A Modular Approach to Büchi Determinization

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CONCUR'15, Madrid

Motivation

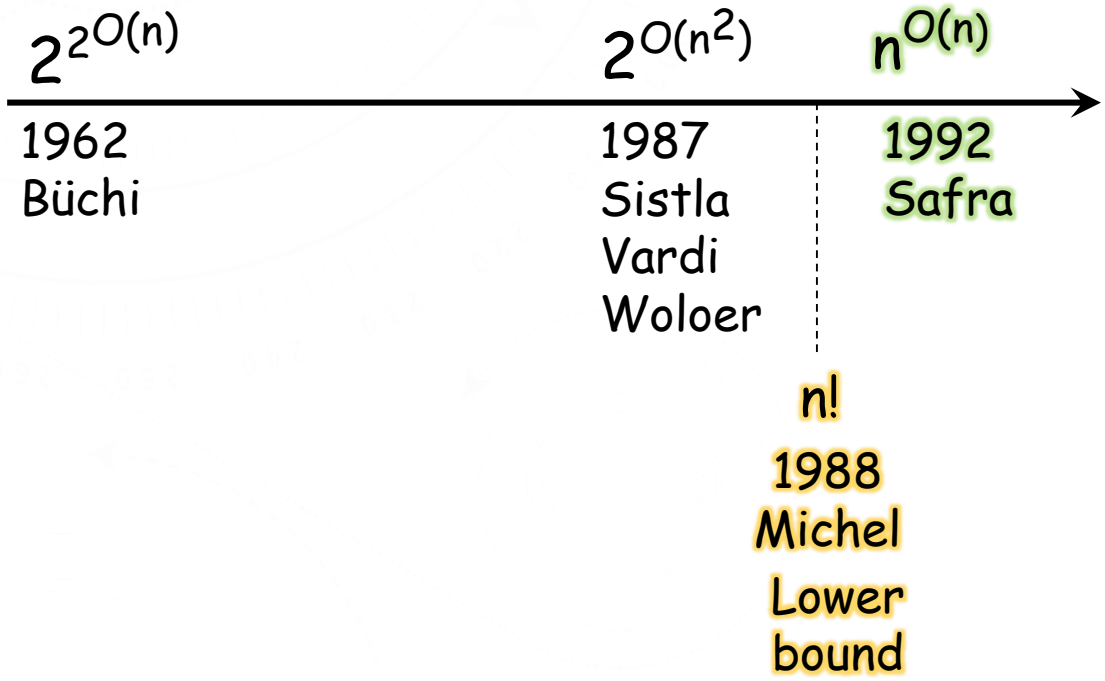
- Büchi determinization is a fundamental problem

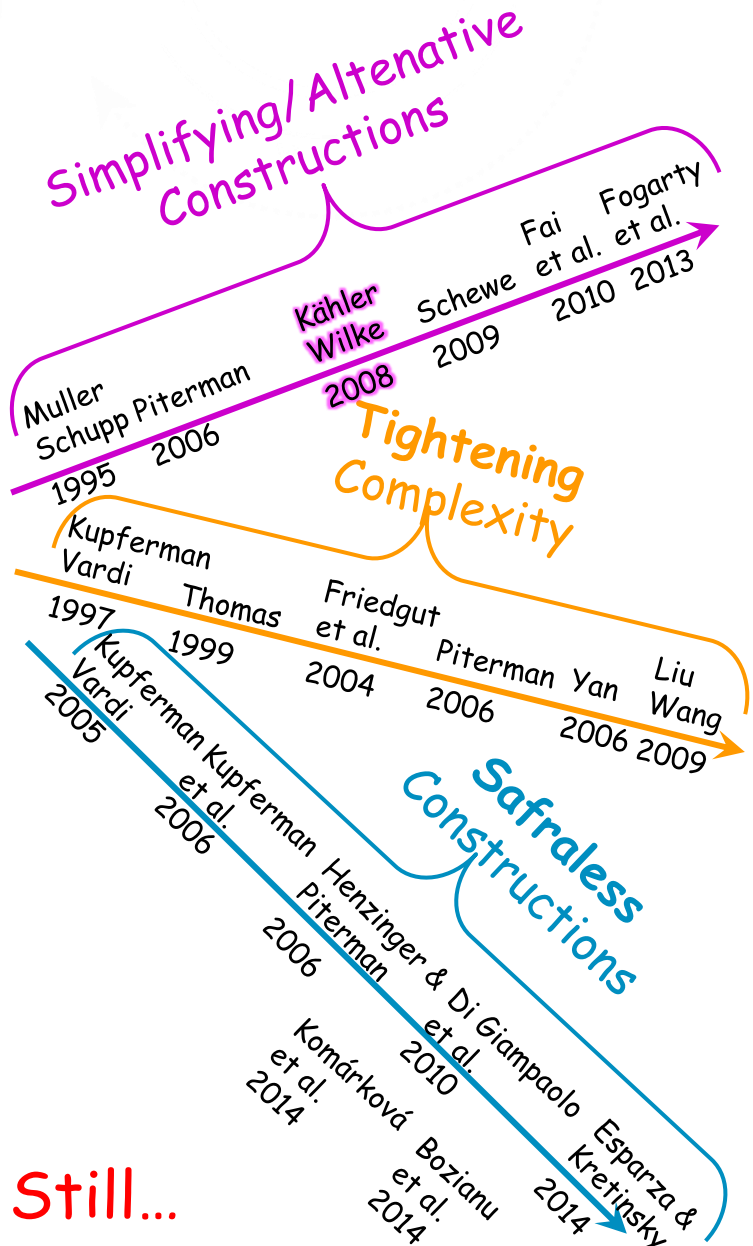
- With applications in
  - Reactive synthesis
  - Multi-agent systems
  - Probabilistic Verification

History (very briefly)

Problem is solved!

Solution achieves the optimal $n^{O(n)}$





Motivation

- Wanted:

  a solution with

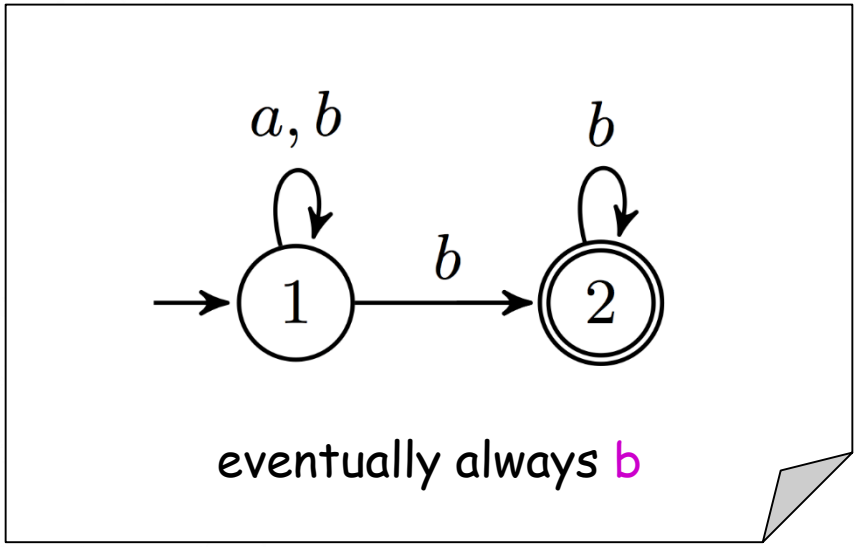
  - Few easy to understand ideas
  - No magic
  - Modular / gradual
  - Complexity in $n^{O(n)}$

Outline

- Some definitions
- Why subset construction and simple approached fail
- Kähler & Wilke's trees (our starting point)
- Overview of our approach
- Construction details

Büchi Automata

- Looks like an NFA, but definition of acceptance is different.
- An infinite word is accepted if it visits one of the accepting states infinitely often.



Other w-Automata

- There are other w-automata differing in the way acceptance is defined
  - Muller
  - Rabin
  - Streett
  - Parity
- We use three letter acronyms

  DPA = deterministic Parity Automaton

  NBA = non-deterministic Büchi Automaton

Determinization

- DBA are less expressive than NBA
- Determinization of Büchi automata refers to transforming them into an equivalent deterministic automaton, using another acceptance criteria (DMA, DRA, DSA, DPA)

Parity Automaton

- In a Parity Automaton:
  - Every state has a color (rank) in [0..k]
  - A run is accepting if the min color visited inf. often is odd.
- Can think of it as having:
  - K buzzers

  - K bells

  - Accept if the min bell to ring inf. often is smaller than the min buzzer to buzz inf. often

A Büchi automaton is a parity automaton with one bell and no buzzers

Determinization on Finite Words

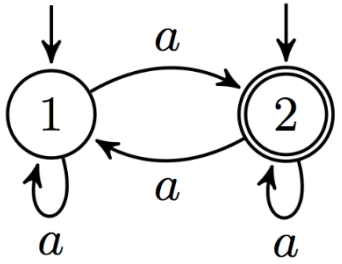
By Subset Construction

- A state in the constructed DFA automaton records the subset of states the NFA can be in after reading the respective word

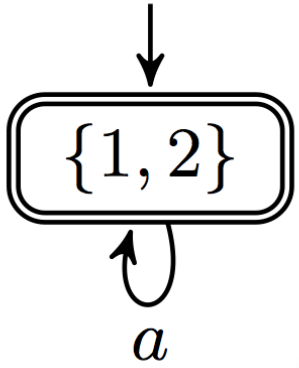
Subset Construction Fails

$\Sigma = \{ a \}$

Only one word to consider: a^ω

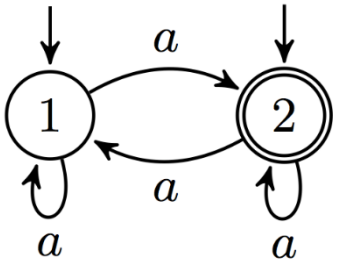


accepts a^ω

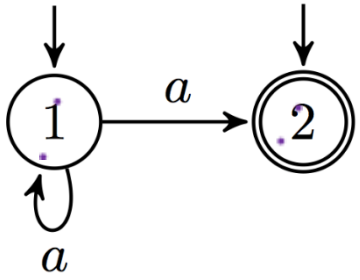


accepts a^ω

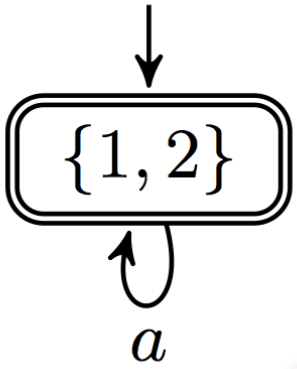
Subset Construction Fails



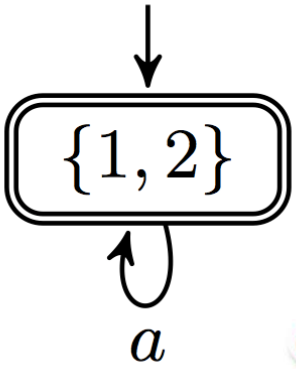
accepts a^ω



accepts nothing



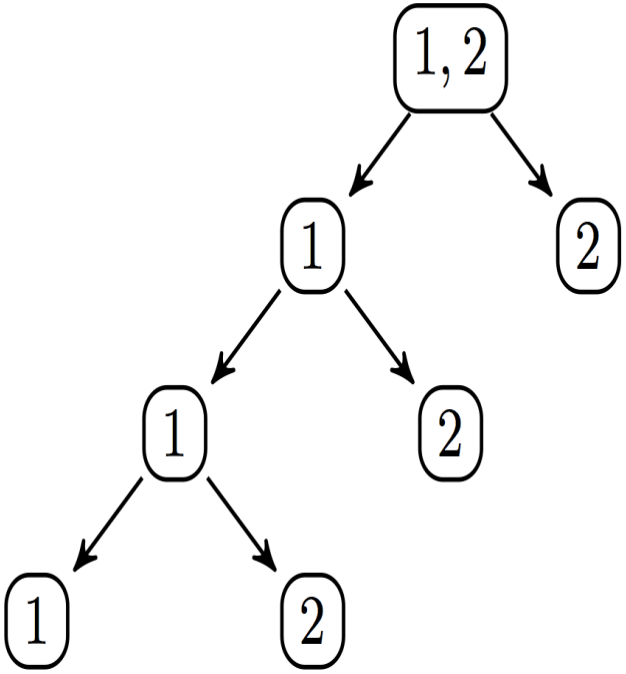
accepts a^ω



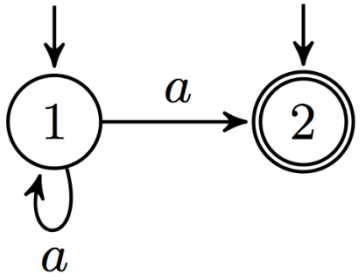
accepts a^ω

What went wrong?

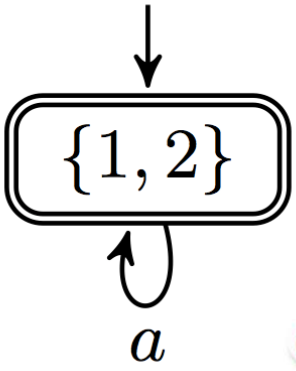
Run Tree:



State 2 is on every layer.
But it is not visited infinitely often.



accepts nothing



accepts a^ω

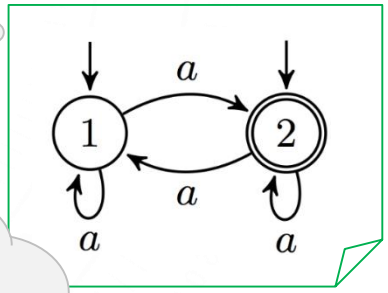
Need to record more!

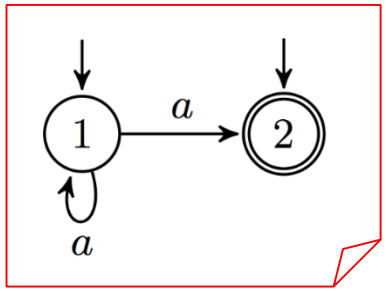
- Kähler and Wilke proposed the following structures to summarize runs of NBAs
  - Split Tree SP^A_w
  - Reduced Tree RD^A_w
  - Skeleton Tree SK^A_w
- All are defined per a given ω-word w and maintain the following property:
  - If there exists an accepting run on w in A then there exists an accepting path in the tree.

Split Tree

Follows the successors, but splits them into two sets:

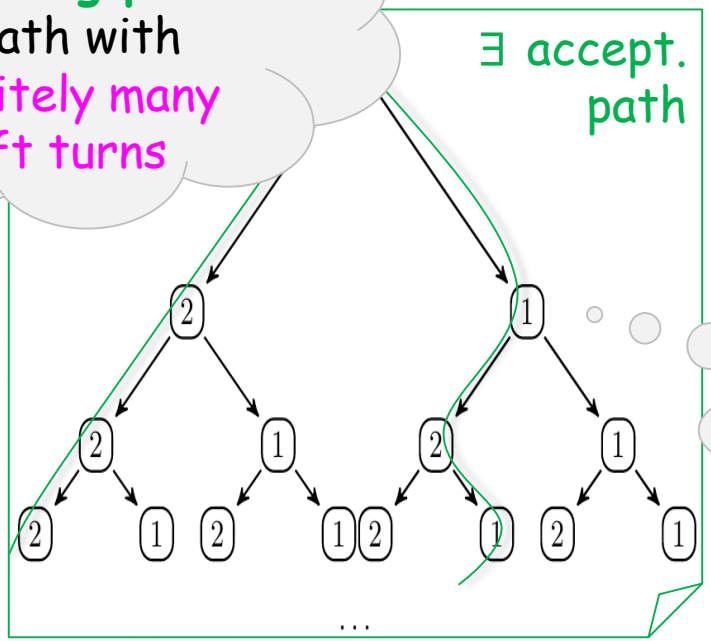
non-accepting on the right and accepting on the left



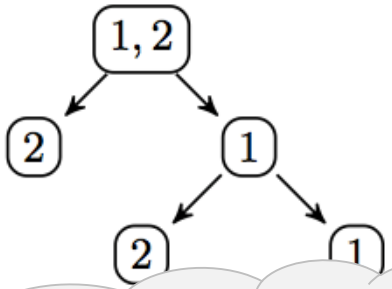


accepting path
= path with
infinitely many
left turns

∃ accept. path



no accept. path



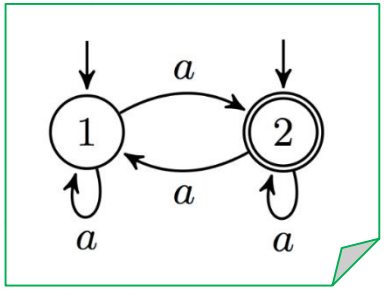
problem:
the width is
unbounded!

solution:
the reduced
tree!

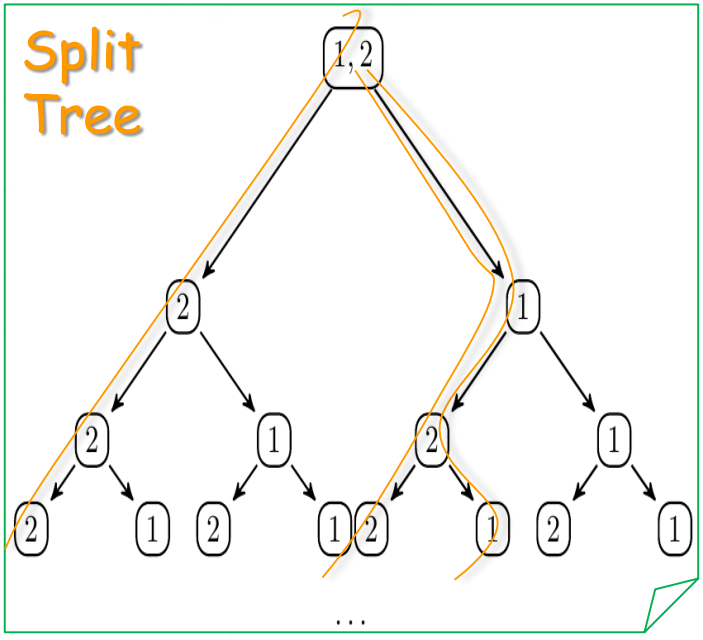
Reduced Tree

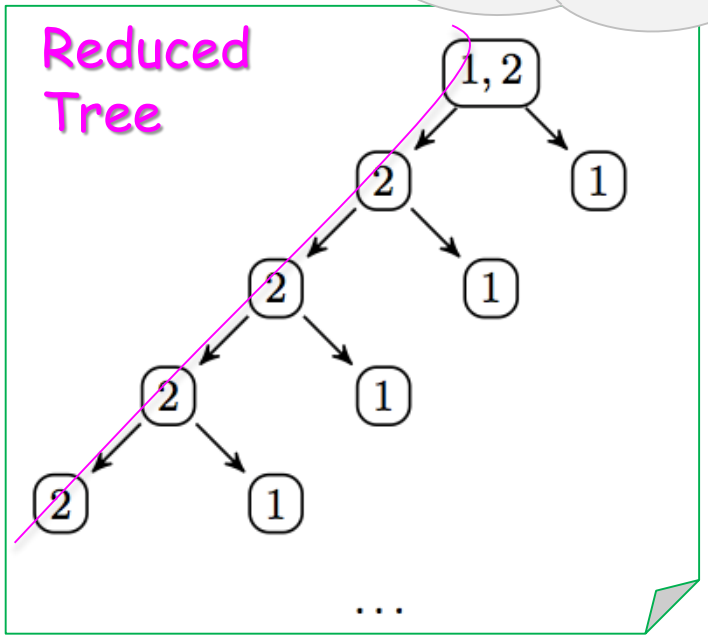
Eliminate from a node states that appeared in a node to its left (on the same layer)

4 3 01/



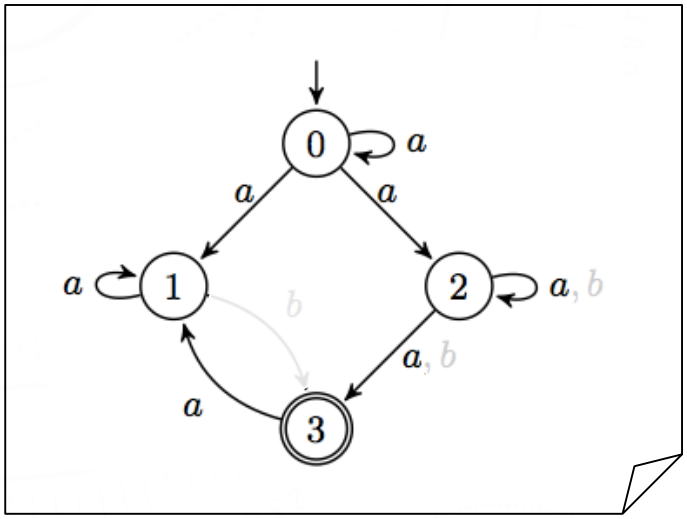
The width of the reduced tree is thus bounded by n

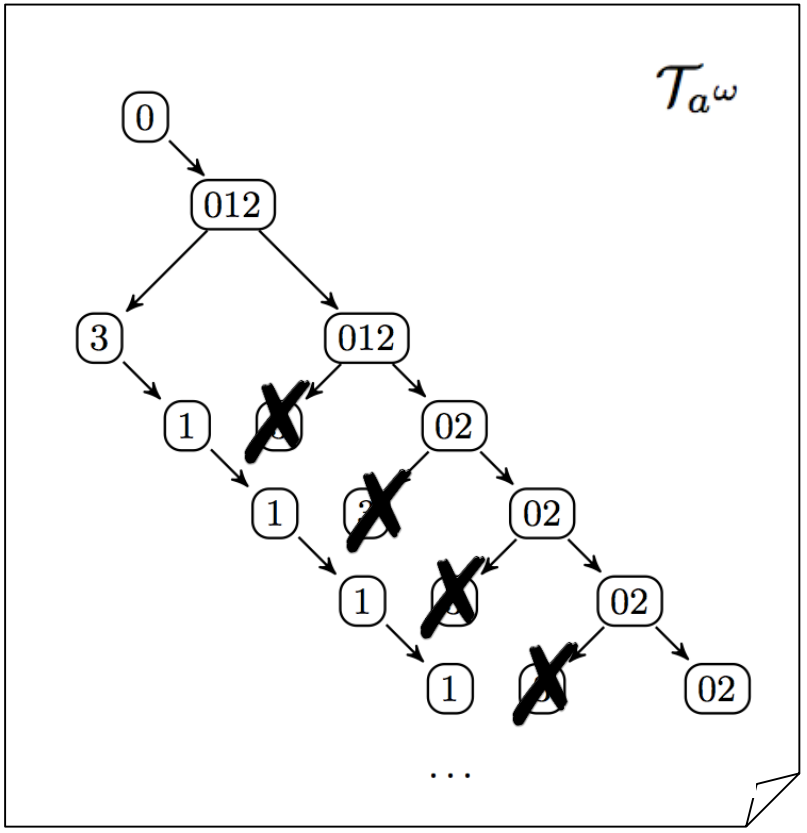




Skeleton Tree

Eliminate dangling nodes (only infinite paths remain)





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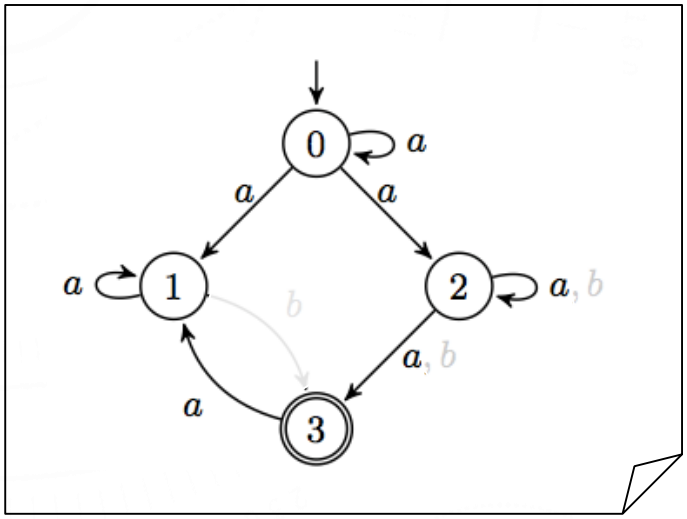
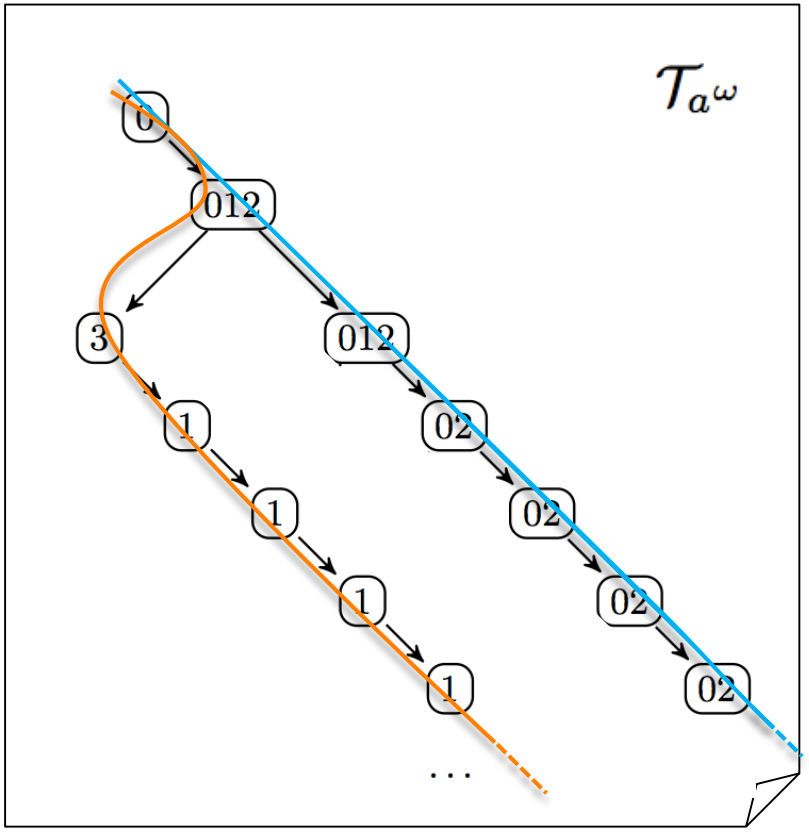
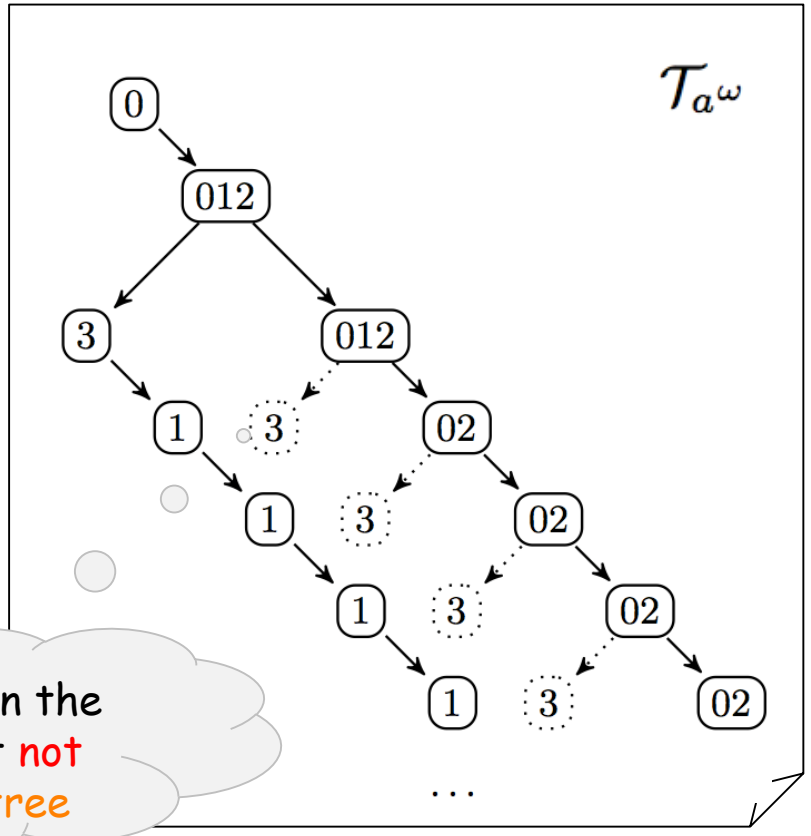


We call the number of paths in the skeleton tree the width.



Here the width is 2.

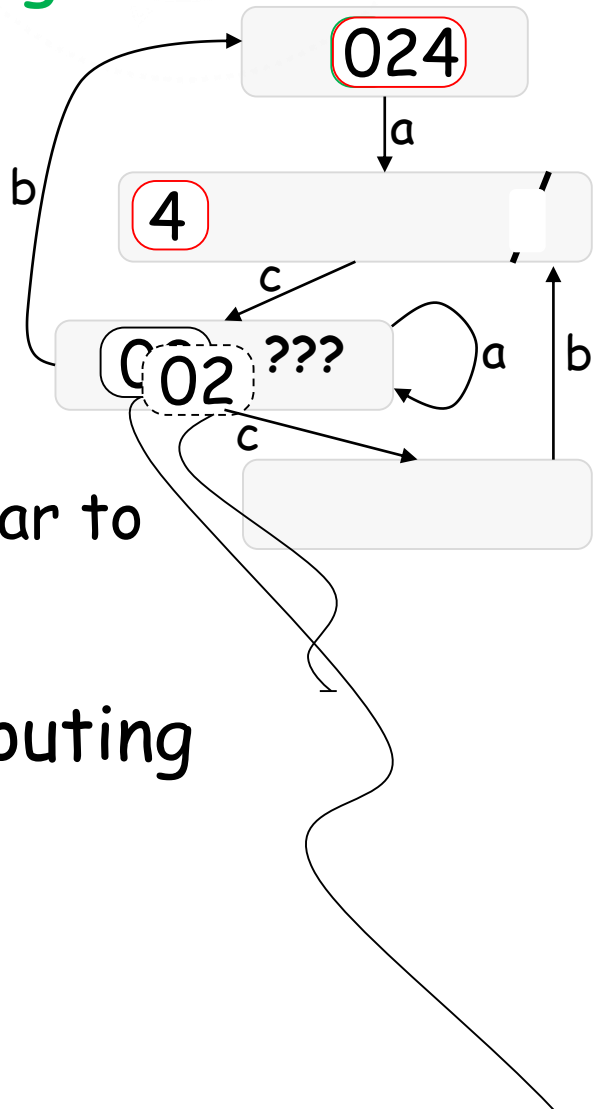
Reduced & Skeleton Tree



A dotted node is in the reduced tree but not in the skeleton tree

The Slice Automaton

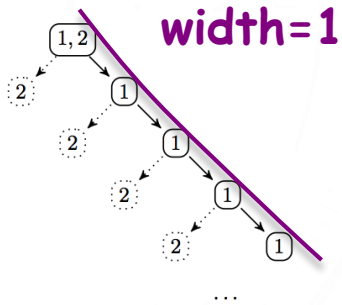
- We can build a det. aut computing the slices of the reduced trees
  - Slice by slice
  - Splitting the successors to accepting/non-accepting
  - Removing states that already appear to the left
- We cannot build a det. aut. computing the slices of the skeleton tree (requires knowing which node eventually has no decedents)



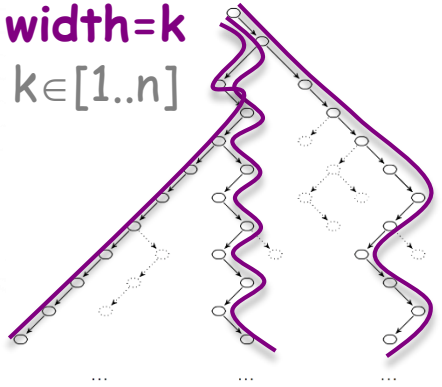
Our contribution starts here

Mission

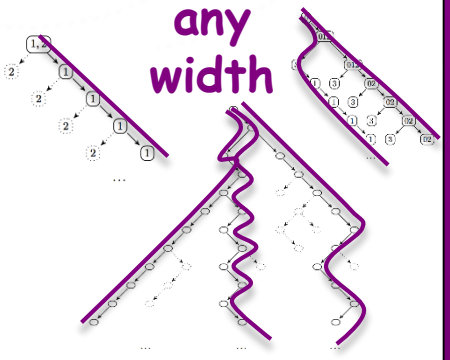
First



Then

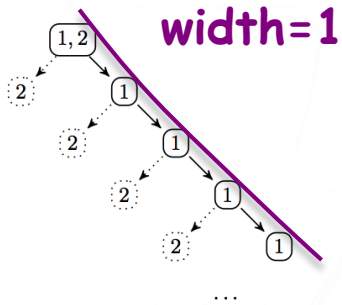


Finally



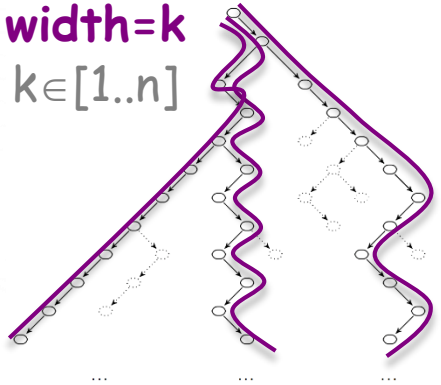
Mission

First



Devise a DPW P_1 that answers correctly only for words of width=1

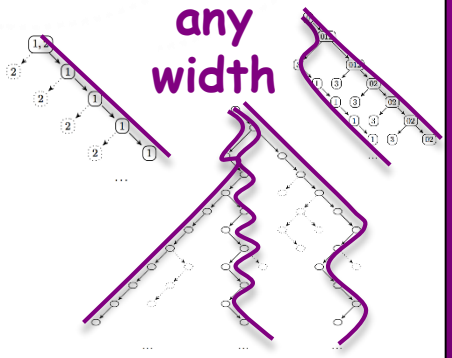
Then



Generalize to a DPW P_k that answers correctly only for words of width=k

given k∈[1..n]

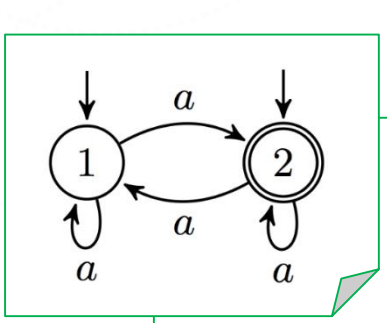
Finally

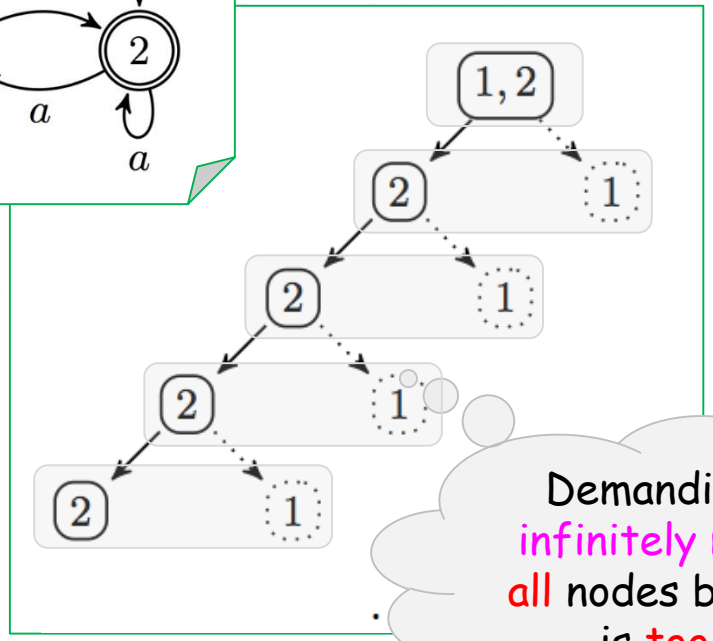


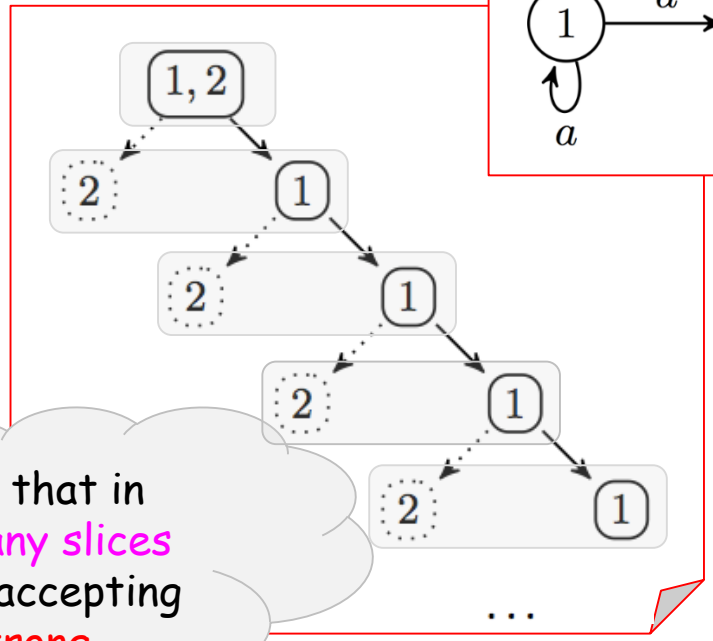
Obtain a DPW P that answers correctly for words of any width by running $P_1 \| P_2 \| \ldots \| P_k$

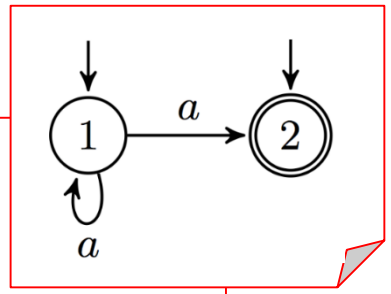
When the width is 1

- Need to answer if the single infinite path is accepting
- We know how to compute the slices ☺
- But we don't know which node is a skeleton node ☹





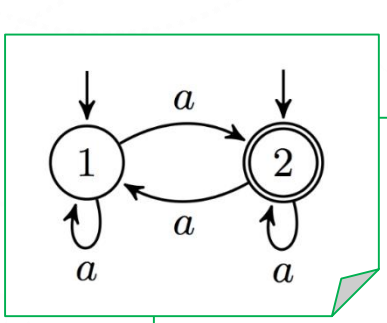


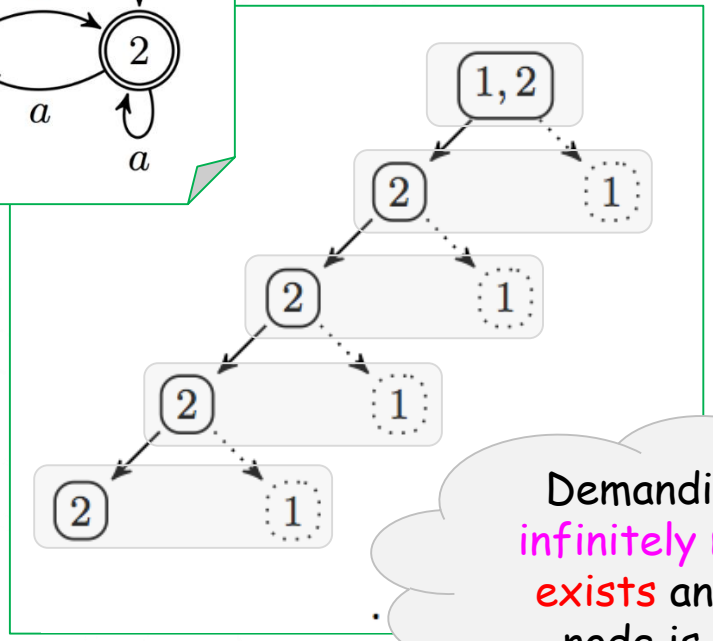


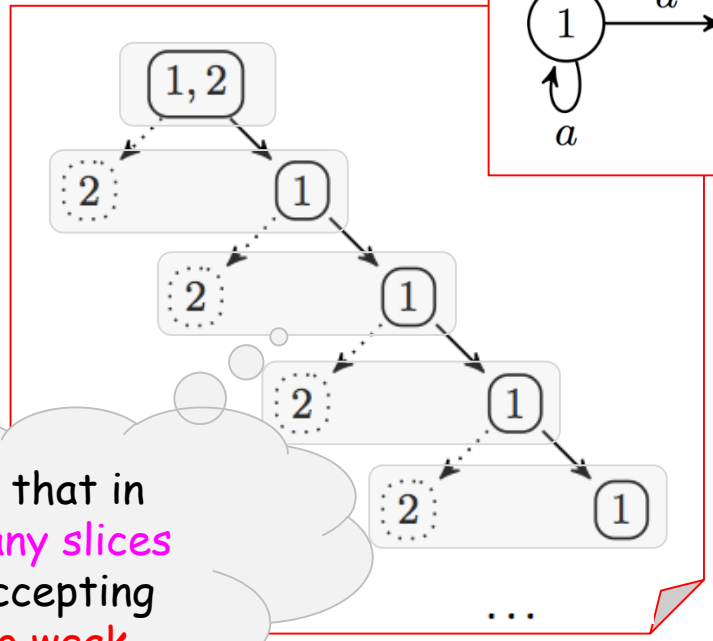
Demanding that in infinitely many slices all nodes be accepting is too strong

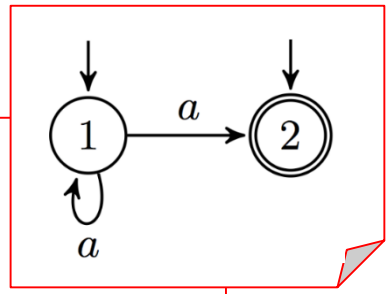
When the width is 1

- Need to answer if the single infinite path is accepting
- We know how to compute the slices ☺
- But we don't know which node is a skeleton node ☹









Demanding that in infinitely many slices exists an accepting node is too weak

When the width is 1

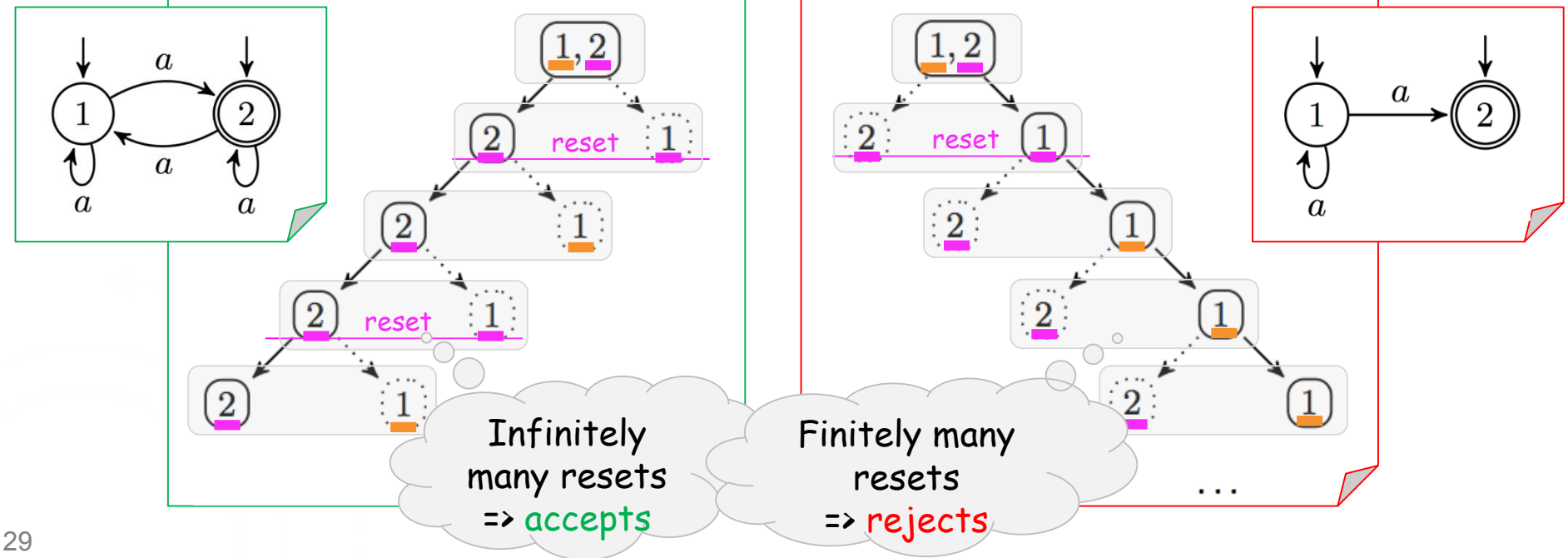
- The Miyano-Hayashi construction asks whether **ALL** infinite paths are accepting
- In the case that there is just **ONE** infinite path, this is the same as asking if **THE** single infinite path is accepting.
- MH-construction:
  - For each node, we track whether it is on a path that visited an accepting state recently or not.
  - We denote it as visited or owes.
  - If all nodes on a layer are marked visited we perform a "reset" – start charging everyone for a visit again

When the width is 1

Fisman Lustig

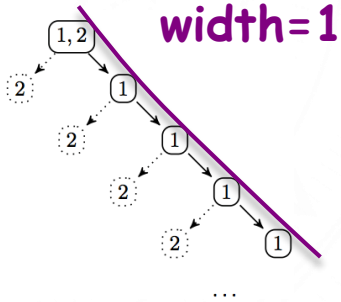
Conclusion:
Desired P_1 can simply be a DBW implementing Miyano-Hayashi on the Reduced Tree

Running Example



Mission

First ✓



Devise a DPW P_1 that answers correctly only for words of width=1

Then

width=k
k∈[1..n]

Generalize to a DPW P_k that answers correctly only for words of width=k

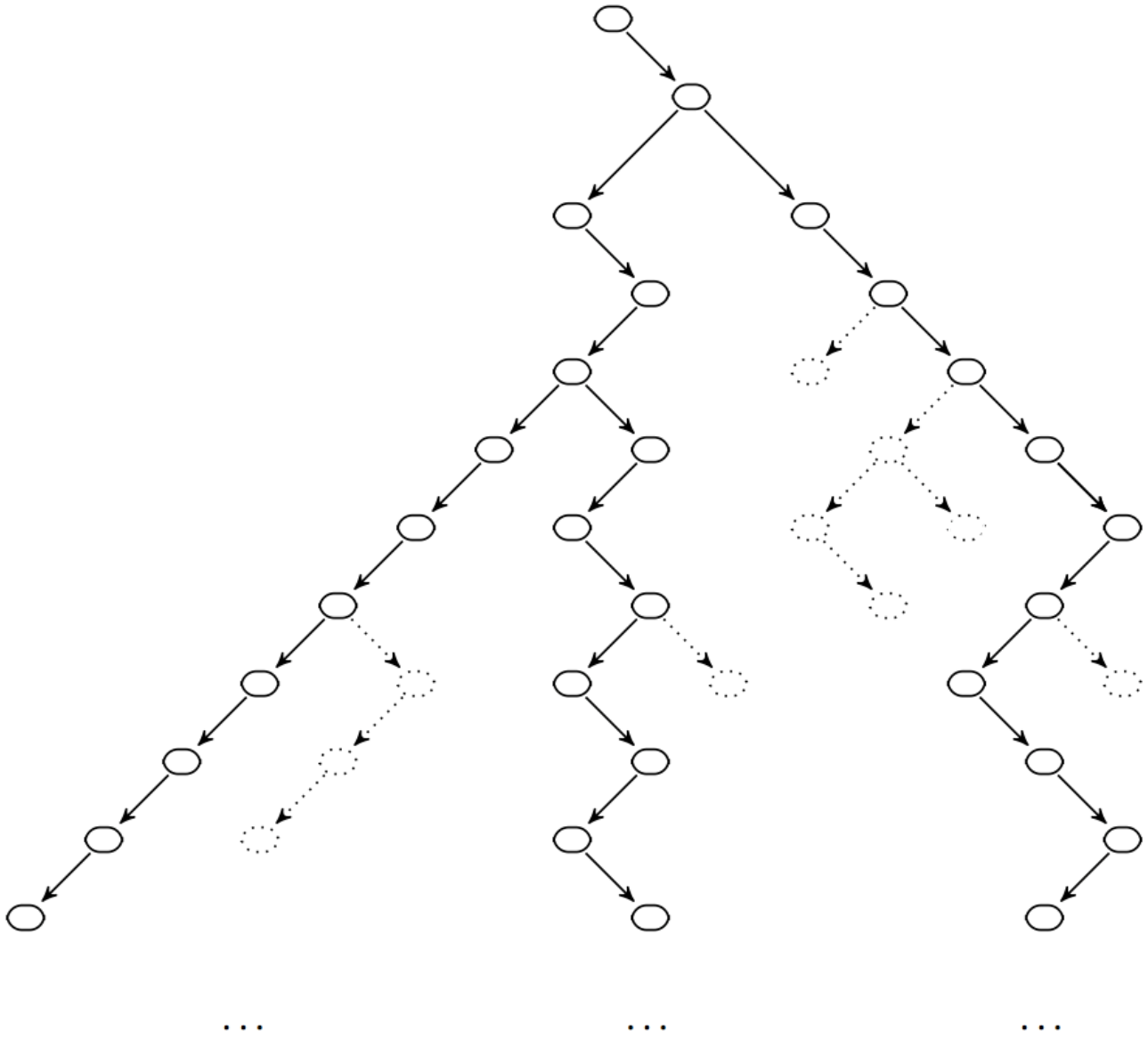
given k∈[1..n]

Finally

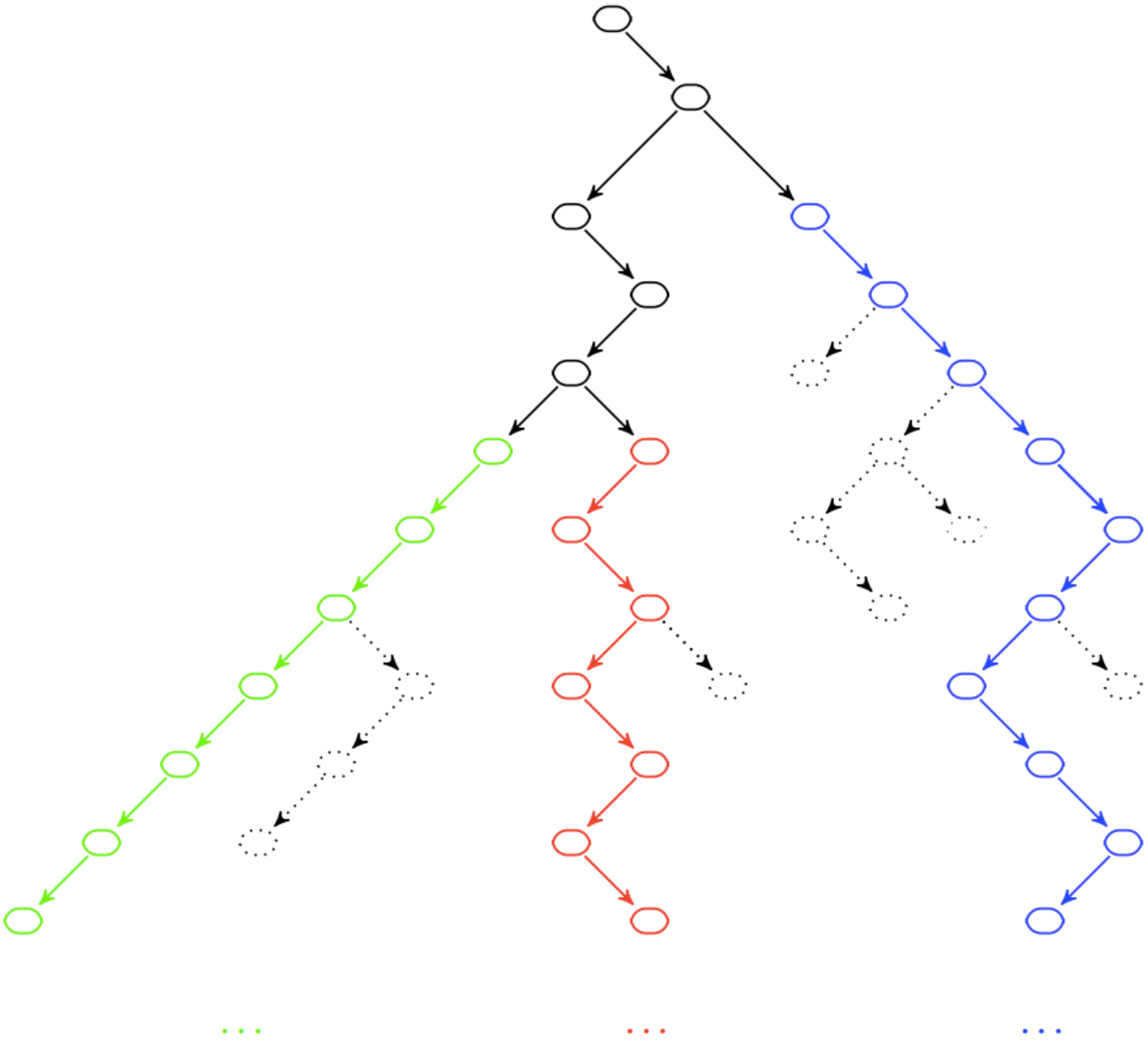
any width

Obtain a DPW P that answers correctly for words of any width by running $P_1 || P_2 || \ldots || P_k$

When the width is not 1 ?

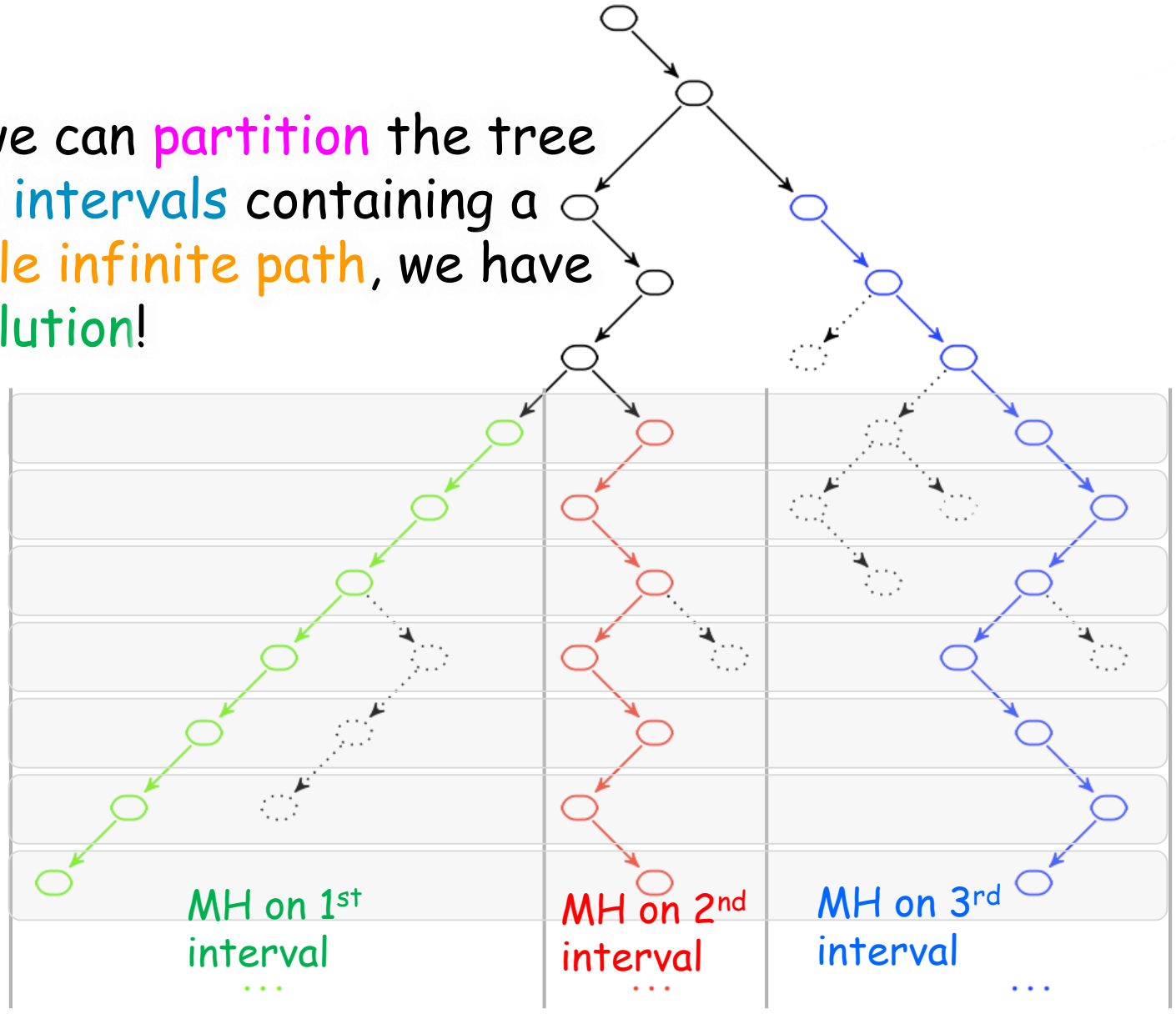


Here the width is 3



Key Observation !

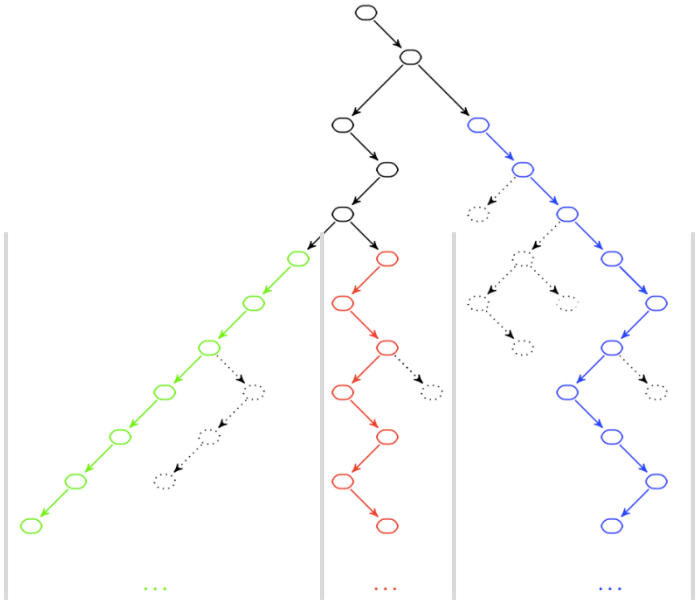
If we can partition the tree into intervals containing a single infinite path, we have a solution!



But how?

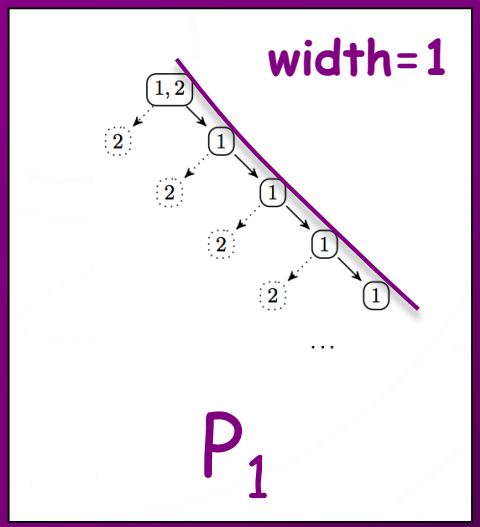
But how can we find this conceptual partition?

- First let's try to answer a simpler question:
  - What is the width?
- Or put as a decision problem:
  - Is the width < k ?

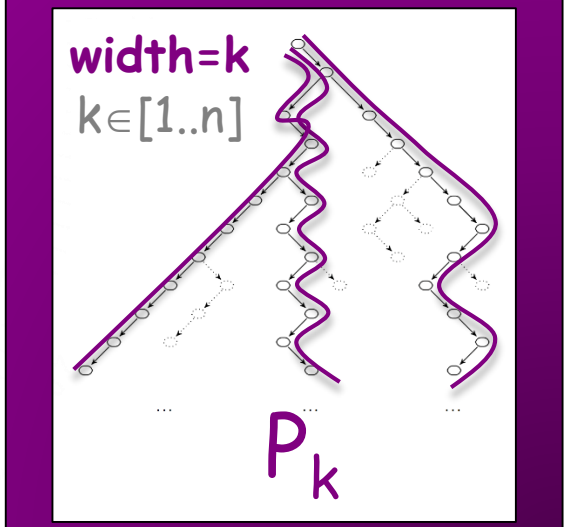


Mission

First



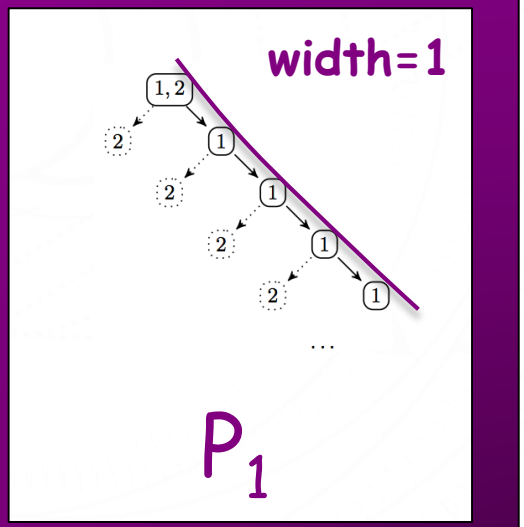
Then



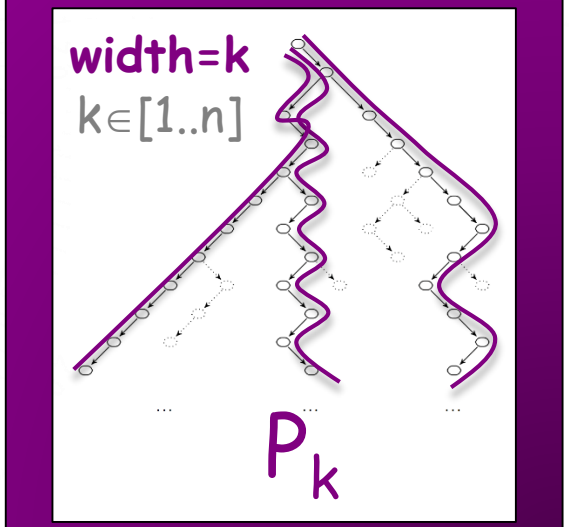
Finally

Mission

First

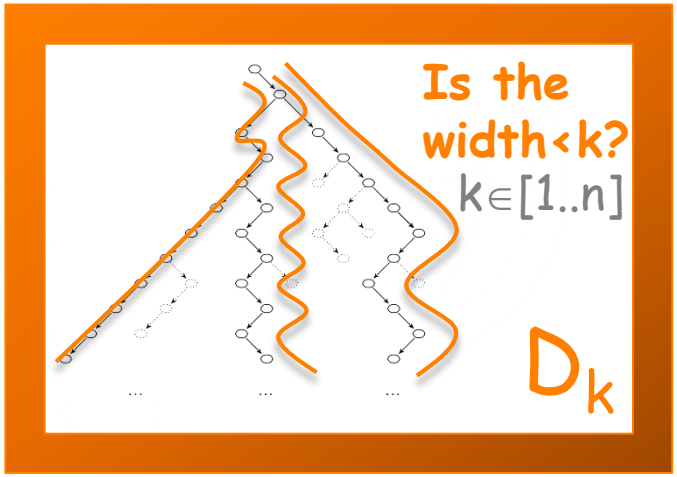


afterwards



Finally

then

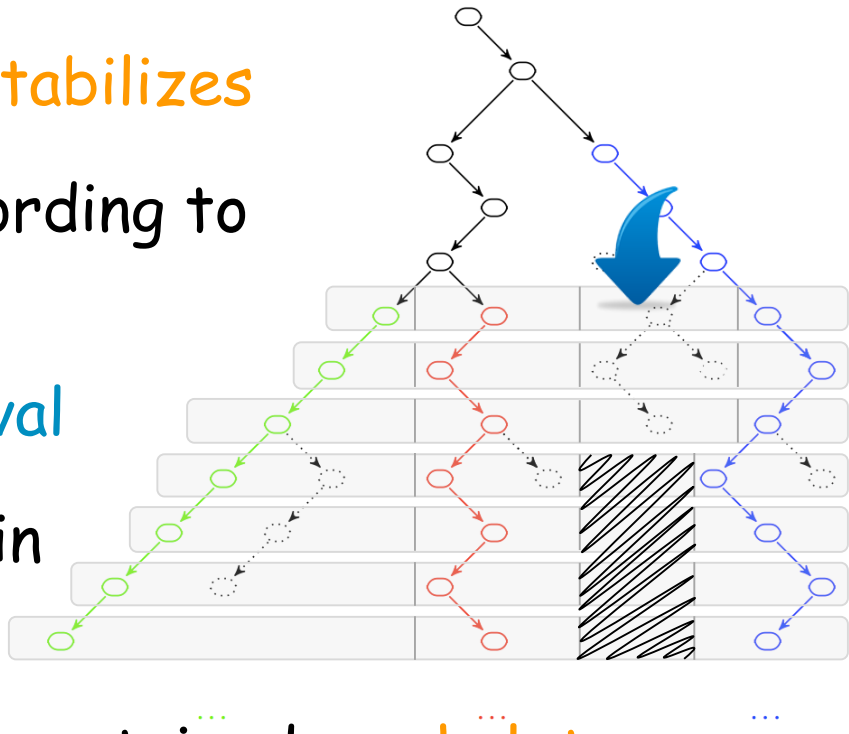


Deciding if width < k (idea)

- The width is monotonically non-decreasing
- Suppose we can start after it stabilizes
- We determine the interval according to the current nodes:

  each node in its own interval

- We develop the successor within the intervals



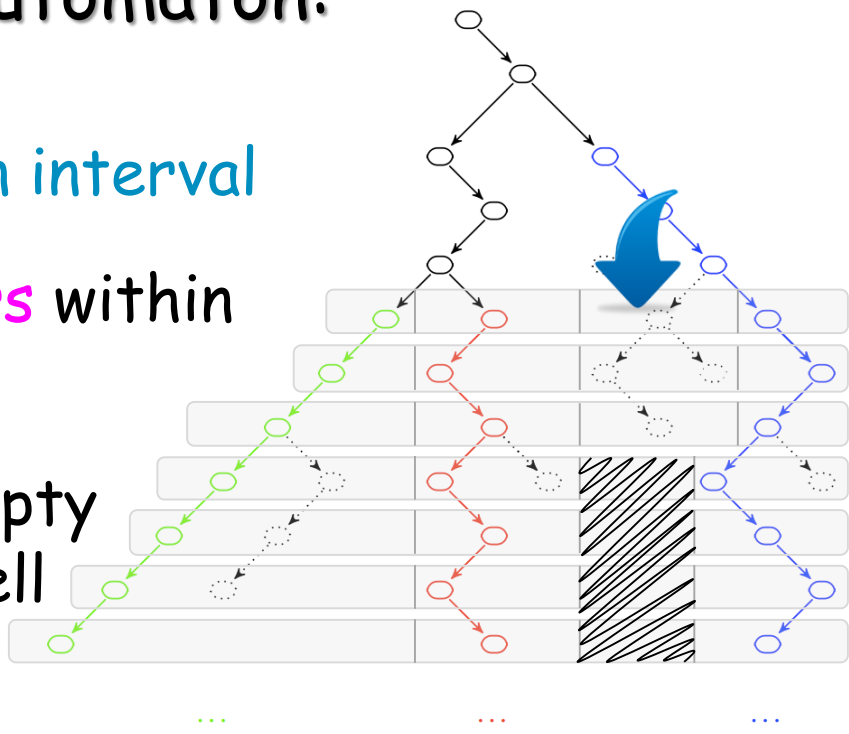
- Intervals that at the beginning contained no skeleton nodes will eventually become empty!

Deciding if width < k (const.)

Fisman Lustig

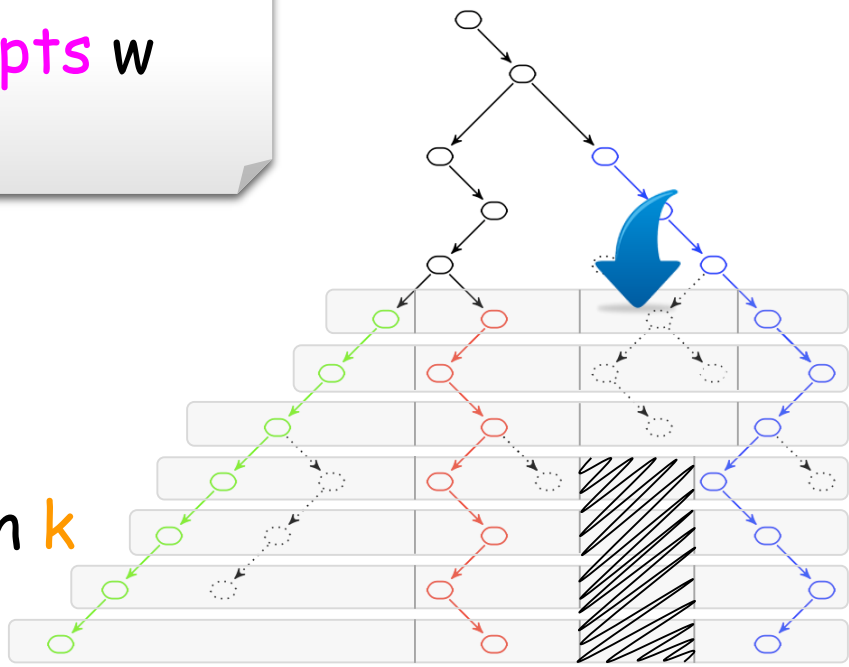
Build a deterministic Büchi automaton:

- **Shred**: put each node in its own interval
- **Develop**: develop the successors within the intervals
- **If** there are less than k non-empty intervals: Shred and ring the bell



Deciding if width < k

Theorem:
The constructed DBW accepts w iff width(w) < k



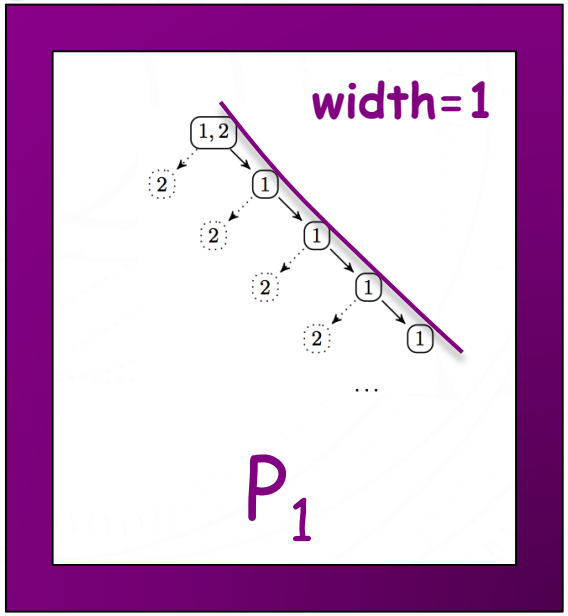
Proof:

- If the width is less than k:
  - Inf. often we'll have less than k intervals and ring the bell
- If the width is k or more, say k*
  - Eventually we will have k* non-empty intervals forever long (and we'll stop ringing the bell)

Claim:
When width=k each skeleton-path gets trapped in its own interval!

Mission

First



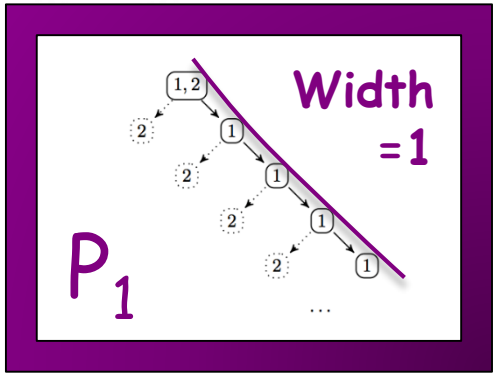
afterwards

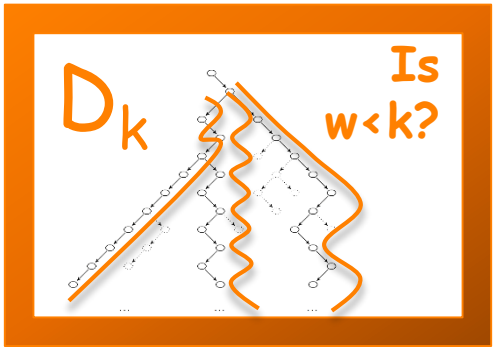
Finally

then

Determinizing for Width = k

- Idea:
  - Combine the construction for determinizing when width =1 (MHoRD)
  - And the construction for deciding if the width is < k





Determinizing for Width = k

Fisman Lustig

Build a deterministic Parity automaton:

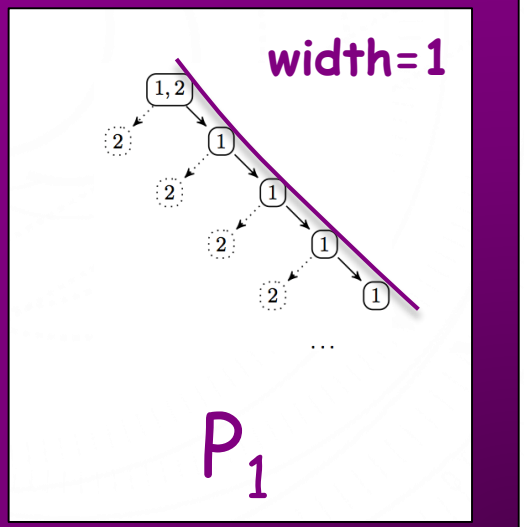
- Compute the next slice of RD
- Annotate with visited/owes
- If less than k intervals – shred and buzz the buzzer
- If in some interval all visited – reset and ring the bell

Theorem:

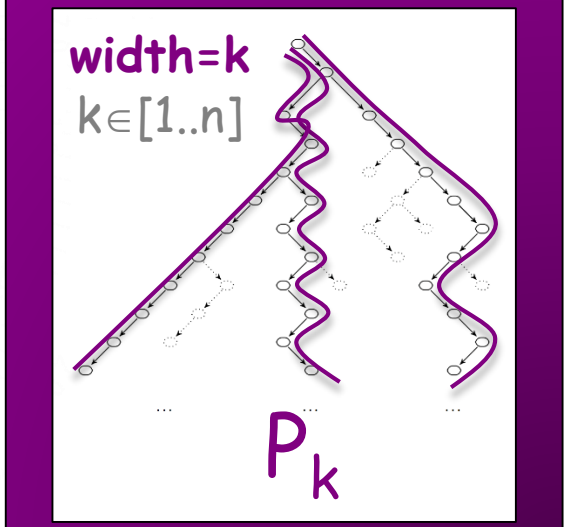
Given the width is k, the constructed DPW accepts w iff the given NBW accepts it.

Mission

First

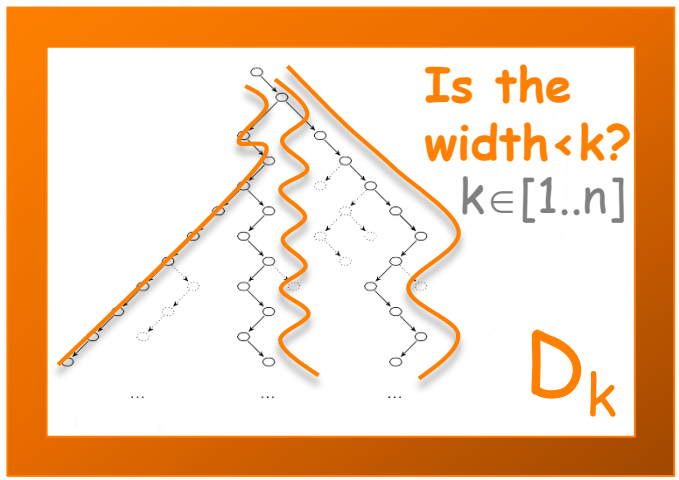


afterwards



Finally

then

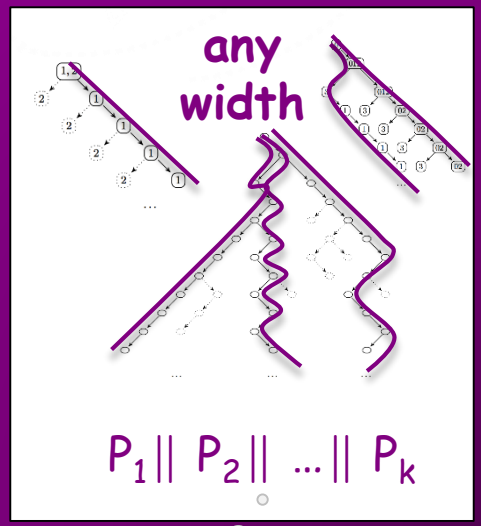


Mission

First

afterwards

Finally



then

How do we retrieve the correct answer?

Determinzing for Any Width

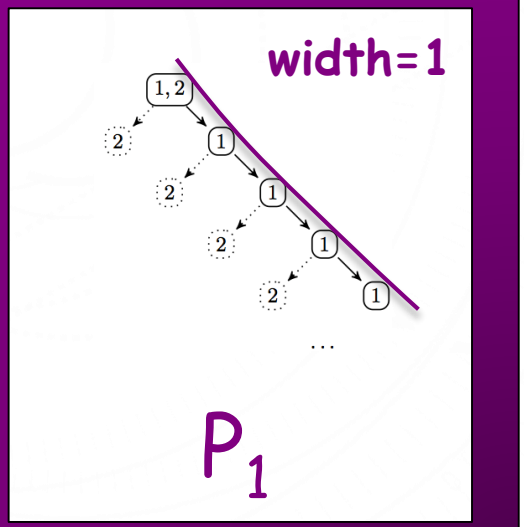
- Run the automata for all the widths in parallel
- Automaton for width =k uses the k-buzzer and the k-bell
- Proof: Assume the width is k*. Then
  - The automaton for width =k* answers correctly: rings the k-bell inf. often iff the NBA accepts (and it doesn't shred inf. often)
  - The automata for width >k* shred (i.e. >k-buzz) inf. often
  - The automata for width <k* don't shred (buzz) i.o. but may reset (<k-ring). If so that is correct –all the paths trapped in one of the intervals accept.

THM:

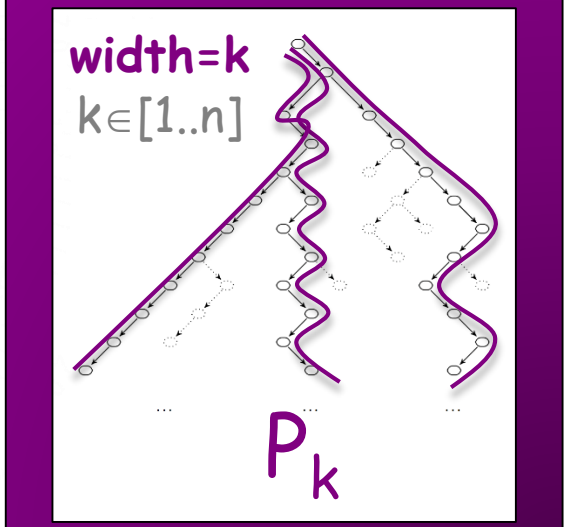
The DPA accepts w **iff** the NBA accepts w

Mission

First

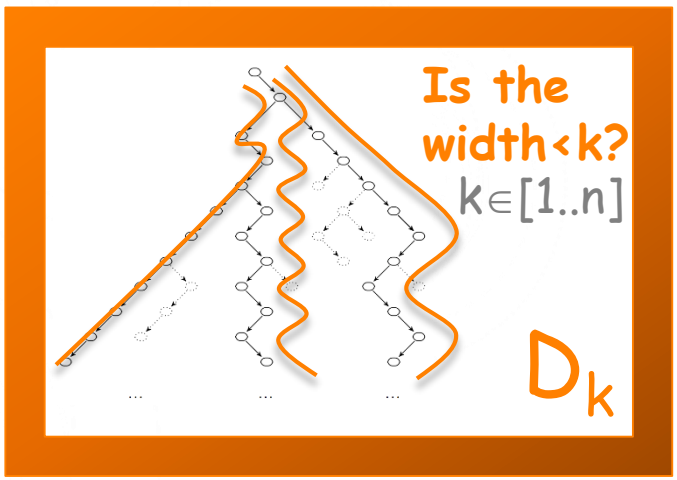


afterwards



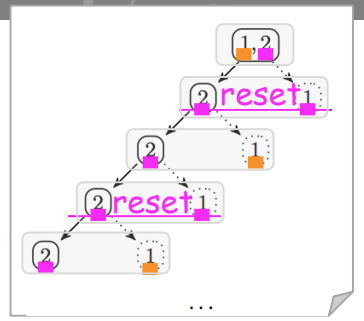
Finally

then

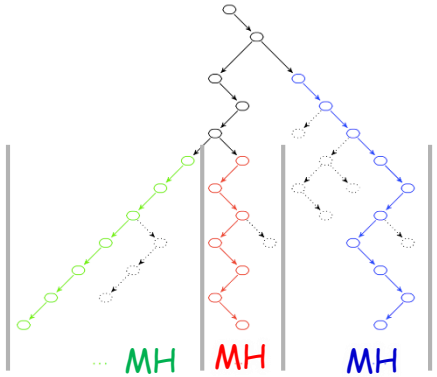


Recap

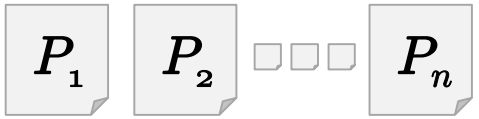
- Determinization for width=1 can be done by applying MH on the RD



- Determinization for width=k is done by trapping the presumably k skeleton-paths each in its own interval and applying MH on each interval separately



- Determinization for any width is obtained by running all width in parallel



- Complexity remains $n^{O(n)}$ (skipped in this talk)

Thank you for your attention!

Questions/ Comments?

Dana Fisman and Yoad Lustig