \leq \theta \leq 1 + t_s$. Although the extended requests in such a rectangle are not randomly located (they are periodic) an examination of the proof of theorem 1 reveals that only local randomness is needed and hence the theorem still holds. In particular by the strong vertical property of f a maximal independent set will w.h.p reside in an ε neighborhood of a line of the form ϕ_b for some b. Since ε can be made arbitrarily small, w.h.p. the interval $[b-\varepsilon, b+\varepsilon]$ will not contain an integer. We can then find a copy (shift by an integer in the θ coordinate) of the maximal independent set in G as required.

The second assertion is proved similarly using almost maximal independent subsets instead of maximal independent subsets. □

THEOREM 8.

1) $mc(\hat{G}) \geq cc(G)$.

2) *If f is linear or $t_s < 1$ then $mc(\hat{G}) = cc(G)$.*

PROOF. Let I be a maximal independent set in $\hat{G}$ and let $I_j = T^j(I)$ be the maximal independent set obtained by shifting all the elements of I by j in the θ coordinate. For each extended request $R = (r, \theta) \in I$ we consider the set $Y_R = \{Q \mid R \leq Q\}$ and $X_R = Y_R - T(Y_R)$. Consider $X = \cup_{R \in I} X_R \cup I^1$. X is a fundamental domain with respect to the operator T in which each request not in I is uniquely represented up to equivalence and requests $R \in I$ are also represented in I^1. Let R_i, $i = 1, .., mc(\hat{G})$ be an enumeration of the elements of I. Consider a minimal chain cover of $\hat{G}$ restricted to X. We construct a directed graph H whose vertices are the R_i. I is a maximal independent set, hence each chain must pass through a single point $R_j \in I$. We denote such a chain by C'_j. Since I^1 is also a maximal independent set, C'_j must also pass through a point of the form $T(R_k) \in I^1$. We let (R_j, R_k) be an edge of H. We also denote by C_j the chain $C'_j - \{T(R_k)\}$. By construction the in degree and out degree of each vertex in H is one hence H decomposes into a union of edge disjoint cycles (or loops). Let S_n, $n = 1, .., r$ be the cycles of H. For a given cycle $S_n = (R_{i_1}, ..., R_{i_k})$, Consider L_n the concatenated chain in $\hat{G}$ formed from the chains $T^{j-1}(C_{i_j})$, $j = 1, ..., k-1$. Also let L'_n be formed by appending $T^k(R_{i_1})$ to L_n. Both L_n and L'_n are indeed chains by the construction of the graph H. Also by construction of X one verifies that each request in G is equivalent to a unique extended request in $\cup_n L_n$ hence each L_n induces through equivalence a cycle U_n of G and the U_n from a cycle cover for G. Moreover it is easy to verify from the construction that the length of U_n is equal to the number of times the chain L'_n crosses a line of the form $\theta = j$ for some integer j. Since the last element and first element of L'_n differ by k in the theta coordinate the length of U_n is k which is also the size of the cycle S_n. Since the in degree of elements in H is 1 summing up over all n yields that the cycle cover length of U_n is simply the size of I as required for the first statement of the theorem.

We will prove the second statement only under the assumption that f is linear. Let $C_1, ..., C_k$ be a chain cover of G whose total length is $cc(G)$. Assume that $l(C_i) > 1$ for some i. Let $P = P_{C_i} : [\theta_0, \theta_0 + l(C_i)] \longrightarrow [0, 1]$ be an extended path corresponding to C_i. Consider the function $\tilde{P}(\theta) = P(\theta) - P(\theta - 1)$. If P attains a minimum at θ_0 then $\tilde{P}(\theta) \leq 0$ and $\tilde{P}(\theta + 1) \leq 0$ hence for some angle θ_1 we have $\tilde{P}(\theta_1) = 0$. Consider the paths P' and P'' which are obtained from P by restricting it to $[\theta_1 - 1, \theta_1]$ and $[\theta_1, \theta_1 - 1]$ respectively.

As already noted in [?] the crucial property of linear seek functions which makes P' and P'' feasible paths is the property that the disk head is allowed to stop at any point along a feasible path while maintaining the feasibility of the path (the *reachability property* in the language of [?]). this is the case since a linear seek function involves no acceleration and deceleration (stopping and resuming motion are instantaneous in this model). By adding $(\theta_1, P(\theta_1))$ as a stopping point (a virtual request) and using the reachability property we see that P' and P'' correspond indeed to feasible disk head motions. By considering the equivalence classes of extended requests in P' and P'' we obtain a decomposition of C_i into two cycles C' and C'' with $l(C'), l(C'') < l(C)$ and $l(C') + l(C'') = l(C)$. Repeating this process we may assume that $l(C_i) = 1$ for all i. Consider the chain $\hat{C}_i$ obtained by the concatenation of $C_i, T(C_i), T^2(C_i), ...$ since the C_i considered as chains form a chain cover of the unit square (since $l(C_i) = 1$) $\hat{C}_i$ form a chain cover of $\hat{G}$ as desired.

□

COROLLARY 1. *If f is strongly vertical then w.h.p $mc(G) = cc(G)$*

We will show in theorem 13 that all seek functions are weakly vertical however we make the following stronger conjecture

conjecture 1: All seek functions are strongly vertical

4.2 From cc(G) to st(G)

We now show that w.h.p we can tie together the cycles that we produced in the previous theorem into one large cycle without adding many rotations. We treat the linear case for which we obtain a very precise combinatorial estimate first.

THEOREM 9. *Let $f(\theta) = c\theta$ be a linear seek function then*

$$st(G) \leq cc(G) + 2(1 + [\frac{1}{c}])$$

where [x] denotes the integral part of x.

PROOF. Consider a cycle C of requests $R_1, ..., R_k$ and it's path P_C. Projecting the graph of P_C into the unit square by $\pi(\theta, P_C(\theta)) = (\theta \ Mod \ 1, P_C(\theta))$, the image will consist of the union of the graphs of $l(C)$ continuous functions $\rho_1(\theta), ..., \rho_{l(C)}(\theta)$. Denote this union by ρ_C. Let $c' = \frac{1}{1+[\frac{1}{c}]}$. Consider the graph of the function $g(\theta) = c'\theta$. Since $c' \leq c$ the head of the disk can trace the path between any two points on the graph. We use the equivalence relation by T to push the graph into the unit square. The result is the union of $\frac{1}{c'} = 1 + [\frac{1}{c}]$ lines. Denote the union of the lines by Δ. Note that Δ is the graph of a (possibly discontinuous) function $h(r)$. Let $C_1, ..., C_m$ be any cycle cover of total length $l = \sum_{i=1}^{m} l(C_i)$. For each cycle C_i let r_i be the minimal value of the r coordinate over all points in the intersection $\rho(C_i) \cap \Delta$ and let Z_i be the point of intersection which realizes the minimal value. We assume for convenience that $r_1 \leq r_2 \leq ... \leq r_m$. We now construct a feasible motion path of the disk head which begins and ends at the point $(0,0)$ and passes through all the requests of length $l + 2(1 + [\frac{1}{c}])$. Starting at $(0,0)$ we follow the graph of h to the point Z_1. Z_1 lies on C_1, we continue through the entire path of C_1 returning to Z_1 $l(C_1)$ rotations later. Upon return we proceed from Z_1 to Z_2 along the graph of h, retrace C_2 and continue in the same fashion. Upon reaching the point Z_m after tracing the path of C_m we proceed to the point $(1,1)$ along the graph of h and slide back to $(0,0)$ along the graph of $1-h$ from $r=1$ down to $r=0$.

As in theorem 8 the reachability property ensures the feasibility of the new path which is obtained from the old ones by gluing the C_i and Δ along the Z_i. The length of the grand tour is simply the sum of cycle lengths and two passes over the graph of h as required. □

THEOREM 10. *If f is weakly vertical then for all $\varepsilon > 0$, w.h.p $st(G) < (1+\varepsilon)cc(G)$.*

Sketch of proof: Fix a large number m. Define also $m' = [\frac{1}{f^{-1}(1/m)}]$. By continuity of f, m' tends to infinity along with m. Consider the rectangles $E_i = \left\{(r,\theta) \mid \frac{i}{m} \leq r \leq \frac{i+1}{m}\right\}$. Let G_i be the disk graph for requests in E_i. Assume for the moment that $st(G_i) \leq (1 + \frac{4}{m'})cc(G_i)$. Since f is weakly vertical, Theorem 1 and the proof of theorem 5 imply that for any ε, w.h.p,

$$\frac{\sum_i mc(G_i) - mc(G)}{mc(G)} \leq \varepsilon$$

In fact we note that $\sum_i mc(G_i) \geq mc(G)$ always holds. If in addition f is weakly vertical an ε neighborhood of ϕ_b will contain a near maximal f-sequences for each G_i. These f-sequences for the G_i can be concatenated into an f-sequence in G after eliminating very few points from each sequence since the ε neighborhoods of ϕ_b restricted to different G_i are nearly f independent.

By the second statement of lemma 1 and by theorem 8 $mc(G)$ and $cc(G)$ are asymptotically equal for weakly vertical f hence

$$\frac{\sum_i cc(G_i) - cc(G)}{cc(G)} < \varepsilon$$

Given cyclic permutations π_i which service all the requests in G_i in $st(G_i)$ rotations we can consider the cyclic permutation π which simply concatenates their cycles into one large cycle. The distance between the last request in π_i and the first request of π_{i+1} is at most t_s rotations and hence $st(G) \leq \sum_i st(G_i) + mt_s$. Putting these facts together we obtain for any ε, w.h.p,

$$\begin{aligned} st(G) - cc(G) &\leq mt_s + (\textstyle\sum_i st(G_i)) - cc(G) \\ &\leq mt_s + \varepsilon cc(G) + \textstyle\sum_i (st(G_i) - cc(G_i)) \\ &\leq mt_s + \varepsilon cc(G) + \tfrac{4}{m'}(\textstyle\sum_i cc(G_i)) \\ &\leq mt_s + \varepsilon cc(G) + \tfrac{4}{m'}(cc(G) + \varepsilon cc(G)) \\ &= mt_s + (2\varepsilon + \tfrac{4}{m'})cc(G) \end{aligned}$$

Choosing m large we obtain the desired conclusion.

It remains to show that w.h.p $st(G_i) < cc(G_i)(1 + \frac{4}{m'})$. This is a combinatorial fact which always holds. Informally the idea is that the requests in G_i are confined to a very narrow radial band. By the definition of m it takes only $\frac{1}{m}$ of a rotation to cross the band from side to side, hence it is possible to move from one cycle to the next at roughly that cost.

Remark 2: We conjecture that in fact for all seek functions f, for all N large enough and for all disk graphs G we have $st(G) \leq (1+\varepsilon)cc(G)$. A polynomial time procedure for converting a minimal cycle decomposition into a Hamiltonian cycle of the same asymptotic size will immediately provide us with a $1+\varepsilon$ approximation algorithm for all f since the minimal cycle decomposition problem is polynomial. The result of [**?**] shows that $st(G) \leq 3/2cc(G)$ always (up to a bounded additive factor).

5. APPLICATIONS

In this section we apply the results of the previous section to prove our main theorems for various classes of seek functions. In all the theorems we assume that the requests are drawn with respect to a distribution of the form $p(r)$ unless stated otherwise.

THEOREM 11. *Assume that $f(\theta) = c\theta$. Let $C_{f,P} = \sqrt{\frac{2}{c}} \int_0^1 \sqrt{p(r)}dr$, then*

1) *w.h.p $|st(G) - C_{f,p}\sqrt{N}| < \varepsilon\sqrt{N}$*
2) *Assume p(r) is the uniform distribution. Let $A = \frac{1}{4}(2c)^{1/6} - \varepsilon$ and $B = 5^{2/3}A + \varepsilon$ then w.h.p*

$$A < \frac{st(G) - \sqrt{\frac{2N}{c}}}{log(N)^{2/3}N^{1/6}} < B$$

3) *There is an algorithm which computes a Hamiltonian cycle of length at most $st(G) + 2(1+[c])$ whose running time is $O(Nlog(N))$*

PROOF. all the statements of the theorem except the running time are an obvious corollary of theorems 6,9 and corollary 1 which together state that w.h.p. G $st(G) - mc(G) \leq 2([c]+1)$. Following the proofs we see that the algorithm must find a minimal chain cover of $mc(\hat{G})$. This requires some sorting operations on a set of size cN and then an application of the patience sort procedure, see [**?**], which has the same $O(Nlog(N))$ running time complexity. Translating the $mc(\hat{G})$ chain cover into a cycle cover for G can be accomplished in a linear number of steps and so can the the lacing of the cycles into a Hamiltonian cycle. Both procedures involve finding the intersections, if they exist of at most a linear number of pairs of line segments. □

Remark 3: It is easy to show that The algorithm of [**?**] which for linear seek functions finds an optimal tour has worst case running time of $\Theta(N^2)$. It's average running time is at most $O(N^{3/2})$ and is likely to be even faster but we have not analyzed it in detail.

THEOREM 12. *Assume f satisfies $f'(0) > 0$. Let $g(\theta) = f'(0)\theta$, then*

1) *f is strongly vertical*

2) *w.h.p* $|st(G) - C_{g,p}\sqrt{N}| < \varepsilon\sqrt{N}$

3) *There is a probabilistic algorithm with running time $O(Nlog(N))$ which w.h.p outputs a $1+\varepsilon$ approximation Hamiltonian cycle.*

PROOF. Since f is convex $f \geq g$ and hence $mc(G_f) \leq mc(G_g)$. We bound $mc(G_f)$ from above by $mc(G_g)$. To bound $mc(G_f)$ from below, choose a small number δ such that for all $0 < \theta < \delta$, $f(\theta < (f'(0) + \varepsilon)\theta = f_\varepsilon$. By theorem A there is a nearly maximal independent set with respect to the f_ε in the $\delta/2$ neighborhood of any line of the form $\theta = b$. By the choice of δ such a set will also be independent with respect to f. Since the sizes of the upper and lower bound both tend asymptotically w.h.p to $C_{f,p}\sqrt{N}$, we conclude that so does $mc(G_f)$, we also obtain strong verticality in the process which allows us using theorem 10 and corollary 1 to deduce the same estimate for $st(G)$. The algorithm for finding w.h.p the $1+\varepsilon$ approximation is simply the algorithm from the previous theorem applied to g. Since chains with respect to g are also chains with respect to f and the gluing procedure is legal with respect to g and hence f we obtain the desired result. □

For a general seek function f we have the following weaker result.

THEOREM 13. *Let f be a general seek function then*

1) *f is weakly vertical.*

2) *Assume f is of the form $f(\theta) = c\theta^a$, $a > 1$. There exist functions $U_a(N)$ such that*

$$\frac{1}{2}(1 - e^{-1}) \leq U_a(N) \leq 2e$$

such that

$$V_{a,c,p}(N) = U_a(N)(\int_0^1 p(r)^{a/(a+1)}dr)c^{-1/(a+1)}$$

satisfies w.h.p

$$|st(G) - V_{a,c,p}(N)N^{a/(a+1)}| < \varepsilon N^{a/(a+1)}$$

3) *There is an algorithm with running time $O(N^3)$ which w.h.p produces a Hamiltonian cycle which is a $1+\varepsilon$ approximation.*

Sketch of proof: The second statement follows from theorems 4 and 7 which provide the concentration property and the size estimate respectively for $mc(G)$.

The main issue is to show weak verticality. For a subset A of the plane define $d(A) = Max_{R\in A}|\theta(R) - 1/2|$ to be the maximal distance between any point in A and the line $\phi_{1/2}$. Fix some $\varepsilon > 0$. Let z^0 be a maximal f-sequence. We will inductively define a sequence z^j of f-sequences which w.h.p will all be nearly maximal and such that $d(z^j) \leq Max(\varepsilon, 1/2 - j\varepsilon)$ hence $z_{1/2\varepsilon}$ will be contained in a ε neighborhood of $\phi_{1/2}$. We consider the following definitions

$\eta^j = (-1)^j$

$W^j = \{i \mid \eta^j r(z_i^j) \leq \eta^j(1/2 + \eta^j\varepsilon/2)\}$

$w^j = (z_i)_{i\in W^j}$

ψ^j - reflection through the line $\phi_{1/2+\eta^j\varepsilon/2}$

$\Gamma^j = \{(r,\theta) \mid w^j \cup \{(r,\theta)\}$ *is an* $f-sequence\}$

$\Omega^j = \Gamma^j \cap \{(r,\theta) \mid \eta^j\theta \leq 1/2 + \eta^j\varepsilon/2\}$

t^j - a maximal f-sequence in $\psi^j(\Omega^j)$

$z^{j+1} = w^j \cup t^j$

We note that using the symmetry of the definition of an f-sequence it is easy to verify that $\psi^j(\Omega^j)$ is also contained in Γ^j hence by induction z^{j+1} is an f-sequence. If $d(\Omega^j \cap z^j) \leq \varepsilon$ we are done, otherwise the continuity of f implies that $\mu(\Omega^j) \geq h_{f,\varepsilon} > 0$ independently of N and the size of the maximal f-function in Ω^j concentrates. Moreover since the density function $p(r)$ is invariant under ψ Ω^j and $\psi^j(\Omega^j)$ with the probability measure induced from $p(r)$ are isomorphic hence by theorem 4 w.h.p t^j which is maximal in $\psi^j(\Omega^j)$ is almost equal in size to $z^j - w^j$ which is nearly maximal in Ω^j, we deduce by induction that z^{j+1} is nearly maximal. Finally if $d(\Omega^j) > \varepsilon$ then $d(\psi^j(\Omega^j)) \leq Max(\varepsilon, d(\Omega^j) - \varepsilon)$. Since $d(w^j) \leq \varepsilon/2$ for $j \geq 1$ the same relation holds for $d(z^j)$ and $d(z^{j+1})$ as required.

Once weak verticality has been established the second and third statements follow from theorems 8 and 10. The run time of the algorithm is dominated by the time required to find a minimal chain cover of G. This can be reduced to the problem of finding a minimal flow in a bipartite graph hence the "generous" run time estimate.

We make the following conjecture which generalizes a well known theorem of J.H.Hammersley, see [?].

Conjecture 2: Let $U_a(N)$ be the constant defined in theorem 13, then $U_a(N)$ may be chosen to be independent of N.

We note that for $a = 1$, $U_a = \sqrt{2}$ by theorem 2. The case $a < 1$ does not correspond to a seek function, but will give the length of the longest chain for the seek function $f(\theta) = \theta^{1/a}$. This issue will be treated in the next section.

Our interest in the conjecture lies in the following generalization of the estimate given in theorem 11. the proof is similar to the proof of theorems 1 or 10.

PROPOSITION 1. *If the conjecture holds, requests are distributed in accordance with a density $p(r)$ and the seek function has the form $f(\theta) = c\theta^a$, then w.h.p the number of rotations needed to service N requests is*

$$(U_a c^{-\frac{1}{a+1}} \int_0^1 p(r)^{\frac{a}{a+1}}dr)N^{\frac{a}{a+1}}$$

6. THE ONLINE PROBLEM

In the online problem requests arrive to the request queue of the disk. the requests are not grouped into batches hence any request may be serviced at any time since it's arrival. The knowledge about newly arriving requests can be used to construct more efficient tours of the ongoing stream of requests. The batched problem that we have studied refers to a particular class of algorithms which simply ignore the incoming requests until the current batch is completely served.

In [?] the authors suggest the following algorithm for the online problem which is called the CHAIN algorithm:

CHAIN: Set the angle of the current head position to be 0. Construct the longest chain in G starting with the current head position. Move to the next request along the chain. Repeat the process with the new request.

The authors show in [?] consider a version of the online problem in which the size of the queue is always N, that is, a new request arrives as soon as service is completed on an old request. They show that CHAIN experimentally outperforms the greedy online algorithm which simply advances to the nearest reachable request.

We are currently unable to analyze the online problem however we briefly examine the following related problem.

The maximal chain problem: Given N requests drawn from the uniform probability distribution, what is the length of the longest chain starting from a point R with $\theta(R) = 0$.

THEOREM 14. *Let $f(\theta) = \theta^a$, $a \geq 1$ be a seek function. The length of the maximal chain constructed by the greedy algorithm is w.h.p asymptotic to* $(\frac{2}{a+1})^{1/(a+1)}\Gamma(1/(a+1))/(a+1)N^{1/(a+1)}$

PROOF. After Poissonizing with density $1/N$ it is easy to see that we are trying to compute the reciprocal of the expectation of the density function
$h(t) = (2t^a/N)Exp(-2t^{a+1}/((a+1)N))$ which is the product of the probabilities of having a reachable request a distance t away and not having anyone closer. The expectation is then seen to be equal to $(\int_0^\infty e^{-t^{a+1}} dt)(\frac{a+1}{2})^{1/(a+1)}N^{1/(a+1)}$ which gives the desired conclusion using the identity

$$\int_0^\infty e^{-t^b} dt = \Gamma(1/b)/b$$

□

Note that for the linear seek function $f(\theta) = \theta$ the greedy algorithm constructs a chain of size $\sqrt{\frac{\pi}{2}}\sqrt{N}$ while the longest chain is of size $\sqrt{2N}$. If $a > 1$ then the size of the maximal chain is unknown however one can easily convince oneself that the size of the largest chain produced by the simple greedy algorithm is asymptotically inferior to the greedy algorithm which looks ahead a bit farther (concatenating chains of length 3). The lengths of the chains which are produced by various algorithms in the maximal chain problem serve as an upper bound on the efficiency of the online process, it is doubtful though that these rates are sustainable. We hope to study these issues further (at least experimentally) in a future paper.

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