A Complexity Measure on Büchi Automata

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Abstract. We define a complexity measure on non-deterministic Büchi automata, based on the notion of the width of the skeleton tree introduced by Kähler and Wilke. We show that the induced hierarchy tightly correlates to the Wagner Hierarchy, a corner stone in the theory of regular ω-languages that is derived from a complexity measure on deterministic Muller automata. The relation between the hierarchies entails, for instance, that a nondeterministic Büchi automaton of width k can be translated to a deterministic parity automaton of degree at most $2k + 1$.

Keywords: automata and logic, automata for system analysis and program verification, classification of regular ω-languages.

1 Introduction

There are various way to define acceptance on infinite words, deriving different types of ω-automata. Büchi automata have the simplest acceptance criterion: a run is declared accepting if the set of states visited infinitely often intersect a designated set of *accepting* states F. Their dual, co-Büchi automata declare a run accepting if the set of states visited infinitely often does not intersect a designated set of *rejecting* states F. Muller automata have the most general acceptance criterion: a run is accepting if the set of states visited infinitely often is exactly one of a set of designated subsets of states $F_1, F_2, \ldots, F_k$. Other types of ω-automata include Rabin, Streett and parity.

The different ω-automata have varying levels of complexity and expressivity. Using the convention that $\mathbb{DT}$ ($\mathbb{NT}$) for $\mathbb{T} \in \{\mathbb{B}, \mathbb{C}, \mathbb{R}, \mathbb{S}, \mathbb{M}, \mathbb{P}\}$ denotes the class of languages accepted by deterministic (nondeterministic) automata of type Büchi, co-Büchi, Rabin, Street, Muller or parity, resp., the following relations are known. The class $\mathbb{DB}$ is less expressive than $\mathbb{NB}$ which is as expressive as $\mathbb{DM}$, $\mathbb{DP}$, $\mathbb{DR}$, $\mathbb{DS}$, which recognize all regular ω-languages. The classes $\mathbb{DB}$ and $\mathbb{DC}$ have incomparable expressive power, in the sense that there exists languages in $\mathbb{DB} \setminus \mathbb{DC}$ and in $\mathbb{DC} \setminus \mathbb{DB}$.

Wagner [13] has suggested a complexity measure on Muller automata, and showed that this complexity measure is language-specific and is invariant over all automata accepting the same language. This result, referred to as the Wagner Hierarchy, is a corner stone in the theory of regular ω-languages. The classes $\mathbb{DB}$

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and $\mathbb{DC}$ coincide with classes of the lower levels of this hierarchy. For a language L the minimal number of colors required by a recognizing deterministic parity automaton, and the minimal number of pairs required by a deterministic Rabin or Streett automaton tightly correlates to the minimal class in the hierarchy in which the recognized language L resides.

In this paper we define a complexity measure on nondeterministic Büchi automata (NBA). The measure is based on the notion of the width of the *skeleton-tree*, a structure for summarizing runs of NBAs, introduced by Kähler and Wilke [7]. A language L is said to be of degree k, or belong to $\mathbb{NB}_k$, if there exists an NBA $\mathcal{N}$ recognizing L such that the width of the skeleton-tree of all words with respect to $\mathcal{N}$ is at most k. We show that this measure induces a strict hierarchy of classes of languages, and that this hierarchy tightly correlates to the Wagner hierarchy. The relation between the hierarchies entails, for instance, that an NBA of width k can be translated to a parity automaton using at most $2k+1$ colors, and a language $L \in \mathbb{NB}_{k+1} \setminus \mathbb{NB}_k$ cannot be translated to a parity automaton using less than $2k$ colors.

We provide definitions for ω-automata, a summary of the Wagner hierarchy, and a definition of the skeleton-tree in Section 2. The contribution of the paper starts in Section 3 where we define the complexity measure on Büchi automata, and state the main theorem of the paper, that the induced hierarchy tightly correlates to the Wagner hierarchy. Section 4 proves the direction from the proposed hierarchy to the Wagner hierarchy and Section 5 the other direction. In a sense, the direction from the Wagner hierarchy to the proposed hierarchy provides an insight on the need for non-determinism in the Büchi model, and a quantification of the amount of non-determinism needed relative to the complexity of the Muller automaton.

2 Preliminaries

Automata on Infinite Words An *automaton* is a tuple $\mathcal{A} = \langle \Sigma, Q, q_0, \delta, \alpha \rangle$ consisting of a finite alphabet Σ of symbols, a finite set Q of states, an initial state q_0, a transition function $\delta : Q \times \Sigma \to 2^Q$, and an acceptance condition α. A run of an automaton on an infinite word $v = a_1 a_2 \dots$ is an infinite sequence of states $p_0, p_1, p_2 \dots$, such that $p_0 = q_0$ and $p_{i+1} \in \delta(p_i, a_i)$ for every $i \in \mathbb{N}$. The transition function can be extended to a function from $Q \times \Sigma^*$ (to Q) by defining $\delta(q, \epsilon) = q$ and $\delta(q, av) = \delta(\delta(q, a), v)$ for $q \in Q$, $a \in \Sigma$ and $v \in \Sigma^*$, where ϵ denotes the empty word. We say that $\mathcal{A}$ is *deterministic* if $|\delta(q, a)| \leq 1$ for every $q \in Q$ and $a \in \Sigma$.

An automaton accepts a word if at least one of the runs on that word is accepting. We use $[\![\mathcal{A}]\!]$ to denote the set of words accepted by $\mathcal{A}$. We use $\overline{L}$ to denote the language $\Sigma^\omega \setminus L$. For finite words the acceptance condition is a set $F \subseteq Q$ and a run on v is accepting if it ends in an accepting state, i.e., if $\delta(q_0, v) \in F$. For infinite words, there are many acceptance conditions in the literature; here we mention four: Büchi, co-Büchi, parity and Muller. In *Büchi* and *co-Büchi* automata the acceptance condition refers to a subset F of the

states set. In *parity automata* the acceptance condition refers to a mapping $\kappa : Q \mapsto [1..k]$ from the set of states to the set of *colors* $[1..k]$.[1] In *Muller* automata the acceptance condition is a function $\tau : 2^Q \to \{+, -\}$ assigning positive/negative polarity to subsets of states.

Let ρ be an infinite sequence of states $s_0, s_1, s_2 \ldots$. We use $\kappa(\rho)$ to denote the respective sequence of colors $\kappa(s_0), \kappa(s_1), \kappa(s_2) \ldots$. We use $\mathsf{Inf}(\rho)$ to denote the set of states that appear infinitely often in ρ. An infinite path ρ satisfies the

- Büchi condition w.r.t F iff $\mathsf{Inf}(\rho) \cap F \neq \emptyset$
- co-Büchi condition w.r.t F iff $\mathsf{Inf}(\rho) \cap F = \emptyset$
- parity condition w.r.t κ iff $\min(\mathsf{Inf}(\kappa(\rho)))$ is odd.
- Muller condition w.r.t τ iff $\tau(\mathsf{Inf}(\rho)) = +$.

We use DBA, DCA, DPA, and DMA, to denote deterministic Büchi, co-Büchi, parity and Muller automata and NBA, NCA, NPA, and NMA to denote the respective non-deterministic automata. Similarly, we use $\mathbb{DB}$, $\mathbb{DC}$, $\mathbb{DP}$ and $\mathbb{DM}$ to denote the class of languages recognized by DBA, DCA, DPA and DMA, respectively, and $\mathbb{NB}$, $\mathbb{NC}$, $\mathbb{NP}$, $\mathbb{NM}$ to denote the class of languages accepted by NBA, NCA, NPA, and NMA, respectively. For parity automata we use $\mathbb{DP}_k$ to denote the set of languages recognized by a DPA with k colors, and refer to it as the $\mathbb{DP}_k$ hierarchy.

The Chain Measure Let $\mathcal{D} = \langle \Sigma, Q, q_0, \delta, \tau \rangle$ be a complete DMA (i.e. a DMA where $|\delta(q, a)| = 1$ for every $q \in Q$ and $a \in \Sigma$). A set of states $S \subseteq Q$ is said to be *admissible* if S is a reachable strongly connected component (SCC) of $\mathcal{D}$. An admissible set S is said to be *positive* or *accepting* (resp. *negative* or *rejecting*) iff $\tau(S) = +$ (resp. $\tau(S) = -$). A chain of admissible sets $S_0 \subset S_1 \subset \cdots \subset S_{m-1}$ is a $\mathcal{D}$*-chain* iff the sets are alternately positive and negative. The *length* of such a chain is m. The polarity of a $\mathcal{D}$-chain is determined according to the polarity of its bottom set. That is, the above $\mathcal{D}$-chain is said to be positive (resp. negative) iff S_0 is positive (resp. negative). We use $k(\mathcal{D})$, $k^+(\mathcal{D})$ and $k^-(\mathcal{D})$ to denote the maximal length of a $\mathcal{D}$-chain, a positive $\mathcal{D}$-chain and a negative $\mathcal{D}$-chain, resp. In the sequel we will focus on $k^+(\mathcal{D})$ to which we refer as the *positive-chain measure*. Note that $0 \leq k(\mathcal{D}) \leq |Q|$. Also $|k^+(\mathcal{D}) - k^-(\mathcal{D})| \leq 1$ simply by omitting the bottom set.

Definition 1 (The Classes $\mathbb{DM}_k^+$ and $\mathbb{DM}_k^-$ [13]) *Let $k \in \mathbb{N}$. The class of languages $\mathbb{DM}_k^+$ and $\mathbb{DM}_k^-$ are defined as follows.*

- $\mathbb{DM}_k^+ = \{\ L \mid \exists\ \text{DMA}\ \mathcal{D} :\quad L = [\![\mathcal{D}]\!],\quad k^+(\mathcal{D}) \leq k\}$
- $\mathbb{DM}_k^- = \{\ L \mid \exists\ \text{DMA}\ \mathcal{D} :\quad L = [\![\mathcal{D}]\!],\quad k^-(\mathcal{D}) \leq k\}$

The Wagner hierarchy consists of an additional measure, the length of the longest sequence of chains that are reachable from each other and have alternating polarities, and takes into account also the polarity of the chains. We omit the details for lack of space and since this measure is of less importance to this paper.

[1] For $j, k \in \mathbb{N}$ s.t. $j \leq k$, we use $[j..k]$ to denote the set $\{j, j{+}1, j{+}2, \ldots, k\}$.

Wagner [13] has shown that the hierarchy is strict and that these measures are invariant overall DMAs accepting the same language. He further showed that the chain hierarchy tightly correlates to the minimal numbers of chains in a Rabin automaton. A similar result regarding parity automata (see f.g. [2, 3]) states that $L \in \mathbb{DM}_k^+$ iff $L \in \mathbb{DP}_k$. The relation between DBAs and the class $\mathbb{DM}_1^+$ was already shown by Landweber [10]. The Wagner hierarchy has been rediscovered several times [8, 1].

Theorem 1 ([13, 10, 2, 3]).

1. $L \in \mathbb{DM}_k^+ \iff \overline{L} \in \mathbb{DM}_k^-$
2. $\mathbb{DM}_k^- \subsetneq \mathbb{DM}_{k+1}^-$ *and* $\mathbb{DM}_k^+ \subsetneq \mathbb{DM}_{k+1}^+$
3. $L \in \mathbb{DM}_1^+ \iff L \in \mathbb{DB}$ *and* $L \in \mathbb{DM}_1^- \iff L \in \mathbb{DC}$.
4. $L \in \mathbb{DM}_k^+ \iff L \in \mathbb{DP}_k$

The Skeleton Tree The complexity measure we propose is based on the notion of the *width* of the *skeleton tree* introduced by Kähler and Wilke [7]. Kähler and Wilke, aiming to provide constructions unifying Büchi determinization, complementation and disambiguation introduced the notions of the *split tree*, the *reduced tree* and the *skeleton tree*, where the latter has been said to be identified from the work of Muller and Schupp [11]. All three are mechanisms to summarize runs of NBAs. For lack of space we suffice here with an informal description; the formal definition of these trees, and an illustrating example, can be found in the appendix.

The *split-*, *reduced-* and *skeleton-trees* are defined per a given word w and w.r.t. a given NBA $\mathcal{N}$. A key invariant that is maintained is that if there exists an accepting run of $\mathcal{N}$ on w then there is an accepting infinite path in all of these trees. Roughly speaking, the split tree refines the subset construction by separating accepting and non-accepting states. From each node of the tree the left son holds its accepting successors and the right son its non-accepting successor. Thus, an accepting path has infinitely many left turns, and is also referred to as *left recurring*. The width of a layer of the split-tree is generally unbounded. The reduced tree bounds the number of nodes on a layer of the tree to n, the number of states of the given Büchi automaton $\mathcal{N}$, by eliminating from a node of the tree all states that appeared in a node to its left. The skeleton-tree is the smallest sub-tree of the reduced-tree that contains all its infinite paths. The *width* of the skeleton tree thus equals the number of infinite paths in the reduced tree. We use $width(\mathcal{N}, w)$ to denote the width of the skeleton tree for w w.r.t to $\mathcal{N}$.

3 A Complexity Measure on NBAs

Given an NBA $\mathcal{N}$ we say that the *width* of $\mathcal{N}$ is the maximal width of a skeleton tree on any given word. Formally

$$width(\mathcal{N}) = \max \{width(\mathcal{N}, w) \mid w \in \Sigma^\omega\}$$

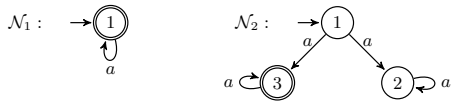

Fig. 1. Two NBAs $\mathcal{N}_1$ and $\mathcal{N}_2$ such that $[\![\mathcal{N}_1]\!] = [\![\mathcal{N}_2]\!]$ yet $width(\mathcal{N}_1) \neq width(\mathcal{N}_2)$.

It is not hard to see that two NBAs $\mathcal{N}_1$ and $\mathcal{N}_2$ may have different widths even if $[\![\mathcal{N}_1]\!] = [\![\mathcal{N}_2]\!]$ since we can add nondeterministic transitions to accepting and non-accepting states without changing the language. For instance, consider the NBAs $\mathcal{N}_1$ and $\mathcal{N}_2$ over alphabet $\Sigma = \{a\}$ depicted in Fig. 1. We have $[\![\mathcal{N}_1]\!] = [\![\mathcal{N}_2]\!] = \{a^\omega\}$, yet $width(\mathcal{N}_1, a^w) = 1$ and $width(\mathcal{N}_2, a^w) = 2$. Moreover, since a^ω is the only word in Σ^ω we have $width(\mathcal{N}_1) = 1$ and $width(\mathcal{N}_2) = 2$.

Definition 2 (The Classes $\mathbb{NB}_k$) *Let $k \geq 1$ be a natural number. The class of languages $\mathbb{NB}_k$ is defined as follows.*

$$\mathbb{NB}_k = \{\ L \mid \exists \text{ NBA } \mathcal{N} :\quad L = [\![\mathcal{N}]\!],\ width(\mathcal{N}) \leq k\}$$

The main contribution of this paper can be summarized by the following two theorems. Theorem 2 states that the hierarchy is strict. For every level k of the hierarchy there exists a language L such that L cannot be recognized by any NBA of width k or smaller. Theorem 3 states that this hierarchy is tightly correlated to the Wagner hierarchy.

Theorem 2. $\mathbb{NB}_k \subsetneq \mathbb{NB}_{k+1}$

Proof. This is a corollary of Theorem 1 and the forthcoming Theorem 3. □

Theorem 3. *Let L be an ω-language, and let $k \geq 0$ be a natural number. Then*

- $L \in \mathbb{NB}_k \Longrightarrow L \in \mathbb{DM}^+_{2k+1}$
- $L \in \mathbb{DM}^+_k \Longrightarrow L \in \mathbb{NB}_{\lceil \frac{k}{2} \rceil + 1}$

Proof. The first and second items are respectively given by the forthcoming Corollary 7 and Proposition 12. □

Deciding the Width In view of these relations, an important question is to find the width of a given NBA $\mathcal{N}$.

Proposition 4 *The problem of finding the width of an* NBA *is solvable in time $n^{O(n)}$ where n is the number of states in the given* NBA.

Proof. Fisman and Lustig [4, Proposition 2] provide a construction for a DBA $\mathcal{B}_k$ that accepts a word w iff $width(\mathcal{N}, w) < k$ when $\mathcal{N}$ is a given NBA. A DBA can be seen as a DPA with 2 colors (accepting states are colored 1 and non-accepting states are colored 2). Given a DPA $\mathcal{P}_k$ one can construct a DPA $\overline{\mathcal{P}_k}$ for

its complement by assigning $\overline{\kappa}(q) = \kappa(q) + 1$ (where κ is the coloring function of $\mathcal{P}_k$ and $\overline{\kappa}$ is the coloring function of $\overline{\mathcal{P}}_k$). The constructed DPA $\overline{\mathcal{P}}_k$ accepts a word w iff $width(\mathcal{N}, w) \geq k$. Thus, $width(\mathcal{N}) < k$ iff $\overline{\mathcal{P}}_k$ is empty. Emptiness of a DPA can be solved in time polynomial in the number of states and colors. Applying a binary search would add a factor of $\log n$ where n is the number of states in $\mathcal{N}$. When n is the number of states in $\mathcal{N}$, the DBA $\mathcal{B}_k$ may have upto $n^{O(n)}$ states. Since the DPA $\overline{\mathcal{P}}_k$ has the same number of states as $\mathcal{B}_k$ (and the time spent to build $\mathcal{B}_k$ is polynomial in the size of $\mathcal{B}_k$), we can decide the width in time $n^{O(n)}$. □

Krishnan, Puri and Brayton [9] show that determining the Rabin index (Streett index) of a given a DRA (DSA) is NP-*complete*, yet it is decidable in polynomial time whether a DRA (DSA) with f pairs has Rabin index (Streett index) f or any constant c.[2] Wilke and Yoo [14] show that the chain measure of a given DMA over an alphabet of size ℓ with n states, m strongly connected components, and f accepting sets can be decided in time $O(f^2 n\ell + m \log m)$. Note that m and f may be exponential in n. Since the width is not an invariant among all NBAs accepting a given language, the question of finding the minimal k for which $[\![\mathcal{N}]\!] \in \mathbb{NB}_k$ where the input is $\mathcal{N}$ is not answered by Proposition 4. It can be answered e.g. by translating it to a DRA or DMA and inferring the result using the above mentioned results and the relations of the hierarchies, but a lower bound remains open.

4 From $\mathbb{NB}_k$ to $\mathbb{DM}_{k'}$

The proof of the first part of Theorem 3 goes via DPAs, using the relations between their rank and the chain measure as stated in Theorem 1.

Proposition 5 $L \in \mathbb{NB}_k \Longrightarrow L \in \mathbb{DP}_{2k+1}$.

Proof. The known constructions from NBA to DPA, given an NBA $\mathcal{N}$ with n states produce a DPA of rank $2n$ [12, 4]. The construction of Fisman and Lustig [4] is based on the notion of the width of the skeleton tree. Given an NBA $\mathcal{N}$, they first show how to construct a DPA $\mathcal{P}_k$ that provides a correct answer only for words of width exactly k w.r.t. $\mathcal{N}$, as stated in Lemma 6.

Lemma 6 ([4, Proposition 4]) *Let $\mathcal{N}$ be an NBA with n states, and let $k \in [1..n]$. There exists a DPA $\mathcal{P}_k$ using colors $\{0, 1, 2\}$ such that for any word w*

- *if $width(\mathcal{N}, w) = k$ then $\mathcal{P}_k$ accepts w iff $\mathcal{N}$ does,*
- *if $width(\mathcal{N}, w) < k$ then $\mathcal{P}_k$ rejects w,*
- *if $\mathcal{P}_k$ accepts w then w is accepted by $\mathcal{N}$, and*
- *$\mathcal{P}_k$ visits 0 infinitely often iff $width(\mathcal{N}, w) < k$.*

[2] The Rabin index (Streett index) is the least possible number of accepting pairs used in a DRA (DSA) recognizing the language.

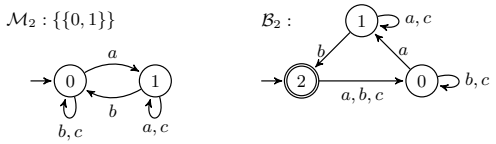

Fig. 2. A DMA $\mathcal{M}_2$ in $\mathbb{DM}_1^+$ and a minimal equivalent DBA $\mathcal{B}_2$.

They then show that a DPA recognizing the same language as $\mathcal{N}$ can be constructed by running the DPAs for width 1 to n in parallel, where n is the size of $\mathcal{N}$. The colors are distributed so that $\mathcal{P}_k$ uses colors $2k$, $2k+1$ and $2n+2$, and the color of the compound state $(s_1, s_2, \ldots, s_n)$ where s_i is the state of $\mathcal{P}_i$ is $\min\{c_1, c_2, \ldots, c_n\}$ where c_i is the color of s_i. The obtained DPA has rank $2n+2$. Clearly the same reasoning shows that given the width of $\mathcal{N}$ is at most k, the DPA obtained by running in parallel the automata $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ provides a correct result, and uses $2k+1$ colors. □

The following is a direct corollary of Proposition 5 and Theorem 1.

Corollary 7 $L \in \mathbb{NB}_k \Longrightarrow L \in \mathbb{DM}_{2k+1}^+$.

5 From $\mathbb{DM}_k^+$ to $\mathbb{NB}_{k'}$

We show here the second item of Theorem 3. We start with the simple cases $\mathbb{DM}_1^+$ and $\mathbb{DM}_1^-$, and then generalize the ideas of these constructions to obtain the construction for $\mathbb{DM}_k^+$ for arbitrary k.

By Theorem 1, $L \in \mathbb{DM}_1^+ \iff L \in \mathbb{DB}$. Any DBA has width 1 when regarded as an NBA. It follows that $L \in \mathbb{DM}_1^+$ implies $L \in \mathbb{NB}_1$. We note, however, that given a DMA $\mathcal{M}$ such that $\mathcal{M} \in \mathbb{DM}_1^+$ it is not always the case that we can define a DBA on the same structure as $\mathcal{M}$. Consider, for instance, the DMA $\mathcal{M}_2$ of Figure 2 defined over alphabet $\Sigma = \{a, b, c\}$. It accepts the language $(\Sigma^* a \Sigma^* b)^\omega$. No rejecting set subsumes an accepting set, thus the maximal length of a positive chain is 1 and $[\![\mathcal{M}_2]\!] \in \mathbb{DM}_1^+$. While $\mathcal{M}_2$ has two states there is no DBA with two states that accepts the same language. A minimal DBA for $[\![\mathcal{M}_2]\!]$ requires at least 3 states. The DBA $\mathcal{B}_2$ of the same figure is a minimal DBA for $[\![\mathcal{M}_2]\!]$.

Proposition 8 below follows from the fact that $L \in \mathbb{DM}_1^+ \iff L \in \mathbb{DB}$. We provide a direct construction based on the LAR (*latest appearance record*) data structure due to Gurevich and Harrington [6], for completeness and to introduce LAR which will be used in the proof of the relations of higher levels of the hierarchy as well.

Proposition 8 $L \in \mathbb{DM}_1^+$ *implies* $L \in \mathbb{NB}_1$

Proof. Let $\mathcal{M}$ be a DMA in $\mathbb{DM}_1^+$. From the definition of the hierarchy class $\mathbb{DM}_1^+$ it follows that no superset of an accepting SCC in $\mathcal{M}$ can be rejecting. On the other hand, a subset of an accepting set may be rejecting. To be able to use a Büchi condition we need to define a set of states F such that a visit to F guarantees that *all* states of some accepting set were visited. The solution uses the LAR (*latest appearance record*) data structure due to Gurevich and Harrington [6]. The idea is to construct a deterministic automaton whose states are permutations of the states of $\mathcal{M}$ augmented by a hit position, denoted $\sharp$. If the current state is $p_1 p_2 \dots p_i \sharp p_{i+1} \dots p_n$ and $\delta(p_n, a) = q$ and $q = p_j$, then

$$\delta_{\mathcal{B}}(\langle p_1 p_2 \dots p_i \sharp p_{i+1} \dots p_n \rangle, a) = \langle p_1 p_2 \dots p_{j-1} \sharp p_{j+1} \dots p_n p_j \rangle$$

That is, the transition relation moves the state p_j currently visited by $\mathcal{M}$ to the rightmost position in the list, and moves the $\sharp$ symbol to the position on the list where p_j resided previously.

We formalize this using the following definitions, that will also be used in later proofs. Formally, let $Q = \{q_0, q_1, \dots, q_{n-1}\}$, $\delta : Q \times \Sigma \to Q$, $\tau : 2^Q \to \{+, -\}$, $\sharp$ a symbol not in Q, and $A \subseteq Q$ a subset of cardinality ℓ with states $p_1, p_2, \dots, p_\ell$ so that if $p_i = q_j$ and $p_{i+1} = q_k$ then $j < k$. Let

- $\textsc{larset}(A) = \{w \in (A \cup \{\sharp\})^* \mid \forall q \in A \cup \{\sharp\}, |w|_q = 1\}$[3]
- $\textsc{larinit}(A, p_i) = \langle p_1 p_2 \dots p_{i-1} \sharp p_{i+1} p_{i+2} \dots p_\ell p_i \rangle$
- $\textsc{lartrans}(A) = \{(\langle p_1 p_2 \dots p_i \sharp p_{i+1} \dots p_\ell \rangle, a, \langle p_1 p_2 \dots p_{j-1} \sharp p_{j+1} \dots p_\ell p_j \rangle) \mid \delta(p_\ell, a) = p_j\}$
- $\textsc{laracc}(A, \tau) = \{u \sharp v \mid set(uv) = A,\ \tau(set(v)) = +\}$ where $set(v)$ denotes the set of states in the word v.

Let $\mathcal{M} = \langle \Sigma, Q, q_0, \delta, \tau \rangle$. Let $\mathcal{B}$ be the DBA $\langle \Sigma, \textsc{larset}(Q), \textsc{larinit}(Q, q_0), \textsc{lartrans}(Q), \textsc{laracc}(Q, \tau) \rangle$.

Lemma 9 ([5, Lemma 1.21]) *Let ρ be a run of $\mathcal{M}$ on a given word w and let $\rho_{\mathcal{B}} = s_0 s_1 s_2 \dots$ be the run of $\mathcal{B}$ on w where $s_i = u_i \sharp v_i$. Then $\mathsf{Inf}(\rho) = S$ iff the following conditions hold*

- *for some $i_0 \in \mathbb{N}$ for all $i > i_0$ we have $set(v_i) \subseteq S$ and*
- *for infinitely many i's we have $set(v_i) = S$*

Since $\mathcal{M} \in \mathbb{DM}_1^+$ guarantees that no superset of an accepting set may be rejecting, it is enough to require that we infinitely often visit a state $u \sharp v$ where all states of v form an accepting set of $\mathcal{M}$, this is exactly the acceptance condition of $\mathcal{B}$. Therefore, $[\![\mathcal{B}]\!] = [\![\mathcal{M}]\!]$. Since $\mathcal{B}$ is deterministic, the width of $\mathcal{B}$ is one. Hence, $L \in \mathbb{NB}_1$. □

For $L \in \mathbb{DM}_1^-$ there is no guarantee that an equivalent DBA exists. We are guaranteed, though, that an equivalent DCA exists. In Proposition 11 we show that this entails that $L \in \mathbb{NB}_2$.

[3] We use $|w|_a$ to denote the number of occurrences of the letter a in w.

First we need some terms and a lemma. Let $\rho = q_0 q_1 q_2 \ldots$ be a run of a given automaton $\mathcal{A}$ on a given word w. We say that the run ρ *gets trapped* in an SCC S if starting from some $z \in \mathbb{N}$ for every $z' > z$ the state $q_{z'}$ of this run belongs to S. In an automaton with finitely many states, every run on an infinite word should eventually get trapped in some SCC. We let $trap(\rho)$ denote the minimal SCC that ρ gets trapped in. If the automaton is non-deterministic, there may be several runs on a given word and each run may get trapped in a different SCC. For a skeleton path $\varrho = Q_0 Q_1 Q_2 \ldots$ we say that it gets trapped in $\{S_1, \ldots, S_k\}$ if starting from some $z \in \mathbb{N}$ for every $z' > z$ we have $Q_{z'} \subseteq \cup_{i \in [1..k]} S_i$. We let $trap(\varrho) = \{S_1, \ldots, S_k\}$ denote the minimal set of minimal SCCs that ϱ gets trapped in.

Lemma 10 *Let $\mathcal{N} = \langle \Sigma, Q, q_0, \delta, F \rangle$ be an* NBA *with an* SCC *S satisfying the following two conditions:*

- *For every letter $a \in \Sigma$ and every state $q \in S$, $|\delta(q, a)| = 1$.*
- *For every pair of states $q, p \in S$ we have $[\![\mathcal{N}_{q_0,q}]\!] \cap [\![\mathcal{N}_{q_0,p}]\!] = \emptyset$.*[4]

Let $\boldsymbol{SK}_w^{\mathcal{A}}$ be the skeleton-tree of w w.r.t $\mathcal{N}$. Let ϱ_1 and ϱ_2 be two skeleton paths in $\boldsymbol{SK}_w^{\mathcal{A}}$. Then $S \in trap(\varrho_1)$ implies $S \notin trap(\varrho_2)$.

Proof. Consider a word $w = a_1 a_2 a_3 \ldots$, and let ρ_1 and ρ_2 be two different runs of $\mathcal{N}$ on w. Assume now both ρ_1 and ρ_2 get trapped in S. We claim that there exists a point in which both runs reach the same state. That is, if $\rho_1 = q_0 q_1 q_2 \ldots$ and $\rho_2 = p_0 p_1 p_2 \ldots$ then there exists $z \in \mathbb{N}$ such that for every $z' > z$ we have $q_{z'} = p_{z'}$. Assume ρ_1, ρ_2 enter the SCC S at time points z_1, z_2, resp. and w.l.o.g. $z_1 \leq z_2$. Then ρ_1 (resp. ρ_2) entered S after reading the prefix $a_1 a_2 \ldots a_{z_1}$ of w (resp. $a_1 a_2 \ldots a_{z_2}$). We claim that $p_{z_2} = q_{z_2}$. If not, then exists two states $p, q \in S$ such that $p = p_{z_2} \neq q_{z_2} = q$ and $a_1 a_2 \ldots a_{z_2} \in [\![\mathcal{N}_{q_0,p}]\!] \cap [\![\mathcal{N}_{q_0,q}]\!]$ contradicting the second premise of the lemma. Now that we established $p_{z_2} = q_{z_2}$, since by the first premise all transitions within S are deterministic, it follows that $p_{z'} = q_{z'}$ for every $z' > z_2$. Since all runs that get trapped in S eventually reach the same state, it follows that they must conjoin to the same path ϱ of $\mathbf{SK}_w^{\mathcal{A}}$. □

Proposition 11 *$L \in \mathbb{DM}_1^-$ implies $L \in \mathbb{NB}_2$*

Proof. Let $\mathcal{M}$ be a DMA in $\mathbb{DM}_1^-$. From the definition of the hierarchy class $\mathbb{DM}_1^-$ it follows that no subset of an accepting SCC in $\mathcal{M}$ can be rejecting. We show that we can build an equivalent NBA $\mathcal{B}$ of width 2 using the following idea. The NBA $\mathcal{B}$ will consists of all of $\mathcal{M}$'s states and transitions. In addition, for each maximal accepting SCC A of M, $\mathcal{B}$ will have the ability to non-deterministically transit to a copy A' of A. The primed copy of A will have all the inner transitions of A, but no transitions out of A'. This way $\mathcal{B}$ can at any point during the run choose to move to a primed copy A' of one of the accepting SCCs A. Once this

[4] The notation $\mathcal{N}_{q,q'}$ is used to denote the NFA $\langle \Sigma, Q, q, \delta, \{q'\} \rangle$ obtained from $\mathcal{N}$ by making q the initial state, q' the final state, and regarding it as a nondeterminsitc automaton on finite words.

choice was made, $\mathcal{B}$ can only remain in A' or fall off the automaton. The set of accepting states of $\mathcal{B}$ consists of all states in the primed copies. If $\mathcal{B}$ visits such a state in say the accepting SCC A'_i infinitely often then the run of $\mathcal{M}$ gets trapped in A_i, and if the run of $\mathcal{M}$ gets trapped in A_i, then there exists a run of $\mathcal{B}$ that will get trapped in A'_i. Therefore, both recognize the same language.

Formally, let $\mathcal{M} = \langle \Sigma, Q, \lambda, \delta, \tau \rangle$. Let $A_1, A_2, \ldots, A_k$ be the maximal accepting SCCs of $\mathcal{M}$.[5] The NBA $\mathcal{B} = \langle \Sigma, Q_{\mathcal{B}}, q_{0\mathcal{B}}, \delta_{\mathcal{B}}, F_{\mathcal{B}} \rangle$ is defined as follows. The states of $Q_{\mathcal{B}}$ are $Q \cup A'_1 \cup A'_2 \ldots \cup A'_k$ where $A'_i = \{q' \mid q \in A_i\}$. The initial state $q_{0\mathcal{B}}$ is q_0. The accepting states $F_{\mathcal{B}}$ are $\cup_{i \in [1..k]} A'_i$. The transitions $\delta_{\mathcal{B}}$ are defined as follows. If $(q, a, p) \in \delta$ then $(q, a, p) \in \delta_{\mathcal{B}}$. In addition, if $q \in Q$ and $p \in A_i$ then $(q, a, p') \in \delta_{\mathcal{B}}$. If $q \in A_i$ and $p \in A_i$ then $(q', a, p') \in \delta_{\mathcal{B}}$.

Suppose a run $q_0 q_1 q_2 \ldots$ of $\mathcal{M}$ on a given word w gets trapped in an accepting SCC A_i. Since no subset of A_i is rejecting, this run is accepting. Let z_0 be such that $q_z \in A_i$ for every $z > z_0$. Then $\mathcal{B}$ can mimic $\mathcal{M}$ up to point $z_0 - 1$, at z_0 it can make a nondeterministic transition to A'_i and then stay there forever long. That is, the run $q_0 q_1 q_2 \ldots q_{z_0-1} q'_{z_0} q'_{z_0+1} q'_{z_0+2} \ldots$ where q'_{z_0+i} is the primed version of q_{z_0+i} is a run of $\mathcal{B}$, and this run is accepting. For the other direction, if there exists an accepting run of $\mathcal{B}$ on a given word w, it means that starting from some point the run moved to one of the A'_i's and stayed there forever long. This entails that when $\mathcal{M}$ reads w, it will get trapped in the SCC A_i. (If this was not the case, that run of $\mathcal{B}$ on w would encounter a letter for which no transitions is available.) Thus $[\![\mathcal{M}]\!] = [\![\mathcal{B}]\!]$.

Next we show that the width of $\mathcal{B}$ is 2. Let ρ_w be the run of $\mathcal{M}$ on a given word w and let $\mathit{trap}(\rho_w) = T$. If T is rejecting then $\mathcal{B}$ can only get trapped in T, since on any choice to move to some A'_i it will end up falling of A'_i since $\mathcal{M} \in \mathbb{DM}_1^-$ implies $T \not\subseteq A_i$. (If this was not the case then, since for any transition in A'_i there is a transition from A_i to A_i, this would entail that $\mathcal{M}$ would have got trapped in A_i which is not the case.) If T is accepting then it equals some A_j for $j \in [1..k]$. Then a run of $\mathcal{B}$ on w can get trapped either in A_j or in A'_j, but it cannot get trapped in any other A_i or A'_i. Since, from the same arguments as in the rejecting case, if at some point $\mathcal{B}$ chooses to transit to A'_i it will eventually fall off of it. We have shown that each run of $\mathcal{B}$ may get trapped in at most 2 MSCCs. Since all MSCCs of $\mathcal{B}$ (the originals of $\mathcal{M}$ and the new MSCCs A'_i) satisfy the premises of Lemma 10, and since for each skeleton path ϱ we have $\mathit{trap}(\varrho) \neq \emptyset$, the maximum width of any run of $\mathcal{B}$ is 2. Thus $[\![\mathcal{M}]\!] \in \mathbb{NB}_2$. □

We can now generalize these two ideas to obtain a construction for $\mathbb{DM}_k^+$ for arbitrary k.

Proposition 12 $L \in \mathbb{DM}_k^+ \Longrightarrow L \in \mathbb{NB}_{\lceil \frac{k}{2} \rceil + 1}$

Proof. Let $\mathcal{M}$ be a DMA in $\mathbb{DM}_k^+$. From the definition of the hierarchy classes $\mathbb{DM}_k^+$ it follows that the maximum length of a positive chain is k. We build

[5] An SCC A of $\mathcal{M}$ is said to be *maximal accepting* if no accepting SCC A' subsumes it. Note that A is not required to be an MSCC. E.g. if $R = A \cup \{q\}$ is an MSCC, and R is rejecting and A is accepting, then A is a maximal accepting SCC, but not an MSCC.

an equivalent NBA $\mathcal{B}$ of width at most $\lceil \frac{k}{2} \rceil + 1$ using the following idea. As in the proof of Proposition 11, $\mathcal{B}$ will consists of all states of the given DMA $\mathcal{M}$, and will have nondeterministic transitions to copies of accepting SCCs. Unlike in that proof, we will need to consider not just the maximal accepting SCCs but all accepting SCCs. As in the proof of Proposition 8, since an accepting SCC A may contain rejecting SCCs, we will use a LAR construction to make sure that all states of an accepting SCC are visited infinitely often.

Formally, let $\mathcal{M} = \langle \Sigma, Q, \lambda, \delta, \tau \rangle$. Assume $A_1, A_2, \ldots, A_m$ are the accepting SCCs of $\mathcal{M}$. We define the NBA $\mathcal{B} = \langle \Sigma, Q_\mathcal{B}, q_{0\mathcal{B}}, \delta_\mathcal{B}, F_\mathcal{B} \rangle$ as follows.

- $q_{0\mathcal{B}} = q_0$
- $Q_\mathcal{B} = Q \uplus_{i \in [1..m]} \textsc{larset}(A_i)$
- $\delta_\mathcal{B} = \delta \uplus_{i \in [1..m]} \textsc{lartrans}(A_i) \cup$
  $\{(q, a, \textsc{larinit}(A_i, p)) \mid (q, a, p) \in \delta,\ q \in Q \setminus A_i,\ p \in A_i\}$
- $F_\mathcal{B} = \uplus_{i \in [1..m]} \textsc{laracc}(A_i, \tau_i)$ where $\tau_i(S) = +$ iff $S = A_i$.

We claim that $\mathcal{B}$ recognizes the same language as $\mathcal{M}$. Let w be a word and let ρ be the run of $\mathcal{M}$ on w. Let $\mathsf{Inf}(\rho) = S$. If $\tau(S) = +$ then $S = A_i$ for some $i \in [1..m]$. Assume ρ gets trapped in S after time point z. Thus $\mathcal{B}$, at some time point after z, can choose a transition of the form $(q, a, \textsc{larinit}(A_i, p))$ and move to $\textsc{larset}(A_i)$. Since $\mathcal{M}$ gets trapped in A_i, $\mathcal{B}$ will not fall off $\textsc{larset}(A_i)$, and by Lemma 9, infinitely often $\mathcal{B}$ will visit a state of the form $\sharp_i v$ such that $\mathit{set}(v) = A_i$. Thus $\mathcal{B}$ will accept. Suppose now $\tau(S) = -$. If $\mathcal{B}$ did not move to any of the $\textsc{larset}(A_i)$, clearly it will not accept. Suppose it did move to $\textsc{larset}(A_i)$ for some i. If A_i does not subsume S then $\mathcal{B}$ will fall off $\textsc{larset}(A_i)$. Assume thus $A_i \supseteq S$. Then by Lemma 9, eventually only states of the form $u\sharp_i v$ where $\mathit{set}(v) \subseteq S$ will be visited. Since $A_i \supsetneq S$, by Lemma 9, it will not be the case that infinitely often states of the form $\sharp_i v$ with $\mathit{set}(v) = A_i$ will be visited, thus $\mathcal{B}$ will reject. Hence, $[\![\mathcal{B}]\!] = [\![\mathcal{M}]\!]$.

We turn to reason about $\mathcal{B}$'s width. Let ρ_w be the run of $\mathcal{M}$ on w and assume $\mathit{trap}(\rho_w) = T$. Let ρ be a run of $\mathcal{B}$, then from the same arguments as in the proof of Proposition 11 we have that either $\mathit{trap}(\rho) = T$ or $\mathit{trap}(\rho) = \textsc{larset}(A_i)$ for some $A_i \supseteq T$. Note that by definition of $\mathcal{B}$, for every $i \in [1..m]$ the set $\textsc{larset}(A_i)$ is an MSCC, and all the MSCCs of $\mathcal{B}$ satisfy the premises of Lemma 10. Thus if ϱ_1 and ϱ_2 are two skeleton paths of $\mathbf{SK}_w^\mathcal{B}$ then $\mathit{trap}(\varrho_1) \cap \mathit{trap}(\varrho_2) = \emptyset$. We claim further that if $\textsc{larset}(A_1) \in \mathit{trap}(\varrho_1)$ and $\textsc{larset}(A_2) \in \mathit{trap}(\varrho_2)$ then either $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$. Assume this is not the case. Note that $\exists \varrho \in \mathbf{SK}_w^\mathcal{B}$ such that $T \in \mathit{trap}(\varrho)$. Since there is a single run leading to T, and all of its states are non-accepting, ϱ must be the rightmost skeleton-path in $\mathbf{SK}_w^\mathcal{B}$. Since Q is the only non sink MSCC of $\mathcal{B}$, all skeleton paths split from ϱ at some point. Assume ϱ_1 and ϱ_2 split from ϱ at time points z_1 and z_2, resp. The nodes at the splits must be accepting (since their sibling in ϱ is non-accepting). Thus at z_1 it must be that ϱ_1 recently visited all states of A_1 and at z_2 it must be that ϱ_2 recently visited all states of A_2. Assume w.l.o.g. $z_1 < z_2$. If at some time point between z_1 and z_2, the original path of $\mathcal{M}$, ρ_w visited a node in $A_2 \setminus A_1$ then ϱ_1 will not have any descendants (since $\textsc{larset}(A_1)$ has no corresponding transitions). Assume thus all nodes of $A_2 \setminus A_1$ have recently been visited before z_1. Then

at z_1 not only all A_1 are visited but also all of A_2, thus there exists a split corresponding to $\textsc{larset}(A_2)$ before or at z_2, which contradicts the split at z_2 since $\textsc{larset}(A_2)$ can only be trapped in one skeleton-path. Therefore if the set of skeleton paths of $\mathbf{SK}_w^{\mathcal{B}}$ is $\{\varrho\} \cup \{\varrho_1, \ldots, \varrho_\ell\}$ where ϱ_i split from ϱ before ϱ_{i+1} then $\mathit{trap}(\varrho_1), \ldots, \mathit{trap}(\varrho_\ell)$ will correspond to accepting sets $A_1, \ldots, A_\ell$ along one inclusion chain of $\mathcal{M}$. Since the length of a positive chain is bounded by k, the number of accepting sets along such a chain is bounded by $\lceil \frac{k}{2} \rceil$. Adding ϱ, the rightmost branch, we obtain that the width of $\mathcal{B}$ is $\lceil \frac{k}{2} \rceil + 1$. □

This completes the proof of Theorem 3.

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A Formal definition of the skeleton tree

We provide here the definitions of the split, reduced and skeleton trees of Kähler and Wilke. In the following we assume a fixed non-deterministic Büchi automaton $\mathcal{A} = (\Sigma, Q, q_0, \delta, F)$ with $|Q| = n$ and an infinite word $w = \sigma_1\sigma_2\sigma_3\ldots$ over Σ. We use $\overline{F}$ to denote the set of non-accepting states, i.e. $Q \setminus F$.

Annotated Binary Trees We first provide some elementary definitions of annotated binary trees. The nodes of such trees are strings in $\{0,1\}^*$. The root node is the empty string ϵ. The left successor of a node t is $t0$ and the right successor is $t1$. An *annotated binary tree* is a function from $\{0,1\}^*$ to some domain D. A node mapped to $\emptyset$ is regarded as absent from the tree. The *depth* of node t, denoted $|t|$, is its distance from the root, and it equals the length of the string t. So the depth of the root is 0, and the depth of its successors are 1. For $t_1, t_2 \in \{0,1\}^*$ we say that $t_1 <_{\text{lex}} t_2$ if the string t_1 is lexicographically smaller than t_2. We say that $t_1 <_{\text{lft}} t_2$ if $|t_1| = |t_2|$ and $t_1 <_{\text{lex}} t_2$. The *i-th level* of an annotated tree T consists of all nodes in depth i. If $t_1 <_{\text{lex}} t_2 <_{\text{lex}} \cdots <_{\text{lex}} t_n$ are all the nodes of the i-th level of T then the *i-th slice* of T is the sequence $\langle T(t_1), T(t_2), \ldots, T(t_n)\rangle$. The width of the i-th level is the number of nodes in that level.

The Split Tree The *split tree* can be thought of as a refinement of the subset construction by differentiating accepting and rejecting states. It is a binary tree, where a node holds a subset of $\mathcal{A}$'s states. If S is a subset in level i of the split-tree, then its left successor holds all the accepting successors of states in S and its right successor holds all the non-accepting successors of states in S. Formally, the split tree of $\mathcal{A}$ on w is a function from $\{0,1\}^*$ to 2^Q satisfying $\mathbf{SP}_w^{\mathcal{A}}(\epsilon) = I$, and for $t \in \{0,1\}^*$ such that $|t| = i$ and $\mathbf{SP}_w^{\mathcal{A}}(t) = S$,

$$\mathbf{SP}_w^{\mathcal{A}}(t0) = \delta(S, \sigma_{i+1}) \cap F \text{ and}$$
$$\mathbf{SP}_w^{\mathcal{A}}(t1) = \delta(S, \sigma_{i+1}) \cap \overline{F}.$$

We say that $\rho \in \{0,1\}^\omega$ is an infinite path of $\mathbf{SP}_w^{\mathcal{A}}(t)$ if $\mathbf{SP}_w^{\mathcal{A}}(t) \neq \emptyset$ for every prefix t of ρ. We say that an infinite path ρ of $\mathbf{SP}_w^{\mathcal{A}}(t)$ is *left-recurring* or *accepting* if ρ has infinitely many zeros (we use the term *left recurring* to emphasize the path has infinitely many left turns, and the term *accepting* to emphasize it has infinitely many accepting nodes).

Claim ([7]).

- $\mathcal{A}$ accepts w iff $\mathbf{SP}_w^{\mathcal{A}}$ has a left recurring path.
- The width of $\mathbf{SP}_w^{\mathcal{A}}$'s layers is unbounded, in general.

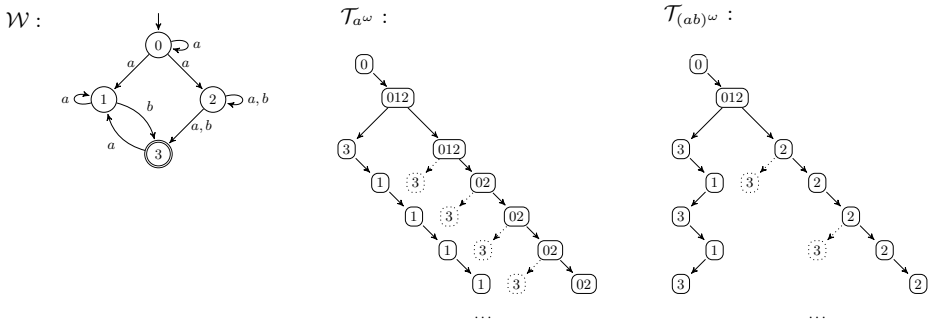

Fig. 3. A Büchi automaton $\mathcal{W}$ and its reduced and skeleton trees for words $w_1 = a^\omega$ and $w_2 = (ab)^\omega$. Solid nodes and edges belong to both the reduced and skeleton trees, and dotted nodes and edges belong to the reduced tree but not to the skeleton tree. A node labeled with multiple digits stands for the subset containing the corresponding states (e.g. 012 stands for $\{0, 1, 2\}$).

The Reduced Split Tree The width of the split-tree is unbounded, in general, which makes it hard to handle. The *reduced split tree* (or simply the *reduced tree*) reduces the split tree by removing from a node all states that appear to the left of that node, in the same level. This reduction eliminates many of the paths of the split-tree, but it keeps the left-most accepting path. And if there are accepting runs of $\mathcal{A}$ on w, then at least one is embedded in the left-most accepting path. Formally, the reduced split tree of $\mathcal{A}$ on w, denoted $\mathbf{RS}_w^{\mathcal{A}}$, is a function from $\{0,1\}^*$ to 2^Q satisfying $\mathbf{RS}_w^{\mathcal{A}}(\epsilon) = I$, and for $t \in \{0,1\}^*$ such that $|t| = i$, $\mathbf{RS}_w^{\mathcal{A}}(t) = S$ and $\tilde{S} = \cup \{S' \mid \exists t' <_{\text{lft}} t,\ \delta(\mathbf{RS}_w^{\mathcal{A}}(t'), \sigma_{i+1}) = S'\}$,

$$\mathbf{RS}_w^{\mathcal{A}}(t0) = \delta(S, \sigma_{i+1}) \cap F \setminus \tilde{S} \text{ and}$$
$$\mathbf{RS}_w^{\mathcal{A}}(t1) = \delta(S, \sigma_{i+1}) \cap \overline{F} \setminus \tilde{S}.$$

Claim ([7]).

- $\mathcal{A}$ accepts w iff $\mathbf{RS}_w^{\mathcal{A}}$ has a left recurring path.
- The width of $\mathbf{RS}_w^{\mathcal{A}}$'s layers is bounded by n.

The Skeleton Tree The reduced tree can be effectively constructed and handled, thanks to its bounded width. It may, however, have nodes with only finitely many descendants. Conceptually, it is easier to think about that tree without these nodes. The *skeleton tree* of $\mathcal{A}$ on w, denoted $\mathbf{SK}_w^{\mathcal{A}}$, is the maximal subtree of $\mathbf{RS}_w^{\mathcal{A}}$ satisfying that every node has infinitely many descendants.

Claim ([7]).

- $\mathcal{A}$ accepts w iff $\mathbf{SK}_w^{\mathcal{A}}$ has a left recurring path.
- The width of $\mathbf{SK}_w^{\mathcal{A}}$'s layers is monotonically non-decreasing and bounded by n.

The second item says that there exists a natural number z_0 such that forall $z \geq z_0$, the width of $\mathbf{SK}_w^{\mathcal{A}}$ at level z is k. This k is thus the number of infinite paths of the skeleton (or reduced) tree. We use $width(\mathcal{A}, w)$ to denote the width of the skeleton tree on w.

Example 13. A Büchi automaton $\mathcal{W}$ and its reduced and skeleton trees for words $w_1 = a^\omega$ and $w_2 = (ab)^\omega$ are depicted in Fig. 3. Both words are of width 2, i.e. $width(\mathcal{W}, w_1) = 2 = width(\mathcal{W}, w_2)$, but $w_1 \notin [\![\mathcal{W}]\!]$ whereas $w_2 \in [\![\mathcal{W}]\!]$. Indeed, the skeleton tree for w_1 has no left-recurring paths, whereas the skeleton tree for w_2 does have a left-recurring path.