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Safety and Liveness, Weakness and Strength, and the Underlying Topological Relations

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We present a characterization that shows what it means for a formula to be a weak or strong version of another formula. We show that the weak version of a formula is not the same as Alpern and Schneider's safety component, but can be achieved by taking the closure in the Cantor topology over an augmented alphabet in which every formula is satisfiable. The resulting characterization allows us to show that the set of semantically weak formulas is exactly the set of non-pathological safety formulas. Furthermore, we use the characterization to show that the original versions of the IEEE standard temporal logics PSL and SVA are broken, and we show that the source of the problem lies in the semantics of the SERE intersection and fusion operators. Finally, we use the topological characterization to show the internal consistency of the alternative semantics adopted by the latest version of the PSL standard.

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1. INTRODUCTION

In recent years, model checking has become a mainstream tool for verification of hardware, with most industrial tools supporting one or both of the IEEE standard temporal logics PSL [Eisner and Fisman 2006; IEEE c; d] and SVA [Cerny et al. 2010; IEEE a; b]. In these logics, as in LTL [Pnueli 1981], temporal logic operators come in *weak* and *strong* versions. Intuitively, a strong operator includes an eventuality requirement, while a weak operator does not. For instance, the LTL formula $[p \mathsf{U} q]$ (read p *strong-until* q) holds if p holds until q holds, and in addition, q eventually holds. The weak version, $[p \mathsf{W} q]$ (read p *weak-until* q), does not require that q eventually hold – it holds if the strong version holds, or if p holds forever.

Most users of model checking have an intuitive understanding that formulas containing only weak operators are easier to check than formulas containing also strong operators, and that furthermore, before checking a formula φ, it can be beneficial to check a weak version

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of it, obtained from a formula in positive normal form by weakening all operators. The user justifies this based on her intuitive understanding of the relation between weak and strong versions of a temporal operator. However, until now the relation between the weak version of a temporal operator and its strong counterpart has not been well understood.

We present a characterization that shows what it means for an operator to be the weak or strong version of another operator, or more generally for a formula to be a weak or strong version of another formula. Our motivating example will be the pair of LTL operators *weak-until* (W) and *strong-until* (U). That is, we present a characterization that lets us say that two formulas, φ and φ', are related in the same way as the pair $\psi = [p \mathsf{W} q]$ and $\psi' = [p \mathsf{U} q]$, and furthermore, that the weak version of any new temporal operator in some extension of LTL is related to its strong version in the same way that W is related to U.

Although our work is theoretical, our motivation is practical. We use the characterization to show that the original versions of the IEEE standard temporal logics PSL and SVA are broken in the sense that they have temporal operators that, intuitively, are expected to form a weak/strong pair, but that are not related semantically in the same way that W is related to U. We show that the source of the problem lies in the semantics of the SERE intersection and fusion operators. Finally, we use the topological characterization to show that an alternative semantics, proposed in [Eisner and Fisman 2008] and adopted by the latest version of the PSL standard [IEEE d], restores the internal consistency of the logic in the sense that all the expected weak/strong pairs are related in the same way.

We begin by noting that the weak/strong dichotomy is related to another dichotomy in temporal logic, that of *safety* vs. *liveness* formulas. Loosely speaking, a safety formula claims that "something bad" does not happen, while a liveness formula claims that "something good" eventually happens. Some formulas are neither safety nor liveness. For instance, the formula $[p \mathsf{W} q]$ is a safety formula, the formula $[\mathit{true} \mathsf{U} q]$ is a liveness formula, and the formula $[p \mathsf{U} q]$ is neither.

A *safety property* is a set of words that can be described by a safety formula, and a *liveness property* is a set of words that can be described by a liveness formula. Alpern and Schneider [Alpern and Schneider 1985] showed that any property can be decomposed into a safety property and a liveness property whose intersection is the original. They work with the Cantor topology on infinite words, and their proof is based on the observation that the safety properties are the closed sets and the liveness properties the dense sets in this topology. In [Alpern and Schneider 1987], they refer to the safety property obtained by taking the closure in the Cantor topology as the *safety component*. Intuitively, we expect the safety component to give a property equivalent to the one obtained by weakening the operators (i.e., replacing U with W, etc.), and usually it does. For example, let p and q be atomic propositions. Then $[p \mathsf{W} q]$ describes the safety component of the property described by $[p \mathsf{U} q]$. However, replacing q by *false* breaks the relation between the safety component and weakening the operators. The safety component of the property described by $[p \mathsf{U} \mathit{false}]$ is the empty set, which is described not by $[p \mathsf{W} \mathit{false}]$ but by *false*.

We show that weakening of a formula can be achieved by taking the closure in the Cantor topology over an augmented alphabet in which all formulas are satisfiable. The resulting framework allows us to show interesting relationships between weak and strong operators and formulas. For example, we show that the set of semantically weak formulas is exactly the set of *non-pathological* safety properties, thus strengthening the result in [Kupferman and Vardi 1999].

Because many verification methods can examine only a finite prefix of an infinite execution [Eisner et al. 2003], we are interested in a characterization that includes finite as well as infinite words. Therefore we will start by generalizing Alpern and Schneider's topological ideas to the set of finite and infinite words.

2. PRELIMINARIES

2.1. Notation

We denote a letter from a given alphabet Γ by ℓ, and an empty, finite, or infinite word from Γ by u, v, or w. The *concatenation* of u and v is denoted by uv. If u is infinite, then $uv = u$. The empty word is denoted by ϵ, so that $w\epsilon = \epsilon w = w$. If $w = uv$, we define $w/u = v$ and we say that u is a *prefix* of w, denoted $u \preceq w$, that v is a *suffix* of w, and that w is an *extension* of u, denoted $w \succeq u$.

We denote the *length* of word v as $|v|$. The empty word ϵ has length 0, a finite non-empty word $v = (\ell_0\ell_1\ell_2\cdots\ell_n)$ has length $n+1$, and an infinite word $v = (\ell_0\ell_1\ell_2\cdots)$ has length ω, where ω denotes the first transfinite ordinal number. We use i, j, and k to denote non-negative integers. For $i < |v|$ we use v^i to denote the $(i+1)^{st}$ letter of v (since counting of letters starts at zero). We denote by $v^{i\cdot\cdot}$ the suffix of v starting at v^i. That is, $v^{i\cdot\cdot} = \ell_i\ell_{i+1}\cdots\ell_n$ or $v^{i\cdot\cdot} = \ell_i\ell_{i+1}\cdots$. We use $v^{i..j}$ to denote the finite word starting at position i and ending at position j. That is, $v^{i..j} = \ell_i\ell_{i+1}\cdots\ell_j$.

We denote a set of finite/infinite words by U, V or W and refer to them as *properties*. The set of finite prefixes of W, denoted *fpref*(W), is the set $\{\text{finite } u \mid \exists w \in W \text{ such that } u \preceq w\}$. The *concatenation* of U and V, denoted UV, is the set $\{uv \mid u \in U,\ v \in V\}$. The *fusion* of U and V, denoted $U \circ V$, is the set $\{u\ell v \mid u\ell \in U,\ \ell v \in V\}$. Define $V^0 = \{\epsilon\}$ and $V^k = VV^{k-1}$ for $k \geq 1$. The *Kleene closure* of V, denoted V^*, is the set $V^* = \bigcup_{k<\omega} V^k$. The notation V^+ is used for the set $\bigcup_{0<k<\omega} V^k$. The infinite concatenation of V to itself is denoted V^ω. The union $V^* \cup V^\omega$ is denoted V^∞. For a letter ℓ we use ℓ^k, ℓ^*, ℓ^+, ℓ^ω, and ℓ^∞ as abbreviations of $\{\ell\}^k$, $\{\ell\}^*$, $\{\ell\}^+$, $\{\ell\}^\omega$, and $\{\ell\}^\infty$, respectively.

We use P to denote a set of atomic propositions, and B to denote a set of Boolean expressions (propositional formulas) over P. We use p and q to denote elements of P and a, b, and c to denote elements of B. We use Σ to denote 2^P, and we use $\widehat{\Sigma}$ to denote the augmented alphabet $\Sigma \cup \{\top, \bot\}$, where $\top$ and $\bot$ are special letters that we will define in the sequel.

The semantics of Boolean expressions over Σ is assumed to be given as a relation $\Vdash \subseteq \Sigma \times B$ relating letters in Σ with Boolean expressions in B. If $(\ell, b) \in \Vdash$ we say that the letter ℓ *satisfies* the Boolean expression b and denote it $\ell \Vdash b$. For a proposition $p \in P$ and a letter $\ell \in \Sigma$, we assume that $\ell \Vdash p$ iff $p \in \ell$. We assume that otherwise the relation $\Vdash$ behaves in the usual manner, and in particular that Boolean disjunction, conjunction and negation behave as usual. Finally, *true* and *false* are the Boolean expressions such that *true* is satisfied by every letter in Σ and *false* is satisfied by no letter.

For $b \in B$, we sometimes abuse notation and use b to denote the property $\{\ell \in \Sigma \mid \ell \Vdash b\}$, and ab to denote the property ab. For $p \in P$, we use $\langle p \rangle$ to denote the letter $\ell \in \Sigma$ such that $\ell \Vdash p$ and $\forall q \in P$ s.t. $q \neq p$, $\ell \nVdash q$. Since an atomic proposition p is also a Boolean expression, we have that p signifies the set of letters on which p holds, whereas $\langle p \rangle$ signifies the single letter on which p and only p holds.

We use φ and ψ to denote formulas, $\models$ to denote formula satisfaction, $[\![\varphi]\!]$ to denote $\{w \in \Sigma^\infty \mid w \models \varphi\}$, and $[\![\varphi]\!]$ to denote $\{w \in \widehat{\Sigma}^\infty \mid w \models \varphi\}$. The *language* of φ over Σ is the property $[\![\varphi]\!]$, and the *language* of φ over $\widehat{\Sigma}$ is the property $[\![\varphi]\!]$. We use $\varphi \equiv \psi$ to denote that $[\![\varphi]\!] = [\![\psi]\!]$, and $\varphi \mathrel{\widehat{\equiv}} \psi$ to denote that $[\![\varphi]\!] = [\![\psi]\!]$.

Let v be a finite word and $L \subseteq \Sigma^\infty$ a property. We say that v is a *bad prefix* of L if for every $w \in \Sigma^\infty$ such that $w \succeq v$ we have $w \notin L$.

We use the term *positive normal form* for formulas in which negation is applied only to Boolean expressions.

We use $cl(\cdot)$, $int(\cdot)$ and $bd(\cdot)$ for topological closure, interior, and boundary, respectively. These topological concepts are defined in Section 2.3.

Definition 1 ELTL Formulas

Let b be a Boolean expression. The set of ELTL *formulas* is recursively defined as follows:

$$\varphi \ ::= \ b! \mid b \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \mathsf{X!}\ \varphi \mid \mathsf{X}\ \varphi \mid [\varphi\ \mathsf{U}\ \varphi] \mid [\varphi\ \mathsf{W}\ \varphi]$$

Definition 2 Semantics of ELTL

Let $v \in \Sigma^\infty$ be a word.

- $v \models b!$ $\iff$ $|v| > 0$ and $v^0 \Vdash b$
  $v \models b$ $\iff$ either $v \models b!$ or $|v| = 0$
- $v \models \neg\varphi$ $\iff$ $v \not\models \varphi$
  $v \models \varphi \vee \psi$ $\iff$ $v \models \varphi$ or $v \models \psi$
  $v \models \varphi \wedge \psi$ $\iff$ $v \models \varphi$ and $v \models \psi$
- $v \models \mathsf{X!}\ \varphi$ $\iff$ $|v| > 1$ and $v^{1\cdot\cdot} \models \varphi$
  $v \models \mathsf{X}\ \varphi$ $\iff$ either $v \models \mathsf{X!}\ \varphi$ or $|v| \leq 1$
- $v \models [\varphi\ \mathsf{U}\ \psi]$ $\iff$ $\exists k < |v| : v^{k\cdot\cdot} \models \psi$ and $\forall j < k : v^{j\cdot\cdot} \models \varphi$
  $v \models [\varphi\ \mathsf{W}\ \psi]$ $\iff$ either $v \models [\varphi\ \mathsf{U}\ \psi]$ or $\forall j < |v| : v^{j\cdot\cdot} \models \varphi$

Definition 3 Additional Syntactic Operators

- $\mathsf{F}\ \varphi \stackrel{\text{def}}{=} [\mathit{true}\ \mathsf{U}\ \varphi]$
- $\mathsf{G}\ \varphi \stackrel{\text{def}}{=} [\varphi\ \mathsf{W}\ \mathit{false}]$

2.2. ELTL

We use a simple extension of LTL over finite as well as infinite and empty as well as non-empty words, shown in Definition 1, that we denote ELTL. To do so, the syntax of LTL is extended to include two *next-time* operators, one weak (X) and one strong ($\mathsf{X!}$) [Manna and Pnueli 1992, pp. 272-273]. The semantics distinguishes between the weak and strong versions only on the last letter of a finite word: $\mathsf{X}\ \varphi$ holds on the last letter of any finite word for any φ, and $\mathsf{X!}\ \varphi$ does not. Similarly, since we interpret the logic over the empty word as well, we define weak and strong Boolean expressions. A strong Boolean expression is satisfied over a word if it is not empty and the first letter satisfies the Boolean expression, and the weak Boolean expression is satisfied also if there is no first letter, i.e. if the word is empty.

The semantics of ELTL is defined inductively as shown in Definition 2. Additional syntactic operators are defined as shown in Definition 3.

2.3. Topology on Finite and Infinite Words

In this section we present the topology that we use in this paper. We assume that the reader has some familiarity with elementary topology and refer to [Kelley 1975] for topological background, or [Hoogeboom and Rozenberg 1986, Section 5] for an overview of the topological characterization of finite and infinite languages.

Given a set S, a *topology* on S is defined by specifying a family $\mathcal{U}$ of subsets of S satisfying the following conditions: (1) $\emptyset \in \mathcal{U}$; (2) $S \in \mathcal{U}$; (3) an arbitrary union of elements of $\mathcal{U}$ is in $\mathcal{U}$; and (4) a finite intersection of elements of $\mathcal{U}$ is in $\mathcal{U}$. The elements of $\mathcal{U}$ are called the *open sets* of S under this topology. Let A be a subset of S. If $S \setminus A$ is an open set, then A is said to be a *closed set*. The *closure* of A in this topology, denoted $cl(A)$, is the intersection of all closed sets containing A. Similarly, the *interior* of A, denoted $int(A)$, is the union of all open sets contained within A. The closure of A is the smallest superset of

Definition 4 The Cantor Topology on Γ^∞

The Cantor topology on Γ^∞ is defined by letting the family of open sets be the collection of all languages of the form $W\Gamma^\infty$, where $W \subseteq \Gamma^*$.

Definition 5 Prefix and Extension Closed Sets

Let $V \subseteq \Gamma^\infty$.

- V is *prefix closed* if $u \preceq v \in V$ implies $u \in V$.
- V is *extension closed* if $w \succeq v \in V$ implies $w \in V$.

Definition 6 Limit Closed and Finite Witnesses

Let $V \subseteq \Gamma^\infty$.

- V has *finite witnesses* if $\forall w \in V$, there exists finite $u \preceq w$ such that $u \in V$.
- V is *limit closed* if $\forall w \in \Gamma^\infty$, if all finite prefixes of w are in V then $w \in V$.

A that is a closed set, while the interior of A is the largest subset of A that is an open set. The *boundary* of A in this topology, denoted $bd(A)$, is the difference $cl(A) \setminus int(A)$.

Let Γ be an arbitrary alphabet. The *Cantor topology* on Γ^ω (see, e.g., [Thomas 1990]) is defined by letting the family of open sets be the collection of all ω-languages of the form $W\Gamma^\omega$, where $W \subseteq \Gamma^*$. We are interested in extending this topology to finite as well as infinite words over Γ. Replacing Γ^ω by Γ^∞, we get as candidate family of open sets the collection of all languages of the form $W\Gamma^\infty$, where $W \subseteq \Gamma^*$. The following proposition confirms that this family satisifies the conditions for a topology.

PROPOSITION 1. *The collection of languages $W\Gamma^\infty$ such that $W \subseteq \Gamma^*$ forms a family of open sets for a topology on Γ^∞.*

We are therefore justified in using this family as the collection of open sets of the Cantor topology extended to Γ^∞ as shown in Definition 4.

It is possible to characterize the Cantor topology on Γ^∞ in terms of prefix/extension/limit closed sets and sets that have finite witnesses. These sets are defined in Definitions 5 and 6. We observe that prefixed closed sets and extension closed sets are dual to each other. That is, V is prefix closed iff $\Gamma^\infty \setminus V$ is extension closed. Similarly, V is limit closed iff $\Gamma^\infty \setminus V$ has finite witnesses. We also note that the notions of prefix/extension closed are stronger than the notions of limit closed and finite witnesses in the following sense: If V is prefix closed then V has finite witnesses, and if V is extension closed then V is limit closed. We are now ready to characterize the closed and open sets in terms of these concepts.

PROPOSITION 2 (CLOSED AND OPEN SETS IN Γ^∞). *Let $V \subseteq \Gamma^\infty$.*

- *V is closed in the Cantor topology iff V is both prefix closed and limit closed.*
- *V is open in the Cantor topology iff V is extension closed and has finite witnesses.*

Finally, we will in the sequel make use of a direct characterization of the closure and interior operators in the Cantor topology.

PROPOSITION 3 (CLOSURE AND INTERIOR IN Γ^∞). *Let $V \subseteq \Gamma^\infty$.*

- $cl(V) = \{w \in \Gamma^\infty \mid \forall \text{ finite } u \preceq w : \exists v \succeq u : v \in V\}$.
- $int(V) = \{w \in \Gamma^\infty \mid \exists \text{ finite } u \preceq w : \forall v \succeq u : v \in V\}$.

2.3.1. Examples. We consider the set of properties shown in Table I, which are slight modifications of Rem's examples [Rem 1990] (see also [Manolios and Trefler 2003]). In these examples, expressions, a, b, and c denote particular Boolean expressions in $B \setminus \{true, false\}$.

i	P_i		φ_i	p.c.	e.c.	l.c.	f.w.
0	The empty set	$\emptyset$	$\mathit{false}!$	✓	✓	✓	✓
1	The empty word	$\{\epsilon\}$	false	✓		✓	✓
2	The set of non-empty finite words	Σ^+	$\mathsf{F}\ \mathsf{X}\ \mathit{false}$			✓	✓
3	The set of finite words	Σ^*	$\mathit{false} \vee$ $\mathsf{F}\ \mathsf{X}\ \mathit{false}$	✓			✓
4	The set of infinite words	Σ^ω	$\mathit{true}! \wedge$ $\mathsf{G}\ \mathsf{X}!\ \mathit{true}$		✓	✓	
5	The set of finite and infinite words	Σ^∞	true	✓	✓	✓	✓
6	There are at most 3 letters. The first (resp., second, third) letter, if it exists, satisfies a (resp., b, c).	$abc \cup ab \cup a \cup \{\epsilon\}$	$a \wedge$ $\mathsf{X}\ b \wedge$ $\mathsf{XX}\ c \wedge$ $\mathsf{XXX}\ \mathit{false}$	✓		✓	✓
7	There are at least 3 letters. The first and third letters satisfy b and the second satisfies c	$bcb\Sigma^\infty$	$b! \wedge$ $\mathsf{X}!\ c \wedge$ $\mathsf{X!X}!\ b$		✓	✓	✓
8	The first letter satisfies a and there exists a letter that does not satisfy a	$a\Sigma^*(\Sigma \setminus a)\Sigma^\infty$	$a! \wedge \mathsf{F}\ \neg a$		✓	✓	✓
9	The number of letters satisfying a is finite	$\Sigma^*(\Sigma \setminus a)^\infty$	$\varphi_3 \vee \mathsf{F}\ \mathsf{G}\ \neg a$	✓			✓
10	The number of letters satisfying a is infinite	$(\Sigma^* a)^\omega$	$\varphi_4 \wedge \mathsf{G}\ \mathsf{F}\ a$		✓	✓	
11	There is a letter satisfying b and every prior letter satisfies a	$a^* b\Sigma^\infty$	$[a\ \mathsf{U}\ b]$		✓	✓	✓
12	Either every letter satisfies a or there is a letter satisfying b and every prior letter satisfies a	$a^* b\Sigma^\infty \cup a^\infty$	$[a\ \mathsf{W}\ b]$	✓		✓	✓

Table I. Rem's examples [Rem 1990; Manolios and Trefler 2003]

For each property P_i we give an ELTL formula φ_i such that $P_i = [\![\varphi_i]\!]$. A ✓ in the appropriate column indicates that the property has the characteristic, while lack of a ✓ indicates that it does not. We use p.c. to indicate prefix closed, e.c. for extension closed, l.c. for limit closed, and f.w. for has finite witnesses.

2.4. Safety and Liveness, a Topological Characterization

Roughly speaking, a formula φ is said to define a *safety* property iff any word violating φ contains a finite prefix all of whose extensions violate φ. A formula φ is said to define a *liveness* property iff any arbitrary finite word has an extension satisfying φ. The traditional definitions for safety and liveness in the literature concentrate on infinite words alone. Definition 7 extends that of [Manna and Pnueli 1992] to finite words as well. The *safety component* of an arbitrary set of infinite words is defined in [Alpern and Schneider 1987] to be its closure in the Cantor topology. Definition 7 extends this notion of safety component

Definition 7 Safety, Liveness, Safety Component

Let $W \subseteq \Sigma^\infty$.
- W is a *safety* property if $\forall w \in (\Sigma^\infty \setminus W)$: $\exists$ finite $u \preceq w$ s.t. $\forall v$: $uv \in (\Sigma^\infty \setminus W)$.
- W is a *liveness* property if $\forall$ finite u: $\exists\ v$ s.t. $uv \in W$.
- The *safety component* of W is its closure, $cl(W)$, in the Cantor topology on Σ^∞.

to the space of finite and infinite words. A formula is said to be a *safety* (*liveness*) formula if its language is a safety (liveness) property. Some formulas are neither safety nor liveness. For instance, the formula $[p\ \mathsf{W}\ q]$ is a safety formula, the formula $\mathsf{F}\ q$ is a liveness formula, and the formula $[p\ \mathsf{U}\ q]$, equivalent to $[p\ \mathsf{W}\ q] \wedge \mathsf{F}\ q$, is neither.

In [Alpern and Schneider 1985], Alpern and Schneider showed that any property on infinite words can be decomposed into a safety property and a liveness property whose intersection is the original. Their proof is based on the observation that the safety properties are the closed sets of the Cantor topology on Σ^ω and the liveness properties are the dense sets in this topology. The following proposition asserts that the same decomposition exists when considering the space Σ^∞ of finite and infinite words, with the above definitions of safety and liveness.

PROPOSITION 4. *The safety properties are the closed sets of the Cantor topology on Σ^∞, and the liveness properties are the dense sets in this topology. Furthermore, every property in Σ^∞ can be decomposed into a safety property and a liveness property whose intersection is the original.*

Recall from elementary topology that a set $U \subseteq \Sigma^\infty$ is closed iff $cl(U) = U$, and it is dense iff $cl(U) = \Sigma^\infty$ or equivalently, $int(\Sigma^\infty \setminus U) = \emptyset$. Referring back to our examples, we have by Proposition 2 that P_0, P_1, P_5, P_6 and P_{12} are closed and P_0, P_5, P_7, P_8 and P_{11} are open. It follows from Proposition 4 that φ_0, φ_1, φ_5, φ_6, φ_{12} are safety. The closure of properties P_2, P_3, P_4, P_5, P_9, and P_{10} is Σ^∞ and thus by Proposition 4, φ_2, φ_3, φ_4, φ_5, φ_9 and φ_{10} are liveness. Properties φ_7, φ_8 and φ_{11} are neither safety nor liveness (the closure of P_7 is $\{\epsilon\} \cup b \cup bc \cup P_7$, the closure of P_8 is $\{\epsilon\} \cup a\Sigma^\infty$, and the closure of P_{11} is $a^\infty \cup P_{11}$). Note that P_0 and P_5 are both open and closed, and that φ_0 and φ_5 are both safety and liveness.

3. CHARACTERIZING WEAKNESS AND STRENGTH

3.1. The Characterization

Our goal is to define the weak version of an ELTL formula, such that $[\varphi\ \mathsf{W}\ \psi]$ is the weak version of $[\varphi\ \mathsf{U}\ \psi]$, $\mathsf{X}\ \varphi$ is the weak version of $\mathsf{X!}\ \varphi$, and such that the weak version of any ELTL formula χ is related to χ in the same way that $[\varphi\ \mathsf{W}\ \psi]$ is related to $[\varphi\ \mathsf{U}\ \psi]$.

First we note that Alpern and Schneider's safety component does not achieve our goal. For example, the safety component of the property defined by $[p\ \mathsf{U}\ \mathit{false}] \equiv \mathit{false}!$ is the empty set, but we would like the weak version of $[p\ \mathsf{U}\ \mathit{false}]$ to be $[p\ \mathsf{W}\ \mathit{false}]$, whose language is non-empty. Eliminating Boolean expressions equivalent to *false* does not solve the problem. For example, let $\varphi = [\mathit{true}\ \mathsf{U}\ \mathsf{G}p] \wedge [\mathit{true}\ \mathsf{U}\ \mathsf{G}\neg p] \equiv \mathit{false}!$. The safety component of the language of φ is the empty set, but we would like the weak version of φ to be $[\mathit{true}\ \mathsf{W}\ \mathsf{G}p] \wedge [\mathit{true}\ \mathsf{W}\ \mathsf{G}\neg p] \equiv \mathit{true}$.

Furthermore we note that it is not possible to define the weak version of a *property* in a way that meets our goal. The reason is that $[p\ \mathsf{U}\ \mathit{false}] \equiv \mathsf{X!}\ \mathit{false}$ (that is, both formulas define the same property), but our goal dictates that their weak versions be different. Nonetheless, we would like to obtain a semantic rather than syntactic characterization, so that it will capture the essence of the relation between weak and strong formulas and so that it can be applied to any new temporal operator.

Definition 8 Boolean Satisfaction for $\top$ and $\bot$

Let b be a Boolean expression.

$$\bullet \top \Vdash b \qquad \bullet \bot \nVdash b$$

Definition 9 Semantics of ELTL over the Augmented Alphabet

Let $v \in \widehat{\Sigma}^\infty$ be a word. The semantics of ELTL over the augmented alphabet is exactly as in Definition 2, but using the augmented Boolean satisfaction relation and using the dual word in the semantics of negation as follows:

$$\bullet\ v \models \neg\varphi \iff \overline{v} \not\models \varphi.$$

Definition 10 Weak and Strong Components of ELTL Formulas

Let φ be an ELTL formula. Its *weak* and *strong components*, denoted $weak(\varphi)$ and $strong(\varphi)$ respectively, are defined as follows:

$$\bullet\ weak(\varphi) = cl([\![\varphi]\!]) \qquad \bullet\ strong(\varphi) = int([\![\varphi]\!])$$

On the one hand, we want a characterization that distinguishes between two formulas which are syntactically distinct, but semantically equivalent. On the other hand, we want a characterization that is semantic, rather than syntactic. It seems that we have set ourselves an impossible task. Our approach is to enhance the set of models over which formulas are interpreted so that previously equivalent formulas for which our goal dictates different weak versions are no longer equivalent, but also in a way that such formulas remain equivalent when restricted to the original set of models.

Our solution is based on noticing that the "problem" with Alpern and Schneider's safety component is that the property described by $[p\ \mathsf{U}\ \mathit{false}]$ is the empty one. We therefore augment the set of models of a formula in such a way that no formula describes the empty property, while ensuring that the satisfiability over the original set of models Σ^∞ is not changed. We accomplish this by augmenting the alphabet with two special letters $\top$ and $\bot$, such that $\top^\omega$ models every formula and $\bot^\omega$ models no formula.

Formally, we define the semantics of ELTL over the augmented alphabet $\widehat{\Sigma} = \Sigma \cup \{\top, \bot\}$. We use $\overline{w}$ to denote the word obtained by replacing every $\top$ with a $\bot$ and vice versa. For a property W, we use $\overline{W}$ to denote the property $\{\overline{w} \mid w \in W\}$. We call $\overline{w}$ ($\overline{W}$) the *dual word (property)* of w (W). For a property W, we denote by W^c the property $\widehat{\Sigma}^\infty \setminus \overline{W}$. We refer to W^c as the *complement* property of W.[1] We augment the Boolean satisfaction relation $\Vdash$ to all of $\widehat{\Sigma}$ as shown in Definition 8, and the formula satisfaction $\models$ as shown in Definition 9.

Note that $\top \models \mathit{false}$! and $\bot \not\models \mathit{true}$. By induction, we get the following:

PROPOSITION 5. *Let φ be an* ELTL *formula. Then $\top^\omega \models \varphi$ and $\bot^\omega \not\models \varphi$.*

The safety component of [Alpern and Schneider 1987] is obtained by taking the closure in the Cantor topology on the set of infinite words over the alphabet Σ, where the closed sets are the safety properties. As shown in Definition 10, our *weak component* is obtained by taking the closure in a topology on both finite and infinite words over the augmented alphabet $\widehat{\Sigma}$. Dually, the *strong component* is obtained by taking the interior in the same topology. From basic results in topology, we observe:

[1] Do not confuse $\overline{W}$ and W^c with the topological closure of W, denoted $cl(W)$.

Definition 11 Syntactically Weak and Strong ELTL Formulas

Let b be a Boolean expression.

- The set of *syntactically weak* ELTL formulas is recursively defined as follows:

$$\varphi ::= b \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \mathsf{X}\,\varphi \mid [\varphi\ \mathsf{W}\ \varphi]$$

- The set of *syntactically strong* ELTL formulas is recursively defined as follows:

$$\varphi ::= b! \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \mathsf{X}!\,\varphi \mid [\varphi\ \mathsf{U}\ \varphi]$$

Definition 12 Semantically Weak and Strong ELTL Formulas

Let φ be an ELTL formula.

- φ is *semantically weak* iff $weak(\varphi) = [\![\varphi]\!]$.
- φ is *semantically strong* iff $strong(\varphi) = [\![\varphi]\!]$.

OBSERVATION 6. *Let φ be an* ELTL *formula.*

$$\bullet\ strong(\varphi) \subseteq [\![\varphi]\!] \subseteq weak(\varphi)$$

To see that Definition 10 captures our intuition, we would like to state that the weak operators are the weak versions of their strong counterparts. The following proposition shows that for each formula in positive normal form, weakening/strengthening each of the operators results in a formula describing the weak/strong component, respectively.

PROPOSITION 7. *Let φ be an* ELTL *formula in positive normal form. Let φ_w be the formula obtained by weakening all operators (replacing $b!$ with b, $\mathsf{X}!$ with X and U with W). Let φ_s be the formula obtained by strengthening all operators. Then*

$$\bullet\ [\![\varphi_w]\!] = weak(\varphi) \qquad \bullet\ [\![\varphi_s]\!] = strong(\varphi)$$

For example, let φ, ψ be ELTL formulas in positive normal form. Then$[\varphi\ \mathsf{U}\ \psi]_w = [\varphi_w\ \mathsf{W}\ \psi_w]$, and so by Definition 10 and Proposition 7,

$$cl([\![[\varphi\ \mathsf{U}\ \psi]]\!]) = weak([\varphi\ \mathsf{U}\ \psi]) = [\![[\varphi_w\ \mathsf{W}\ \psi_w]]\!]$$

By working over the augmented alphabet $\widehat{\Sigma}$ we have obtained the desired correspondence between closure in the Cantor topology (our weak component) and weakening of the operators. As observed in the introduction, this correspondence breaks down over the original alphabet Σ using Alpern and Schneider's safety component. From another perspective, Proposition 7 shows that what is syntactically weak (strong) is also semantically weak (strong), where syntactically/semantically weak/strong are defined in Definitions 11 (similar to [Sistla 1994]) and 12. The converses do not hold. For example, $(\mathsf{G}p) \wedge (\mathsf{F}q \vee \mathsf{F}\neg q \vee \mathit{false})$ is semantically, but not syntactically, weak. And $(\mathsf{F}p) \vee (\mathsf{G}q \wedge \mathsf{G}\neg q \wedge \mathit{true}!)$ is semantically, but not syntactically, strong.

We say that word w is *finitely consistent with* φ if for every finite prefix $u \preceq w$, $u\top^\omega$ satisfies φ. For finite w, it follows that w is finitely consistent with φ iff $w\top^\omega \models \varphi$. For example, bc is finitely consistent with φ_7 of Table I, and both aaa and a^ω are finitely consistent with φ_{11}. Similarly we say that word w *finitely establishes* φ if w has a finite prefix u such that $u\bot^\omega$, satisfies φ. For example bcb and $bcbaa$ finitely establish φ_7, and aab finitely establishes φ_{11}.

In analogy with the notions of weak and strong satisfaction in [Eisner et al. 2003], the following proposition shows that the weak component of a formula φ consists of all words that are finitely consistent with φ, and the strong component consists of all words that finitely establish φ.

Definition 13 Informative Bad Prefix

Let $v \in \widehat{\Sigma}^*$ be a finite word and φ an ELTL formula. Then v is an informative bad prefix for φ iff $v\bot^\omega \models \neg\varphi$.

PROPOSITION 8. *Let φ be an* ELTL *formula.*

- $weak(\varphi) = \{w \in \widehat{\Sigma}^\infty \mid \forall \text{ finite } u \preceq w : u\top^\omega \in [\![\varphi]\!]\}$
- $strong(\varphi) = \{w \in \widehat{\Sigma}^\infty \mid \exists \text{ finite } u \preceq w : u\bot^\omega \in [\![\varphi]\!]\}$

As an additional check on our intuition, we have that if a word is a counterexample of a strong property but not its weak counterpart, then its entirety is needed – in other words no proper prefix of it is a finite counterexample of its weak counterpart. This is stated formally by the following proposition.

PROPOSITION 9. *Let φ be an* ELTL *formula. Then*

$$fpref((\widehat{\Sigma}^\infty \setminus strong(\varphi)) \setminus (\widehat{\Sigma}^\infty \setminus weak(\varphi))) \cap (\widehat{\Sigma}^\infty \setminus weak(\varphi)) = \emptyset$$

The following propositions, the proofs of which are topological, show how weakness and strength interplay with negation and complementation. Proposition 10 can be thought of as generalizing duality relations involving negation and strength, such as the equivalences $\mathsf{X}\neg\varphi \equiv \neg(\mathsf{X!}\ \varphi)$ and $\mathsf{F}\neg\varphi \equiv \neg(\mathsf{G}\varphi)$. Proposition 11 states that semantically weak and strong formulas are dual to each other.

PROPOSITION 10. *Let φ be an* ELTL *formula. Then*

- $weak(\neg\varphi) = (strong(\varphi))^c$
- $strong(\neg\varphi) = (weak(\varphi))^c$

PROPOSITION 11. *Let φ be an* ELTL *formula. Then φ is semantically strong iff $\neg\varphi$ is semantically weak.*

3.2. The Relation to Classification of Safety Formulas

It follows from Proposition 2 and Definition 10 that $weak(\varphi)$ is both prefix closed and limit closed and $strong(\varphi)$ is both extension closed and has finite witnesses. The finite witnesses quality of $strong(\varphi)$ can be understood in terms of the definitive prefix of [Eisner et al. 2003] and the informative bad prefixes of [Kupferman and Vardi 1999]. The definitive prefix of a word w with respect to a formula φ is the shortest finite prefix $u \preceq w$ such that either u and all its extensions satisfy φ or u and all its extensions fail to satisfy φ. Such a finite prefix need not exist.

The concept of an informative bad prefix is defined syntactically in [Kupferman and Vardi 1999]. In [Eisner et al. 2003] we gave a semantic definition, which, as shown in [Eisner et al. 2006], can be restated in terms of the alphabet $\widehat{\Sigma}$ as shown in Definition 13. It follows that the finite words in $strong(\neg\varphi)$ are exactly the informative bad prefixes for φ. It is shown in [Eisner et al. 2003] that the definitive prefix of w with respect to φ (if it exists) is the shortest informative bad prefix for either φ or $\neg\varphi$.

The relation to informative bad prefixes is interesting since it helps us show that semantically weak formulas are not only safety formulas, but are also of a good kind in the sense of admitting efficient model checking. Kupferman and Vardi [Kupferman and Vardi 1999] use the notion of informative bad prefixes as the base for a classification of safety properties into three distinct levels. A formula is *intentionally safe* if all of its bad prefixes are informative. For example, the formula $\mathsf{G}p$ is intentionally safe. A formula is *accidentally safe* if every computation that violates it has an informative bad prefix. For example, the formula $\mathsf{G}(p \vee (\mathsf{X}q \wedge \mathsf{X}\neg q))$ is accidentally safe. A formula is *pathologically safe* if there is a

Definition 14 Weak and Strong Satisfaction

Let $\varphi \in \text{LTL}^{sere}$.

- $w \models^{-} \varphi$ denotes $w\top^\omega \models \varphi$ and $[\![\varphi]\!]^-$ denotes the set $\{w \in \widehat{\Sigma}^\infty \mid w\top^\omega \models \varphi\}$
- $w \models^{+} \varphi$ denotes $w\bot^\omega \models \varphi$ and $[\![\varphi]\!]^+$ denotes the set $\{w \in \widehat{\Sigma}^\infty \mid w\bot^\omega \models \varphi\}$

computation that violates it and has no informative bad prefix. For example, the formula $(\mathsf{G}(p_1 \vee \mathsf{FG}q) \wedge \mathsf{G}(p_2 \vee \mathsf{FG}\neg q)) \vee \mathsf{G}p_1 \vee \mathsf{G}p_2$ is pathologically safe.

It is shown in [Kupferman and Vardi 1999] that for non-pathologically safe formulas (the set of formulas that are intentionally or accidentally safe), it is sufficient to limit the search for bad prefixes to informative bad prefixes, and searching for informative bad prefixes is exponentially easier than searching for any bad prefix ($2^{O(n)}$ for only informative bad prefixes and $2^{2^{O(n)}}$ for all bad prefixes). Furthermore, [Kupferman and Vardi 1999] show that a syntactically safe formula is non-pathologically safe. The proposition below strengthens this result by showing that the same and the converse hold for semantically weak formulas.

PROPOSITION 12. *Let φ be an* ELTL *formula.*

- *φ is semantically weak iff φ is non-pathologically safe.*
- *φ is semantically strong iff $\neg\varphi$ is non-pathologically safe.*

Thus the notion of weakness captures semantically exactly the set of "good" safety properties − that is, those that are computationally easy to verify.

Looking for informative bad prefixes of safety formula φ is easier or as hard as model checking φ. Based on that, and on the assumption that no one intentionally writes a pathologically safe formula, [Kupferman and Vardi 1999] propose the following methodology for model checking safety formulas: First search for informative bad prefixes, and only if none are found, check whether the formula φ is pathologically safe (a decision which is PSPACE-complete [Kupferman and Vardi 1999]). If it is not, then return that φ holds, otherwise notify the user that φ is pathologically safe. This ignores the fact that in order to use the method, we must first determine whether φ is safety, a decision that is also PSPACE-complete [Sistla 1994]. Using the notion of weakness, we can generalize [Kupferman and Vardi 1999]'s methodology for any formula: First search for informative bad prefixes, and only if none are found check whether the formula φ is safety. If it is, it must be pathologically safe (otherwise we would have found an informative bad prefix), so notify the user that φ is pathologically safe, otherwise model check φ.

Automating our generalization depends on a method of searching for informative bad prefixes of φ, even if φ is not a safety formula. The following proposition gives us a method to do that by checking the formula φ' obtained from φ by converting to positive normal form and weakening all operators.

PROPOSITION 13. *Let φ be an* ELTL *formula, and let $M \subseteq \Sigma^\infty$. Searching for words in M that have informative bad prefixes for φ is equivalent to searching for words in M that are not in weak(φ).*

3.3. The Relation to Weak and Strong Satisfaction of PSL and SVA

PSL and SVA define *weak* and *strong satisfaction* ($\models^{-}$ and $\models^{+}$, respectively – see also [Eisner et al. 2003]) and as shown in [Eisner et al. 2006], those definitions are equivalent to that of Definition 14. The sets of words in $[\![\varphi]\!]^-$ and $cl([\![\varphi]\!])$ are closely related (and indeed the present work was inspired in large part by [Eisner et al. 2003]), but they are not necessarily the same. For example, let $\varphi = [p \mathsf{U} q]$, and let $w = \langle p \rangle^\omega$. Then $w \in cl([\![\varphi]\!])$ because for

every finite prefix $u \preceq w$, $uq \models \varphi$ (also $u\top \models \varphi$). However, $w\top^\omega = w$, which does not satisfy φ. Therefore $w \notin [\![\varphi]\!]^-$. Similarly, $\langle p \rangle^\omega \in [\![\neg\varphi]\!]^+$, but $\langle p \rangle^\omega \notin int([\![\neg\varphi]\!])$.

Note that if φ is semantically weak, then $[\![\varphi]\!] = cl([\![\varphi]\!])$ and thus by Theorem 14 we have that $[\![\varphi]\!] = [\![\varphi]\!]^- = cl([\![\varphi]\!])$. Similary, if φ is semantically strong, then $[\![\varphi]\!] = [\![\varphi]\!]^+ = int([\![\varphi]\!])$. The following propositions state the relationship between $[\![\varphi]\!]$, $[\![\varphi]\!]^-$, and $cl([\![\varphi]\!])$ and between $[\![\varphi]\!]$, $[\![\varphi]\!]^+$, and $int([\![\varphi]\!])$ in the general case.

PROPOSITION 14. *Let $\varphi \in$ ELTL. Then*

1. $[\![\varphi]\!] \subseteq [\![\varphi]\!]^- \subseteq cl([\![\varphi]\!])$
2. $[\![\varphi]\!] \supseteq [\![\varphi]\!]^+ \supseteq int([\![\varphi]\!])$

PROPOSITION 15. *Let φ in* ELTL*. The sets of finite words in $[\![\varphi]\!]^-$ and $cl([\![\varphi]\!])$ are the same, i.e.,*

$$[\![\varphi]\!]^- \cap \widehat{\Sigma}^* = cl([\![\varphi]\!]) \cap \widehat{\Sigma}^*$$

By Proposition 15, the difference between the set of words weakly (strongly) satisfied by φ and the weak (strong) component of φ contains only infinite words. This stems from the differing motivations of [Eisner et al. 2003] and the present paper. In [Eisner et al. 2003], the motivation was to examine the use of temporal logic in incomplete verification methods. In such methods (e.g., simulation or bounded model checking), we are presented with a *truncated word*, a finite word that is not necessarily maximal. The purpose of weak (strong) satisfaction is to characterize what can be known about the validity of φ on the maximal word from the validity of φ on a prefix of it. In the present paper our motivation is different. We have the maximal word, but we would like to use a computationally less expensive verification method on it. Thus we want to understand the relation between weak and strong operators in order to characterize what can be known about the validity of φ on a maximal word from the validity of $weak(\varphi)$ on it.

4. AN APPLICATION

Our motivation is to provide a characterization of weakness that allows us to check whether a proposed weak version of some new temporal operator is related to its strong version in a way that maintains the internal consistency of the logic. In this section, we use the characterization to show that the original versions of the IEEE standard temporal logics PSL [Eisner and Fisman 2006; IEEE c] and SVA [Cerny et al. 2010; IEEE a; b] are broken in the sense that they have temporal operators that are expected to form a weak/strong pair, but that are not related semantically in the same way as W/U. We show that the source of the problem lies in the semantics of the SERE intersection and fusion operators. We then use the topological characterization to show that the alternative semantics proposed in [Eisner and Fisman 2008] and adopted by the latest version of the PSL standard [IEEE d] restores the internal consistency of the logic in the sense that all the expected weak/strong pairs are related in the same way.

4.1. The Logic LTL^{sere}

The logic LTL^{sere} lies at the core of the standard temporal logics PSL and SVA. It extends ELTL with semi-extended regular expressions (SEREs). SEREs in turn extend regular expressions with intersection and fusion (a kind of overlapping concatenation). SEREs and LTL^{sere} formulas are defined inductively as shown in Definitions 15 and 16.

Note that every Boolean expression is a SERE, thus the weak and strong Booleans of ELTL and of [IEEE c] are included by means of r and $r!$, which form the base of the inductive definition. As previously, we also make use of $\mathsf{F}\ \varphi$ and $\mathsf{G}\ \varphi$, defined to be $[\mathit{true}\ \mathsf{U}\ \varphi]$ and $[\varphi\ \mathsf{W}\ \mathit{false}]$, respectively.

Definition 15 Semi-Extended Regular Expressions (SEREs)

Let $b \in B$ be a Boolean expression. The set of SEREs is recursively defined as follows:

$$r ::= \lambda \mid b \mid r \cdot r \mid r \circ r \mid r^+ \mid r \cup r \mid r \cap r$$

Definition 16 LTL^{sere} Formulas

Let r be a SERE. The set of LTL^{sere} *formulas* is recursively defined as follows:

$$\varphi ::= r! \mid r \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \mathsf{X}!\, \varphi \mid \mathsf{X}\, \varphi \mid [\varphi\ \mathsf{U}\ \varphi] \mid [\varphi\ \mathsf{W}\ \varphi]$$

Definition 17 The Language of SERE r

$$\mathcal{L}(\lambda) = \{\epsilon\} \qquad \mathcal{L}(r_1 \cdot r_2) = \mathcal{L}(r_1)\mathcal{L}(r_2) \qquad \mathcal{L}(r_1 \cup r_2) = \mathcal{L}(r_1) \cup \mathcal{L}(r_2)$$
$$\mathcal{L}(b) = \{\ell \in \widehat{\Sigma} \mid \ell \Vdash b\} \qquad \mathcal{L}(r_1 \circ r_2) = \mathcal{L}(r_1) \circ \mathcal{L}(r_2) \qquad \mathcal{L}(r_1 \cap r_2) = \mathcal{L}(r_1) \cap \mathcal{L}(r_2)$$
$$\mathcal{L}(r^+) = \mathcal{L}(r)^+$$

4.2. The $\top,\bot$ Approach to the Semantics of LTL^{sere}

Because of their use of the special letters $\top$ and $\bot$, we term the semantics as in the original versions of the PSL [Eisner and Fisman 2006; IEEE c] and SVA [Cerny et al. 2010; IEEE a; b] standards *the* $\top,\bot$ *approach to the semantics of* LTL^{sere} (usually we say simply *the* $\top,\bot$ *approach*). The $\top,\bot$ approach is defined inductively, using as the base case the semantics of Boolean expressions over letters in $\widehat{\Sigma}$. We augment the Boolean relation $\Vdash$ as before, so that $\top \Vdash b$ and $\bot \nVdash b$ for any Boolean expression b.

The language of SERE r (the words on which r *holds tightly* [IEEE c], or is *tightly satisfied* [IEEE a; b]), denoted $\mathcal{L}(r)$, is defined formally as shown in Definition 17.

We use $w \models_{\top\bot} \varphi$ to denote that w models φ under the $\top,\bot$ approach. The semantics is given in Definition 18. The semantics of the ELTL operators are exactly that used in Section 3. The semantics of $r!$ is straightforward: $r!$ holds on a word w if there exists a non-empty finite prefix of w that is in $\mathcal{L}(r)$. For example, $(p \cdot q^+ \cdot p)!$ holds on the word $u = \langle p\rangle\langle q\rangle\langle p\rangle\bot$.

Intuitively, r is intended to be a weak version of $r!$. Thus, if we do not find a prefix of w that is in $\mathcal{L}(r)$, then at least we never reach the point where things have gone so wrong that appending some number of $\top$'s won't get us there. For example, $(p \cdot q^+ \cdot p)$ holds on the word $v_1 = \langle p\rangle\langle q\rangle\langle p\rangle\bot$, and also on the "too short" word $v_2 = \langle p\rangle\langle q\rangle$ and on the "too long" word $v_3 = \langle p\rangle\langle q\rangle^\omega$, in which we "get stuck" in the starred sub-expression q. In a similar manner, the semantics of $[\varphi\ \mathsf{W}\ \psi]$ can be understood intuitively (and only intuitively, see below) as $w \models_{\top\bot} [\varphi\ \mathsf{W}\ \psi] \iff \forall j < |w|, w^{0..j}\top^\omega \models_{\top\bot} [\varphi\ \mathsf{U}\ \psi]$.

The intuitive explanation is not accurate, however. While it is what we are striving for, and holds for the semantics of ELTL given in Definition 2, it does not hold for the $\top,\bot$ approach due to the issues to be discussed in the next section.

4.3. Structural Contradictions

In [Eisner and Fisman 2008] we argued that the $\top,\bot$ approach to the semantics of LTL^{sere} is "broken" because of the presence of *structural contradictions*, SEREs that are unsatisfiable (i.e., whose language is empty) due to their structure. That is, for any replacement of the propositions in the SERE, the language of the SERE remains empty. For example, $p \cap (p \cdot q)$ is a structural contradiction, while $(p \cdot q) \cap (p \cdot \neg q)$ is a contradiction, but not a structural one. The $\cap$ operator is not the only source of structural contradictions in the $\top,\bot$ approach. A structural contradiction can also be formed from the fusion operator $\circ$. The SERE $\lambda \circ p$ is a structural contradiction formed without the $\cap$ operator.

Our argument in [Eisner and Fisman 2008] was intuitive, and we did not provide topological evidence. Using a topological characterization, we can make a stronger case, as follows:

Definition 18 The $\top,\bot$ Approach to the Semantics of LTL^{sere}

Let $w \in \widehat{\Sigma}^\infty$ be a word.

- $w \models_{\top\bot} r! \iff \exists j < |w|$ s.t. $w^{0..j} \in \mathcal{L}(r)$
  $w \models_{\top\bot} r \iff \forall j < |w|, w^{0..j}\top^\omega \models_{\top\bot} r!$
- $w \models_{\top\bot} \neg\varphi \iff \overline{w} \not\models_{\top\bot} \varphi$
  $w \models_{\top\bot} \varphi \wedge \psi \iff w \models_{\top\bot} \varphi$ and $w \models_{\top\bot} \psi$
  $w \models_{\top\bot} \varphi \vee \psi \iff w \models_{\top\bot} \varphi$ or $w \models_{\top\bot} \psi$
- $w \models_{\top\bot} \mathsf{X!}\ \varphi \iff |w| > 1$ and $w^{1..} \models_{\top\bot} \varphi$
  $w \models_{\top\bot} \mathsf{X}\ \varphi \iff$ either $w \models_{\top\bot} \mathsf{X!}\ \varphi$ or $|w| \leq 1$
- $w \models_{\top\bot} [\varphi\ \mathsf{U}\ \psi] \iff \exists k < |w|$ such that $w^{k..} \models_{\top\bot} \psi$ and $\forall j < k,\ w^{j..} \models_{\top\bot} \varphi$
  $w \models_{\top\bot} [\varphi\ \mathsf{W}\ \psi] \iff$ either $w \models_{\top\bot} [\varphi\ \mathsf{U}\ \psi]$ or $\forall j < |w|,\ w^{j..} \models_{\top\bot} \varphi$

Although r is intended to be a weak verison of $r!$, our topological characterization shows that it is not. The proposition for $\models_{\top\bot}$ analogous to Proposition 5 fails for the $\top,\bot$ approach. For example, the formula $[p\ \mathsf{U}\ (\{q\}\cap\{q\cdot q\})]$ does not hold on $w = \top^\omega$. As a result, the proposition analogous to Proposition 7 fails, and we get, for instance, that the weak component of $[p\ \mathsf{U}\ (\{q\}\cap\{q\cdot q\})]$ is the empty set rather than $[\![[p\ \mathsf{W}\ (\{q\}\cap\{q\cdot q\})]]\!]$.

In a preliminary version of this paper [Eisner et al. 2005], we conjectured that unsatisfiability of structural contradictions could be fixed by defining $\mathcal{L}(b)$ to be $\{\ell \in \widehat{\Sigma} \mid \ell \models b\} \cup \top^+$. The problem with this approach is that it leaves the formula r formed from the structural contradiction $r = \lambda \circ p$ unsatisfiable.

In a similar approach, called the *flexible letter approach* in [Eisner and Fisman 2008], we define that $\mathcal{L}(\lambda) = \top^*$ and that $\mathcal{L}(b) = \{\ell \in \widehat{\Sigma} \mid \ell \models b\}\top^*$. The problem with the flexible letter approach is that it changes the semantics of formulas that do not contain structural contradictions. Let $r_1 = (p \cdot p \cdot p)$, let $r_2 = (q \cdot q \cdot q)$, and let $\varphi = r_1 \circ r_2$. Let $w = \langle p\rangle\langle p\rangle\langle p\rangle$. Then under the $\top,\bot$ approach, using the original semantics of $\mathcal{L}(r)$ as in Definition 17, we have that φ does not hold on w. To see this, note that $|w| = 3$. Thus $w \models_{\top\bot} \varphi$ iff $\forall j < 3$ we have that $w^{0..j}\top^\omega \models_{\top\bot} (r_1 \circ r_2)!$. Taking $j = 2$ we get that $w^{0..2}\top^\omega \not\models_{\top\bot} (r_1 \circ r_2)!$, because that requires one letter overlap between $(p \cdot p \cdot p)$ and $(q \cdot q \cdot q)$, but we do not have that q holds on the third letter.

Under the flexible letter approach, using its modified definition of $\mathcal{L}(b)$, we can use more than three letters to form a word in the language of $(p \cdot p \cdot p)$. So take the first four letters of $w^{0..2}\top^\omega$ to be in $\mathcal{L}(p \cdot p \cdot p)$, and then, starting from the last letter of those four (which is $\top$), take some number of letters to be in $\mathcal{L}(q \cdot q \cdot q)$. We can take three $\top$s for that. Then the overlap happens on a $\top$, and thus under the flexible letter approach we get that φ holds on w.

In [Eisner and Fisman 2008] we proposed instead an alternative semantics based on the natural alphabet $\Sigma = 2^P$, thus relegating the special letters $\top$ and $\bot$ back to a supporting role in the topological characterization. Following [Eisner and Fisman 2008], we term these the *truncated semantics of* LTL^{sere}. In the next section, we present the truncated semantics of LTL^{sere}, adopted by the latest version of PSL [IEEE d], and use our topological characterization to show that in it, r is a weak version of $r!$ and that the semantics behaves as intuitively expected with respect to the topological characterization.

Definition 19 $\mathcal{F}(r)$ and $\mathcal{I}(r)$

The language of finite proper prefixes of SERE r, denoted $\mathcal{F}(r)$, and the loop language of SERE r, denoted $\mathcal{I}(r)$, are defined as follows:

- $\mathcal{F}(\lambda) = \emptyset$
  $\mathcal{F}(b) = \epsilon$
- $\mathcal{F}(r_1 \cdot r_2) = \mathcal{F}(r_1) \cup (\mathcal{L}(r_1)\mathcal{F}(r_2))$
  $\mathcal{F}(r_1 \circ r_2) = \mathcal{F}(r_1) \cup (\mathcal{L}(r_1) \circ \mathcal{F}(r_2))$
  $\mathcal{F}(r^+) = \mathcal{L}(r)^*\mathcal{F}(r)$
- $\mathcal{F}(r_1 \cap r_2) = \mathcal{F}(r_1) \cap \mathcal{F}(r_2)$
  $\mathcal{F}(r_1 \cup r_2) = \mathcal{F}(r_1) \cup \mathcal{F}(r_2)$

- $\mathcal{I}(\lambda) = \emptyset$
  $\mathcal{I}(b) = \emptyset$
- $\mathcal{I}(r_1 \cdot r_2) = \mathcal{I}(r_1) \cup (\mathcal{L}(r_1)\mathcal{I}(r_2))$
  $\mathcal{I}(r_1 \circ r_2) = \mathcal{I}(r_1) \cup (\mathcal{L}(r_1) \circ \mathcal{I}(r_2))$
  $\mathcal{I}(r^+) = (\mathcal{L}(r)^*\mathcal{I}(r)) \cup (\mathcal{L}(r) \setminus \{\epsilon\})^\omega$
- $\mathcal{I}(r_1 \cap r_2) = \mathcal{I}(r_1) \cap \mathcal{I}(r_2)$
  $\mathcal{I}(r_1 \cup r_2) = \mathcal{I}(r_1) \cup \mathcal{I}(r_2)$

Definition 20 Truncated Semantics of LTL^{sere}

Let $w \in \Sigma^\infty$ be a word.

- $w \models r!$ $\iff$ $\exists j < |w|$ s.t. $w^{0..j} \in \mathcal{L}(r)$
  $w \models r$ $\iff$ either $w \models r!$ or $w \in \mathcal{I}(r) \cup \mathcal{F}(r) \cup \{\epsilon\}$
- $v \models \neg\varphi$ $\iff$ $v \not\models \varphi$
  $v \models \varphi \vee \psi$ $\iff$ $v \models \varphi$ or $v \models \psi$
  $v \models \varphi \wedge \psi$ $\iff$ $v \models \varphi$ and $v \models \psi$
- $v \models \mathsf{X!}\ \varphi$ $\iff$ $|v| > 1$ and $v^{1\cdot\cdot} \models \varphi$
  $v \models \mathsf{X}\ \varphi$ $\iff$ either $v \models \mathsf{X!}\ \varphi$ or $|v| \leq 1$
- $v \models [\varphi\ \mathsf{U}\ \psi]$ $\iff$ $\exists k < |v| : v^{k\cdot\cdot} \models \psi$ and $\forall j < k : v^{j\cdot\cdot} \models \varphi$
  $v \models [\varphi\ \mathsf{W}\ \psi]$ $\iff$ either $v \models [\varphi\ \mathsf{U}\ \psi]$ or $\forall j < |v| : v^{j\cdot\cdot} \models \varphi$

4.4. The Truncated Semantics of LTL^{sere}

In addition to the language of a SERE, $\mathcal{L}(r)$ (Definition 17), the truncated semantics of LTL^{sere} makes use of two additional languages. Intuitively, the language of proper prefixes of a SERE consists of *finite* proper prefixes of words in $\mathcal{L}(r)$, except that logical and structural contradictions are considered satisfiable. The loop language of a SERE extends the *loop*$(\cdot)$ of [Harel and Sherman 1982] to the intersection and fusion operators. Intuitively, it consists of *infinite* words in which we get stuck forever in a starred sub-expression of r. Formally, these languages are given in Defintion 19.[2]

The truncated semantics of LTL^{sere} is shown in Definition 20. It uses the language $\mathcal{L}(r)$ to define the semantics of $r!$, and uses the languages $\mathcal{F}(r)$ and $\mathcal{I}(r)$ to define the semantics of r. Other than that, it is identical to the semantics of ELTL given in Definition 2.

Note that the truncated semantics of LTL^{sere} is defined over Σ, while $\mathcal{L}(r)$ is defined over $\widehat{\Sigma}$. We will make use of the extra words in $\mathcal{L}(r)$, those not relevant for Definition 20, later, in the topological characterization. Furthermore, do not confuse $\mathcal{L}(r) \cap \Sigma^\infty$ with $[\![r!]\!]$ or $[\![r]\!]$. From Definition 20,

$$[\![r!]\!] = ((\mathcal{L}(r) \cap \Sigma^\infty) \setminus \{\epsilon\})\Sigma^\infty$$

$$[\![r]\!] = [\![r!]\!] \cup (\mathcal{I}(r) \cap \Sigma^\infty) \cup (\mathcal{F}(r) \cap \Sigma^\infty) \cup \{\epsilon\}$$

[2] In [IEEE d], the loop language of r^+ is defined as $\mathcal{I}(r^+) = (\mathcal{L}(r)^*\mathcal{I}(r)) \cup \mathcal{L}(r)^\omega$. If $\epsilon \in \mathcal{L}(r)$, then this definition admits the empty word to $\mathcal{I}(r)$ and thus admits finite words through the recursive definition. This was not the intent and is corrected here.

Definition 21 $\widehat{\mathcal{L}}(r)$

$\widehat{\mathcal{L}}(r) = \mathcal{L}(r) \cup (\mathcal{F}(r)\top^+) \cup \top^+$

Definition 22 $\mathcal{L}_w(r)$

$\mathcal{L}_w(r) = \mathcal{L}(r) \cup \mathcal{I}(r) \cup (\mathcal{F}(r)\top^\infty) \cup \top^\infty$

Definition 23 Truncated Semantics of LTL^{sere} over the Augmented Alphabet

Let $w \in \widehat{\Sigma}^\infty$ be a word. The semantics of LTL^{sere} over the augmented alphabet is exactly as in Definition 20, except that the semantics of $r!$ and $\neg\varphi$ are as follows:

- $w \models r! \iff \exists j < |w|$ s.t. $w^{0..j} \in \widehat{\mathcal{L}}(r)$
- $w \models \neg\varphi \iff \overline{w} \not\models \varphi$

4.5. A Topological Characterization of the Proposed Semantics

As previously, our topological characterization will be based on the augmented alphabet $\widehat{\Sigma}$, and we augment the Boolean relation $\Vdash$ as before, so that $\top \Vdash b$ and $\bot \not\Vdash b$ for any Boolean expression b.

We must extend the semantics of LTL^{sere} to the alphabet $\widehat{\Sigma}$ in such a way that (1) the semantics of Definition 20 is preserved on the original alphabet Σ and (2) the characterization of Section 3 holds for LTL^{sere} using the semantics on the augmented alphabet. The heart of the topological characterization of weakness and strength in ELTL is Proposition 7, which states that the denotation of the syntactic weakening of a formula in positive normal form is the topological closure of the denotation of the formula, and dually that the denotation of the syntactic strengthening of a formula in positive normal form is the topological interior of the denotation of the formula. The key to establishing this relationship for LTL^{sere} is proving it for the base case of SERE formulas, namely that

$$\bullet\ [\![r]\!] = cl([\![r!]\!]) \qquad \bullet\ [\![r!]\!] = int([\![r]\!]) \tag{1}$$

These equalities are therefore primary goals of the extension of the semantics of LTL^{sere} to $\widehat{\Sigma}$.

We accomplish the extension and proof of the topological characterization using two auxiliary languages, $\widehat{\mathcal{L}}(\cdot)$ and $\mathcal{L}_w(\cdot)$, shown in Definitions 21 and 22. $\widehat{\mathcal{L}}(\cdot)$ replaces $\mathcal{L}(\cdot)$ in defining the semantics of LTL^{sere} over the alphabet $\widehat{\Sigma}$, shown in Definition 23, while $\mathcal{L}_w(\cdot)$ is used in the proof of the topological characterization.

It is easy to see that Definition 23 preserves the semantics of Definition 20 when restricted to the original alphabet Σ. $\widehat{\mathcal{L}}(r)$ is obtained from $\mathcal{L}(r)$ by adding $(\mathcal{F}(r)\top^+) \cup \top^+$, which consists of non-empty finite words that either are proper prefixes of words in $\mathcal{L}(r)$ followed by at least one letter $\top$ or are made up entirely of letters $\top$. These additions are to eliminate structural contradictions and ensure that all LTL^{sere} formulas are satisfiable over $\widehat{\Sigma}$, which is key to proving the Equalities (1). This fact is established by the following generalization of Proposition 5, the proof of which is by induction.

THEOREM 16. *Let φ be an* LTL^{sere} *formula. Then $\top^\omega \models \varphi$ and $\bot^\omega \not\models \varphi$.*

For the topological characterization we extend the definitions of weak and strong components and syntactically and semantically weak and strong formulas in the obvious ways, shown in Definitions 24, 25 and 26. Proof of the topological relationship of SERE formulas as expressed in the Equalities (1) proceeds from analysis of the relationships of the auxiliary

Definition 24 Weak and Strong Components of LTL^{sere} Formulas

Let φ be an LTL^{sere} formula. Its weak and strong components, denoted $weak(\varphi)$ and $strong(\varphi)$ respectively, are defined as follows.

- $weak(\varphi) = cl([\![\varphi]\!])$
- $strong(\varphi) = int([\![\varphi]\!])$

Definition 25 Syntactically Weak and Strong LTL^{sere} Formulas

Let r be a SERE.

- The set of *syntactically weak* LTL^{sere} formulas is recursively defined as follows:

$$\varphi ::= r \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \mathsf{X}\ \varphi \mid [\varphi\ \mathsf{W}\ \varphi]$$

- The set of *syntactically strong* LTL^{sere} formulas is recursively defined as follows:

$$\varphi ::= r! \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \mathsf{X!}\ \varphi \mid [\varphi\ \mathsf{U}\ \varphi]$$

Definition 26 Semantically Weak and Strong LTL^{sere} Formulas

Let φ be an LTL^{sere} formula.

- φ is *semantically weak* iff $weak(\varphi) = [\![\varphi]\!]$
- φ is *semantically strong* iff $strong(\varphi) = [\![\varphi]\!]$

languages $\widehat{\mathcal{L}}(r)$ and $\mathcal{L}_w(r)$. We first show that $\mathcal{L}_w(r)$ is prefix closed and that if a word $w \in \widehat{\Sigma}^\infty$ has infinitely many prefixes in $\mathcal{F}(r)$ then it is in $\mathcal{I}(r)$, from which it is proved that $\mathcal{L}_w(r)$ is limit closed.

THEOREM 17. *$\mathcal{L}_w(r)$ is prefix closed.*

THEOREM 18. *Let $w \in \widehat{\Sigma}^\omega$ and assume that w has infinitely many prefixes in $\mathcal{F}(r)$. Then $w \in \mathcal{I}(r)$.*

THEOREM 19. *$\mathcal{L}_w(r)$ is limit closed.*

These results are used to prove the following:

THEOREM 20 (RECURSIVE CHARACTERIZATION OF $cl(\widehat{\mathcal{L}}(r))$). *Let r be a* SERE. *Then $cl(\widehat{\mathcal{L}}(r)) = \mathcal{L}_w(r)$.*

From Definition 20 (and 23, which points to it), satisfaction of the weak SERE formula r is defined in terms of satisfaction of the strong SERE formula $r!$ and the language $\mathcal{I}(r) \cup \mathcal{F}(r) \cup \{\epsilon\}$. Straightforward manipulation shows that:

THEOREM 21. *$w \models r$ iff $w \models r!$ or $w \in \mathcal{L}_w(r)$.*

Using Proposition 3, Theorems 20, 21, and some properties resulting from the fact that $\widehat{\mathcal{L}}(r)$ is closed under switching letters to $\top$, we get that:

THEOREM 22. *Let r be a* SERE *and $w \in \widehat{\Sigma}^\infty$. The following are equivalent:*

(1) For every finite $u \preceq w : u\top^\omega \models r!$.
(2) $w \models r$.

Theorems 16 and 22 and induction yield the Strength Relation Theorem for LTL^{sere}, so named for the analogous theorem for LTL appearing in [Eisner et al. 2003]; the term *strength* refers to [Eisner et al. 2003]'s *weak* and *strong satisfaction*, discussed in Section 3.3.

THEOREM 23 (STRENGTH RELATION). *Let $f \in \text{LTL}^{sere}$ and let $w \in \widehat{\Sigma}^\infty$. Then*

1. $w \models f \implies w\top^\omega \models f$
2. $w\bot^\omega \models f \implies w \models f$

Using this result, we obtain a representation of closure and interior of the denotation of an LTL^{sere} formula in terms of $\top$ and $\bot$:

THEOREM 24 ($\top, \bot$ CHARACTERIZATION OF $cl([\![f]\!])$, $int([\![f]\!])$). *Let $f \in \text{LTL}^{sere}$. Then*

1. $cl([\![f]\!]) = \{w \in \widehat{\Sigma}^\infty \mid \forall \text{ finite } u \preceq w : u\top^\omega \in [\![f]\!]\}$
2. $int([\![f]\!]) = \{w \in \widehat{\Sigma}^\infty \mid \exists \text{ finite } u \preceq w : u\bot^\omega \in [\![f]\!]\}$

From these pieces the Equalities (1) then follow as part of:

THEOREM 25 (TOPOLOGICAL RELATIONSHIP OF SERE FORMULAS). *Let r be a* SERE. *Then:*

- $[\![r]\!] = cl([\![r]\!]) = cl([\![r!]\!])$
- $[\![r!]\!] = int([\![r]\!]) = int([\![r!]\!])$

In particular, r is semantically weak and $r!$ is semantically strong.

With the topological relationships of SERE formulas established, proofs of the remainder of the results of Section 3 carry over in a straightforward way from those for ELTL. In particular, the generalization of Proposition 7 provides the main result of the topological characterization for LTL^{sere}:

THEOREM 26. *Let φ be an LTL^{sere} formula in positive normal form. Let φ_w be the formula obtained by weakening all operators (replacing* $\mathsf{X!}$ *with* X, U *with* W, *and $r!$ with r.). Let φ_s be the formula obtained by strengthening all operators. Then*

- $[\![\varphi_w]\!] = weak(\varphi)$
- $[\![\varphi_s]\!] = strong(\varphi)$

We state the generalizations to LTL^{sere} of the remaining results of Section 3 below.

THEOREM 27. *Let φ be an LTL^{sere} formula.*

- $weak(\varphi) = \{w \in \widehat{\Sigma}^\infty \mid \forall \text{ finite } u \preceq w : u\top^\omega \in [\![\varphi]\!]\}$
- $strong(\varphi) = \{w \in \widehat{\Sigma}^\infty \mid \exists \text{ finite } u \preceq w : u\bot^\omega \in [\![\varphi]\!]\}$

THEOREM 28. *Let φ be an LTL^{sere} formula. Then*

$$fpref((\widehat{\Sigma}^\infty \setminus strong(\varphi)) \setminus (\widehat{\Sigma}^\infty \setminus weak(\varphi))) \cap (\widehat{\Sigma}^\infty \setminus weak(\varphi)) = \emptyset$$

THEOREM 29. *Let φ be an LTL^{sere} formula. Then*

- $weak(\neg\varphi) = (strong(\varphi))^c$
- $strong(\neg\varphi) = (weak(\varphi))^c$

THEOREM 30. *Let φ be an LTL^{sere} formula. Then φ is semantically strong iff $\neg\varphi$ is semantically weak.*

THEOREM 31. *Let φ be an LTL^{sere} formula.*

- *φ is semantically weak iff φ is non-pathologically safe.*
- *φ is semantically strong iff $\neg\varphi$ is non-pathologically safe.*

THEOREM 32. *Let φ be an LTL^{sere} formula, and let $M \subseteq \Sigma^\infty$. Searching for words in M that have informative bad prefixes for φ is equivalent to searching for words in M that are not in $weak(\varphi)$.*

THEOREM 33. *Let* $\varphi \in \text{LTL}^{sere}$. *Then*

1. $[\![\varphi]\!] \subseteq [\![\varphi]\!]^- \subseteq cl([\![\varphi]\!])$
2. $[\![\varphi]\!] \supseteq [\![\varphi]\!]^+ \supseteq int([\![\varphi]\!])$

THEOREM 34. *Let* φ *in* LTL^{sere}. *The sets of finite words in* $[\![\varphi]\!]^-$ *and* $cl([\![\varphi]\!])$ *are the same, i.e.,*

$$[\![\varphi]\!]^- \cap \widehat{\Sigma}^* = cl([\![\varphi]\!]) \cap \widehat{\Sigma}^*$$

5. DISCUSSION

We have achieved a topological characterization of weakness (strength) by moving to an extended domain that adds solutions that do not exist in the original domain, then taking closure (interior) in the extended domain. Taking closure in the extended domain can add not only solutions from the extended domain, but also solutions from the original domain. For example, the closure of $[p \mathsf{U} \mathit{false}]$ in the original domain is the empty set, but in the extended domain it includes words from the original domain, such as $\langle p \rangle \langle p \rangle \langle p \rangle$ and $\langle p \rangle^\omega$.

A similar effect happens in the move from the real numbers $\mathbb{R}$ to the complex numbers $\mathbb{C}$, and the behavior of our special letters $\top$ and $\bot$ is analogous to the behavior of the imaginary number i. Intuitively, $\leq$ is a weak form of $<$. But over $\mathbb{R}$, the topological characterization using closure does not work, because $cl(\{x \in \mathbb{R} \mid x^2 < 0\}) = \emptyset$, which is not the same as $\{x \in \mathbb{R} \mid x^2 \leq 0\} = \{0\}$. As in our observation that the "problem" with Alpern and Schneider's safety component is that the property described by $[p \mathsf{U} \mathit{false}]$ is the empty one, the "problem" here is that $x^2 < 0$ has no solution for $x \in \mathbb{R}$. Moving to the extended domain $\mathbb{C}$, we get that $cl(\{x \in \mathbb{C} \mid x^2 < 0\}) = i \cdot \mathbb{R} = \{x \in \mathbb{C} \mid x^2 \leq 0\}$. Taking closure in the extended domain adds the solution 0, which belongs to the original domain $\mathbb{R}$.

Many techniques for the analysis of programs and specifications rely on checking language emptiness, while our topological characterizaton uses an augmented alphabet over which any LTL^{sere} formula is satisfiable. Our approach does not preclude those techniques. The augmented alphabet is needed only to provide a suitable domain in which topological closure achieves the full set of intuitively desired models for the weak component, including models over the original alphabet. Analyses can proceed as usual, restricted to the original alphabet, where the satisfiability question remains non-trivial. In particular, as long as the models being analyzed do not use the special letters $\top$ and $\bot$, existing algorithms stand.

The usual definitions of safety and liveness (e.g., [Alpern and Schneider 1985; 1987; Harel and Sherman 1982]) typically use an alphabet such that some LTL formulas describe the empty property, and those are the definitions that we have used here. Alternatively, it is possible to define safety and liveness with respect to an arbitrary alphabet. In that case, whether or not a formula is safety or liveness depends on the alphabet under which it is being examined. For example, the formula $[p \mathsf{U} \mathit{false}]$ is a safety formula for the alphabet Σ, but is not a safety formula for the alphabet $\widehat{\Sigma}$. Similarly, the formula $\mathsf{F}\, \mathit{false}$ is a liveness formula for the alphabet $\widehat{\Sigma}$, but not for Σ.

If we define safety and liveness with respect to an arbitrary alphabet, then our weak component is exactly the safety component over the extended alphabet $\widehat{\Sigma}$, but our strong component is not Alpern and Schneider's liveness component in that alphabet. The strong component of φ consists of all words that finitely establish φ, whereas the liveness component is formed from the union of the original property with the complement of its safety component.

6. RELATED WORK

The *finite acceptance* of [Wolper et al. 1983; Vardi and Wolper 1994] is analogous to our strong component. However, the analogy between their *looping acceptance* and our weak

component is imperfect. At the level of automata, both Alpern and Schneider's safety component and our weak component can be achieved by considering all states of a Büchi automaton to be accepting. Looping acceptance does exactly this, and [Vardi and Wolper 1994] observes that an automaton with looping acceptance defines a safety property. However, the particular safety property that results from applying looping acceptance depends on the form of the automaton, not just on its original language. For instance, consider the Büchi automaton consisting of a single (not final) state and no transitions. Obviously, its language is empty. Now add a single transition on the letter ℓ. Since the single state is not final, we have not changed the language of the automaton, which remains empty. However, applying the looping acceptance condition of [Vardi and Wolper 1994] to the original and the modified automata will result in different languages. Thus looping acceptance can be made equivalent to Alpern and Schneider's safety component, or to our weak component, for logics which associate a specific Büchi automaton with each temporal operator, that is, for logics whose semantics are presented in the form of automata construction (and of course the automaton used to get the safety component will be equivalent to but structurally different than the one used to get the weak component). The topological approach eliminates the need to specify the particular automaton associated with each operator, and thus allows the characterization of weakness to be applied to logics without regard to the form in which their semantics are presented, as we have done above for the denotational semantics of ELTL and LTL^{sere}.

By considering the augmented alphabet $\widehat{\Sigma}$, our weak component distinguishes between formulas such as $\varphi = [p \mathsf{U} \mathit{false}]$ and $\psi = \mathit{false}!$, which are equivalent under the orignal alphabet Σ, while the safety component of [Alpern and Schneider 1987] does not. In a similar manner, φ and ψ are not equivalent in Armoni et al.'s *reset semantics* [Armoni et al. 2003]. As we showed in [Eisner et al. 2006], their reset semantics for LTL are exactly those of Definition 9, which we use in our topological characterization of ELTL. Thus the difference between Alpern and Schneider's safety component, the closure over alphabet Σ, and our weak component, the closure over alphabet $\widehat{\Sigma}$, is reminiscent of the difference between the abort semantics and the reset semantics, examined in [Armoni et al. 2003]. Their goal was a complexity analysis, whereas ours is a characterization of weakness.

In [Eisner and Fisman 2008], we argue that the problem with the current semantics of LTL^{sere} is that it treats structural contradictions such as $\{p \cup \{p \cdot p\}\}$ differently than logical contradictions such as *false*. In this paper, we provide topological evidence in support of that argument, and in support of the logical consistency of the fix proposed by [Eisner and Fisman 2008] and adopted by [IEEE d].

The ideas of weakness and strength examined in this paper are very closely related to the ideas of weak and strong satisfaction that we first examined in [Eisner et al. 2003]. In that paper, we were concerned with the problem of reasoning on truncated paths, that is, over paths that are finite, but not necessarily maximal. In this paper, we are concerned with providing a semantic characterization of weakness for the purposes of verifying the internal consistency of a logic with weak and strong operators.

In [Maier 2004] Maier gives a characterization of intuitionistic safety and liveness properties for LTL interpreted over finite as well as infinite words. Maier works with prefix closed sets of non-empty words. His safety properties are the subsets of $\Sigma^+ \cup \Sigma^\omega$ that are closed in the Scott topology defined from the prefix partial order. The Scott topology coincides with our topology on Σ^∞, and so Maier's characterization of safety for properties over Σ is obtained from ours by eliminating the empty word. Since Maier's properties are all prefix closed, his notion of liveness is distinct from ours. Maier does not consider the relationship of weakness to safety, and his framework of prefix closed properties is not suited to discussion of the dual notion of strength.

In [Manolios and Trefler 2003] Manolios and Trefler gave an elegant characterization of safety and liveness using lattices, which unifies previous results on safety and liveness, including the results for the linear time and the branching time frameworks and for ω-regular string and tree languages. Our Proposition 4 can be proved by observing that $cl(\cdot)$ is a lattice closure on $\langle \Sigma^\infty, \cap, \cup \rangle$. Thus, it conforms with the uniform characterization.

The particular extension of LTL with regular expressions that we present in Section 4 is intended only to illustrate the utility of our characterization of weakness. Previous works [Armoni et al. 2002; Bustan et al. 2005; Bustan and Havlicek 2006; Lange 2007; Wolper et al. 1983; Vardi and Wolper 1994; Wolper 1983] have examined the complexity and expressivity issues of extending LTL with regular expressions in various manners.

A preliminary version of this work (considering regular expressions rather than semi-extended regular expressions) appeared in [Eisner et al. 2005].

7. CONCLUSIONS

We have been concerned with finding a semantic characterization that relates weak and strong linear temporal operators. We have shown that the weak version of a strong temporal operator is not the same as Alpern and Schneider's safety component. We have shown that the desired weak component can be obtained by taking closure in an alphabet augmented with two special letters $\top$ and $\bot$ in the Cantor topology on the set of finite and infinite words.

The resulting framework establishes a convenient way to study relations between weak and strong operators, and syntactically and semantically weak/strong formulas. For example, we have shown that semantically weak formulas characterize exactly the set of "good" safety properties, thus strengthening the result in [Kupferman and Vardi 1999].

The practical importance of our work lies in the fact that users of temporal logic typically assume the relation between weak and strong versions of an operator that we have characterized here, thus it is important that any new pair of operators maintain the same relation. We have used our characterization to demonstrate that the extensions of ELTL used by the original versions of the IEEE standards PSL and SVA are broken with respect to weak and strong semi-extended regular expressions r and $r!$. We have shown that the semantics proposed by [Eisner and Fisman 2008] and adopted by the latest version of PSL [IEEE d] fix this problem, and that it is sound in the sense that r is the weak version of $r!$.

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A. PROOF STRUCTURE

Propositions 1-4 are proved in Section B. In Section C we show that Propositions 5 and 7-15 are subsumed by Theorems 16 and 26-34. Therefore providing independent proofs of them is superfluous. In Section D we prove some lemmas about SEREs. Starting from Section E, we prove Theorems 16-34, one or two per section.

B. PROOFS OF PROPOSITIONS 1-4

PROPOSITION 1. *The collection of languages $W\Gamma^\infty$ such that $W \subseteq \Gamma^*$ forms a family of open sets for a topology on Γ^∞.*

PROOF.

(1) $\emptyset = W\Gamma^\infty$ for $W = \emptyset \subseteq \Gamma^*$.
(2) $\Gamma^\infty = W\Gamma^\infty$ for $W = \{\epsilon\} \subseteq \Gamma^*$.
(3) Let $W_\alpha \subseteq \Gamma^*$. Then $\bigcup(W_\alpha\Gamma^\infty) = W\Gamma^\infty$ for $W = \bigcup W_\alpha \subseteq \Gamma^*$.
(4) Let $U, V \subseteq \Gamma^*$. Define ${}^V U = \{u \in U \mid \exists v \in V : v \preceq u\}$, and let ${}^U V$ be defined similarly. Clearly ${}^V U \subseteq U$ and ${}^U V \subseteq V$. Now,

$$\begin{array}{ll} & w \in U\Gamma^\infty \cap V\Gamma^\infty \\ \text{iff} & w = ux = vy \text{ for some } u \in U,\ v \in V, \text{ and } x, y \in \Gamma^\infty \\ \text{iff} & [u \preceq v \text{ or } v \preceq u]\ w \in {}^U V\Gamma^\infty \text{ or } w \in {}^V U\Gamma^\infty \end{array}$$

This shows that $U\Gamma^\infty \cap V\Gamma^\infty = {}^U V\Gamma^\infty \cup {}^V U\Gamma^\infty$. Since ${}^U V \subseteq U \subseteq \Gamma^*$ and similarly ${}^V U \subseteq V \subseteq \Gamma^*$, we have that $U\Gamma^\infty \cap V\Gamma^\infty = W\Gamma^\infty$ for $W = {}^U V \cup {}^V U \subseteq \Gamma^*$.

□

LEMMA B.1. *$L \subseteq \Gamma^\infty$ is closed iff the following condition holds for all $w \in \Gamma^\infty$:*

(*) *If for every finite $u \preceq w$ there exists $v \succeq u$ such that $v \in L$, then $w \in L$.*

PROOF. ($\Rightarrow$) Assume that $L \subseteq \Gamma^\infty$ is closed. Then there exists $M \subseteq \Gamma^*$ such that $L = \Gamma^\infty \setminus M\Gamma^\infty$. Let $w \in \Gamma^\infty$. We prove the contrapositive of (*). Assume $w \notin L$. Then $w \in M\Gamma^\infty$, so there is a finite prefix $u \preceq w$ such that $u \in M$. Then every extension of u is in $M\Gamma^\infty$, hence not in L.

($\Leftarrow$) Assume (*) holds for all $w \in \Gamma^\infty$. Let $M = \{u \in \Gamma^* \mid \forall v \succeq u : v \notin L\}$. Then $M\Gamma^\infty \subseteq \Gamma^\infty \setminus L$. Suppose $w \in \Gamma^\infty \setminus L$. Then $w \notin L$ and by (*), there exists finite $u \preceq w$ such that $\forall v \succeq u : v \notin L$. Therefore $u \in M$, hence $w \in M\Gamma^\infty$. We have that $M\Gamma^\infty \subseteq \Gamma^\infty \setminus L$ and also that $\Gamma^\infty \setminus L \subseteq M\Gamma^\infty$. Thus $\Gamma^\infty \setminus L = M\Gamma^\infty$, and hence $L = \Gamma^\infty \setminus M\Gamma^\infty$. Since $M\Gamma^\infty$ is open by Definition 4, we have that its complement, L, is closed. □

LEMMA B.2. *$L \subseteq \Sigma^\infty$ is safety iff L is closed.*

PROOF. Let $L \subseteq \Sigma^\infty$.

$$\begin{array}{ll} & L \text{ is safety} \\ \text{iff} & \forall w \in \Sigma^\infty \setminus L : \exists \text{ finite } u \preceq w : \forall v : uv \in \Sigma^\infty \setminus L \\ \text{iff} & \forall w \in \Sigma^\infty \setminus L : \exists \text{ finite } u \preceq w : \forall u' \succeq u : u' \in \Sigma^\infty \setminus L \\ \text{iff} & \forall w \in \Sigma^\infty : w \notin L \Rightarrow \exists \text{ finite } u \preceq w : \forall u' \succeq u : u' \in \Sigma^\infty \setminus L \\ \text{iff} & [\text{contrapositive}]\ \forall w \in \Sigma^\infty : (\forall \text{ finite } u \preceq w : \exists u' \succeq u : u' \in L) \Rightarrow w \in L \\ \text{iff} & [\text{Lemma B.1}]\ L \text{ is closed.} \end{array}$$

□

LEMMA B.3. *$L \subseteq \Sigma^\infty$ is a liveness property iff L is dense in Σ^∞.*

PROOF. ($\Rightarrow$) Let L be a liveness property and let $M \subseteq \Sigma^\infty$ be a closed set containing L. Suppose there exists $w \in \Sigma^\infty \setminus M$. Then $w \notin M$ and by Lemma B.1, there exists a finite prefix u of w such that for all $v \succeq u$, $v \notin M$. However, since L is a liveness property, there exists $v \succeq u$ such that $v \in L \subseteq M$, a contradiction. Therefore, $M = \Sigma^\infty$. This proves that L is dense.

($\Leftarrow$) Let L be dense in Σ^∞. Suppose L is not a liveness property. Then there exists finite u such that for all v, $uv \notin L$. Therefore $u\Sigma^\infty \subseteq \Sigma^\infty \setminus L$, and so $L \subseteq \Sigma^\infty \setminus u\Sigma^\infty$. Since $\Sigma^\infty \setminus u\Sigma^\infty$ is closed by Definition 4 and L is dense, $\Sigma^\infty \setminus u\Sigma^\infty = \Sigma^\infty$, and therefore $u\Sigma^\infty$ is empty, a contradiction. □

LEMMA B.4. *Let X be a topological space and let $P \subseteq X$. Then $X \setminus (cl(P) \setminus P)$ is dense. Furthermore,*

$$P = cl(P) \cap (X \setminus (cl(P) \setminus P)) .$$

PROOF. Let Y be a closed set containing $X \setminus (cl(P) \setminus P)$. We must show that $Y = X$. Consider the open set $X \setminus Y$. Since Y contains $X \setminus (cl(P) \setminus P)$,

$$X \setminus Y \subseteq cl(P) \setminus P$$

Since $P \subseteq cl(P)$,

$$P \subseteq cl(P) \setminus (X \setminus Y)$$

and since $cl(P) \setminus (X \setminus Y)$ is closed, we must have $X \setminus Y$ empty. The proof of the equality is exactly as in [Alpern and Schneider 1985]. □

PROPOSITION 2. *Let $V \subseteq \Gamma^\infty$.*

(1) *V is closed in the Cantor topology iff V is both prefix closed and limit closed.*
(2) *V is open in the Cantor topology iff V is extension closed and has finite witnesses.*

PROOF.

(1) By Lemma B.1, X is closed iff the following property holds for all $w \in \Gamma^\infty$:

(*) If for every finite $u \preceq w$ there exists $x \succeq u$ such that $x \in X$, then $w \in X$.

($\Rightarrow$) Assume that X is closed, so that (*) holds for all $w \in \Gamma^\infty$. Suppose that $w \in \Gamma^\infty$ and that all finite prefixes of w are in X. By (*), it follows that $w \in X$. This proves that X is limit closed. Suppose now that $u \preceq v \in X$. Then for all finite $u' \preceq u$, $u' \preceq v \in X$. From (*), $u \in X$. This proves that X is prefix closed.
($\Leftarrow$) Assume that X is both prefix closed and limit closed. We will show that (*) holds for all $w \in \Gamma^\infty$. Let $w \in \Gamma^\infty$ and assume that for every finite $u \preceq w$ there exists $x \succeq u$ such that $x \in X$. Since X is prefix closed, it follows that for every finite $u \preceq w$, $u \in X$. Since X is limit closed, $w \in X$.

(2) Follows from Part (1) by duality.

□

PROPOSITION 3.

(1) $cl(V) = \{w \in \Gamma^\infty \mid \forall$ *finite* $u \preceq w : \exists v \succeq u : v \in V\}$.
(2) $int(V) = \{w \in \Gamma^\infty \mid \exists$ *finite* $u \preceq w : \forall v \succeq u : v \in V\}$.

PROOF.

(1) Let $C(X) = \{w \in \Gamma^\infty \mid \forall \text{ finite } u \preceq w : \exists v \succeq u : v \in X\}$. First we show that $C(X)$ is closed. Let $w \in \Gamma^\infty$. Assume that $\forall$ finite $u \preceq w : \exists v \succeq u : v \in C(X)$. Using the definition of $C(X)$, it follows that $\forall$ finite $u \preceq w : \exists x \succeq u : x \in X$. Therefore $w \in C(X)$. By Lemma B.1, $C(X)$ is closed.

Suppose now that $X \subseteq L \subseteq \Gamma^\infty$ and L is closed. We show that $C(X) \subseteq L$. Let $w \in C(X)$. Then $\forall$ finite $u \preceq w : \exists x \succeq u : x \in X \subseteq L$. Since L is closed, Lemma B.1 implies that $w \in L$.

(2) Let $I(X) = \{w \in \Gamma^\infty \mid \exists \text{ finite } u \preceq w : \forall v \succeq u : v \in X\}$. First, we show that $I(X)$ is open. Let $w \in I(X)$. Then $\exists$ finite $u \preceq w : \forall x \succeq u$: $x \in X$. From the definition of $I(X)$, it follows that $\forall v \succeq u$, $v \in I(X)$. Therefore $w \in u\Gamma^\infty \subseteq I(X)$. Since $u\Gamma^\infty$ is open by Definition 4, this shows that $I(X)$ is open.
Suppose now that M is open and $M \subseteq X$. We show that $M \subseteq I(X)$. Let $w \in M$. Since M is open, there exists $U \subseteq \Gamma^*$ such that $w \in U\Gamma^\infty \subseteq M \subseteq X$. Then $\exists$ finite $u \preceq w$ $(u \in U) : \forall v \succeq u : v \in U\Gamma^\infty \subseteq X$, hence $w \in I(X)$.

□

PROPOSITION 4. *The safety properties are the closed sets of the Cantor topology on Σ^∞, and the liveness properties are the dense sets in this topology. Furthermore, every property on Σ^∞ can be decomposed into a safety property and a liveness property whose intersection is the original.*

PROOF. It follows from Lemma B.2 that the safety properties are the closed sets of the Cantor topology on Σ^∞. It follows from Lemma B.3 that the liveness properties are the dense sets. Finally, Lemma B.4 shows that any property can be decomposed into a safety property and a liveness property whose intersection is the original. □

C. PROOFS OF PROPOSITIONS 5 AND 7-15

Lemma C.1 shows that ELTL is faithfully embedded as a sublogic of LTL^{sere}. Thus the statements of Theorems 16 and 26-34 are generalizations of Propositions 5 and 7-15 and the proofs of Theorems 16 and 26-34 imply Propositions 5 and 7-15.

LEMMA C.1. *Let φ be an* ELTL *formula, let $w \in \widehat{\Sigma}^\infty$ and let $\models_9$ and $\models_{23}$ denote the satisfaction relations of Definitions 9 and 23, respectively. Then $w \models_9 \varphi$ iff $w \models_{23} \varphi$.*

PROOF. The statement of the semantics for all ELTL operators except for strong and weak Booleans is identical in Definitions 9 and 23. It remains to prove the equivalence in the cases of strong and weak Booleans. Recall that in LTL^{sere}, a Boolean expression is a SERE, thus the semantics of $b!$ and b are obtained through those of $r!$ and r, respectively.

- $w \models_{23} b!$
  iff $\exists j < |w|$ s.t. $w^{0..j} \in \widehat{\mathcal{L}}(b)$
  iff $\exists j < |w|$ s.t. $w^{0..j} \in \mathcal{L}(b) \cup (\mathcal{F}(b)\top^+) \cup \top^+$
  iff $\exists j < |w|$ s.t. $w^{0..j} \in \{\ell \in \widehat{\Sigma} \mid \ell \Vdash b\} \cup (\epsilon\top^+) \cup \top^+$
  iff $\exists j < |w|$ s.t. either $(j = 0$ and $w^0 \Vdash b)$ or $(w^{0..j} \in \top^+)$
  iff $|w| > 0$ and $w^0 \Vdash b$
  iff $w \models_9 b!$

- $w \models_{23} b$
  iff either $w \models_{23} b!$ or $w \in \mathcal{I}(b) \cup \mathcal{F}(b) \cup \{\epsilon\}$
  iff either $w \models_9 b!$ [first bullet] or $w \in \emptyset \cup \{\epsilon\} \cup \{\epsilon\}$
  iff either $w \models_9 b!$ or $w = \epsilon$
  iff either $w \models_9 b!$ or $|w| = 0$
  iff $w \models_9 b$

□

D. LEMMAS ABOUT SEREs

LEMMA D.1. *Every word in $\mathcal{I}(r)$ is infinite.*

PROOF. By induction on the structure of r.

λ: $\mathcal{I}(\lambda) = \emptyset$.
b: $\mathcal{I}(b) = \emptyset$.
$r \cdot s$: $\mathcal{I}(r \cdot s) = \mathcal{I}(r) \cup (\mathcal{L}(r)\mathcal{I}(s))$. By induction, $\mathcal{I}(r)$ and $\mathcal{L}(r)\mathcal{I}(s)$ consist only of infinite words.
$r \circ s$: $\mathcal{I}(r \circ s) = \mathcal{I}(r) \cup (\mathcal{L}(r) \circ \mathcal{I}(s))$. By induction, $\mathcal{I}(r)$ and $\mathcal{L}(r) \circ \mathcal{I}(s)$ consist only of infinite words.
r^+: $\mathcal{I}(r^+) = (\mathcal{L}(r)^* \mathcal{I}(r)) \cup (\mathcal{L}(r) \setminus \{\epsilon\})^\omega$. By induction, $\mathcal{L}(r)^* \mathcal{I}(r)$ consists only of infinite words. Also, the exclusion of ϵ ensures that $(\mathcal{L}(r) \setminus \{\epsilon\})^\omega$ consists only of infinite words.
$r \cup s$: $\mathcal{I}(r \cup s) = \mathcal{I}(r) \cup \mathcal{I}(s)$. By induction, $\mathcal{I}(r)$ and $\mathcal{I}(s)$ consist only of infinite words.
$r \cap s$: $\mathcal{I}(r \cap s) = \mathcal{I}(r) \cap \mathcal{I}(s)$. By induction, $\mathcal{I}(r)$ and $\mathcal{I}(s)$ consist only of infinite words.

□

LEMMA D.2. *$\mathcal{L}(\cdot)$, $\mathcal{F}(\cdot)$, $\mathcal{I}(\cdot)$ and $\widehat{\mathcal{L}}(\cdot)$ are all closed under switching (0 or more) letters to $\top$.*

PROOF.

(1) $\mathcal{L}(\cdot)$: Assume that $\mathcal{L}(r)$ and $\mathcal{L}(s)$ are closed under switching letters to $\top$. Let u result from v by switching letters to $\top$.
   λ: $\mathcal{L}(\lambda) = \{\epsilon\}$, which is closed under switching letters to $\top$.
   b: $\mathcal{L}(b) = \{\ell \in \widehat{\Sigma} \mid \ell \Vdash b\}$, which is closed under switching letters to $\top$.
   $r \cdot s$: Let $v \in \mathcal{L}(r \cdot s) = \mathcal{L}(r)\mathcal{L}(s)$. Then $v = xy$, where $x \in \mathcal{L}(r)$ and $y \in \mathcal{L}(s)$. Then $u = x'y'$, where x', y' result from x, y (resp.) by switching letters to $\top$. By induction, $x' \in \mathcal{L}(r)$, $y' \in \mathcal{L}(s)$, hence $u \in \mathcal{L}(r \cdot s)$.
   $r \circ s$: Let $v \in \mathcal{L}(r \circ s) = \mathcal{L}(r) \circ \mathcal{L}(s)$. Then $v = x\ell y$, where $x\ell \in \mathcal{L}(r)$ and $\ell y \in \mathcal{L}(s)$. Then $u = x'\ell' y'$, where x', ℓ', y' result from x, ℓ, y (resp.) by switching letters to $\top$. By induction, $x'\ell' \in \mathcal{L}(r)$, $\ell' y' \in \mathcal{L}(s)$, hence $u \in \mathcal{L}(r \circ s)$.
   r^+: Let $v \in \mathcal{L}(r^+) = \mathcal{L}(r)^+$. Then $v = x_1 \cdots x_n$, where each $x_i \in \mathcal{L}(r)$. Then $u = x'_1 \cdots x'_n$, where x'_i results from x_i by switching letters to $\top$. By induction, $x'_i \in \mathcal{L}(r)$, hence $u \in \mathcal{L}(r^+)$.
   $r \cup s$: Let $v \in \mathcal{L}(r \cup s) = \mathcal{L}(r) \cup \mathcal{L}(s)$. Without loss of generality, $v \in \mathcal{L}(r)$, so by induction $u \in \mathcal{L}(r) \subseteq \mathcal{L}(r \cup s)$.
   $r \cap s$: Let $v \in \mathcal{L}(r \cap s) = \mathcal{L}(r) \cap \mathcal{L}(s)$. Then $v \in \mathcal{L}(r)$ and $v \in \mathcal{L}(s)$, so by induction $u \in \mathcal{L}(r)$ and $u \in \mathcal{L}(s)$, hence $u \in \mathcal{L}(r \cap s)$.
(2) $\mathcal{F}(\cdot)$: Assume that $\mathcal{F}(r)$ and $\mathcal{F}(s)$ are closed under switching letters to $\top$. Let u result from v by switching letters to $\top$.
   λ: $\mathcal{F}(\lambda) = \emptyset$, which is closed under switching letters to $\top$.
   b: $\mathcal{F}(b) = \{\epsilon\}$, which is closed under switching letters to $\top$.
   $r \cdot s$: Let $v \in \mathcal{F}(r \cdot s) = \mathcal{F}(r) \cup (\mathcal{L}(r)\mathcal{F}(s))$. If $v \in \mathcal{F}(r)$, then by induction $u \in \mathcal{F}(r) \subseteq \mathcal{F}(r \cdot s)$. Otherwise, $v = xy$, where $x \in \mathcal{L}(r)$ and $y \in \mathcal{F}(s)$. Then $u = x'y'$, where x', y' result from x, y (resp.) by switching letters to $\top$. By induction, $x' \in \mathcal{L}(r)$, and $y' \in \mathcal{F}(s)$, hence $u \in \mathcal{F}(r \cdot s)$.
   $r \circ s$: Let $v \in \mathcal{F}(r \circ s) = \mathcal{F}(r) \cup (\mathcal{L}(r) \circ \mathcal{F}(s))$. If $v \in \mathcal{F}(r)$, then by induction $u \in \mathcal{F}(r) \subseteq \mathcal{F}(r \circ s)$. Otherwise, $v = x\ell y$, where $x\ell \in \mathcal{L}(r)$ and $\ell y \in \mathcal{F}(s)$. Then $u = x'\ell' y'$, where x', ℓ', y' result from x, ℓ, y (resp.) by switching letters to $\top$. By induction, $x'\ell' \in \mathcal{L}(r)$, and by induction, $\ell' y' \in \mathcal{F}(s)$, hence $u \in \mathcal{F}(r \circ s)$.

r^+: Let $v \in \mathcal{F}(r^+) = \mathcal{L}(r)^*\mathcal{F}(r)$. Then $v = x_1 \cdots x_n$, where $n \geq 1$ and $x_i \in \mathcal{L}(r)$ for $1 \leq i < n$ and $x_n \in \mathcal{F}(r)$. Then $u = x'_1 \cdots x'_n$, where x'_i results from x_i by switching letters to $\top$. By induction, $x'_i \in \mathcal{L}(r)$ for $1 \leq i < n$, and by induction $x_n \in \mathcal{F}(r)$, hence $u \in \mathcal{F}(r^+)$.
$r \cup s$: Let $v \in \mathcal{F}(r \cup s) = \mathcal{F}(r) \cup \mathcal{F}(s)$. Without loss of generality, $v \in \mathcal{F}(r)$, so by induction $u \in \mathcal{F}(r) \subseteq \mathcal{F}(r \cup s)$.
$r \cap s$: Let $v \in \mathcal{F}(r \cap s) = \mathcal{F}(r) \cap \mathcal{F}(s)$. Then $v \in \mathcal{F}(r)$ and $v \in \mathcal{F}(s)$, so by induction $u \in \mathcal{F}(r)$ and $u \in \mathcal{F}(s)$, hence $u \in \mathcal{F}(r \cap s)$.

(3) $\mathcal{I}(\cdot)$: Assume that $\mathcal{I}(r)$ and $\mathcal{I}(s)$ are closed under switching letters to $\top$. Let u result from v by switching letters to $\top$.
λ: $\mathcal{I}(\lambda) = \emptyset$, which is closed under switching letters to $\top$.
b: $\mathcal{I}(b) = \emptyset$, which is closed under switching letters to $\top$.
$r \cdot s$: Let $v \in \mathcal{I}(r \cdot s) = \mathcal{I}(r) \cup (\mathcal{L}(r)\mathcal{I}(s))$. If $v \in \mathcal{I}(r)$, then by induction $u \in \mathcal{I}(r) \subseteq \mathcal{I}(r \cdot s)$. Otherwise, $v = xy$, where $x \in \mathcal{L}(r)$ and $y \in \mathcal{I}(s)$. Then $u = x'y'$, where x', y' result from x, y (resp.) by switching letters to $\top$. By induction, $x' \in \mathcal{L}(r)$, and $y' \in \mathcal{I}(s)$, hence $u \in \mathcal{I}(r \cdot s)$.
$r \circ s$: Let $v \in \mathcal{I}(r \circ s) = \mathcal{I}(r) \cup (\mathcal{L}(r) \circ \mathcal{I}(s))$. If $v \in \mathcal{I}(r)$, then by induction $u \in \mathcal{I}(r) \subseteq \mathcal{I}(r \circ s)$. Otherwise, $v = x\ell y$, where $x\ell \in \mathcal{L}(r)$ and $\ell y \in \mathcal{I}(s)$. Then $u = x'\ell'y'$, where x', ℓ', y' result from x, ℓ, y (resp.) by switching letters to $\top$. By induction, $x'\ell' \in \mathcal{L}(r)$, and by induction, $\ell'y' \in \mathcal{I}(s)$, hence $u \in \mathcal{I}(r \circ s)$.
r^+: Let $v \in \mathcal{I}(r^+) = (\mathcal{L}(r)^*\mathcal{I}(r)) \cup (\mathcal{L}(r) \setminus \{\epsilon\})^\omega$. Suppose first that $v \in \mathcal{L}(r)^*\mathcal{I}(r)$. Then $v = x_1 \cdots x_n$, where $n \geq 1$ and $x_i \in \mathcal{L}(r)$ for $1 \leq i < n$ and $x_n \in \mathcal{I}(r)$. Then $u = x'_1 \cdots x'_n$, where x'_i results from x_i by switching letters to $\top$. By induction, $x'_i \in \mathcal{L}(r)$ for $1 \leq i < n$, and by induction $x_n \in \mathcal{I}(r)$, hence $u \in \mathcal{I}(r^+)$. Suppose now that $v \in (\mathcal{L}(r) \setminus \{\epsilon\})^\omega$. Then $v = x_1 \cdots$, where each $x_i \in \mathcal{L}(r) \setminus \{\epsilon\}$. Then $u = x'_1 \cdots$, where x'_i results from x_i by switching letters to $\top$. By induction, $x'_i \in \mathcal{L}(r) \setminus \{\epsilon\}$, so $u \in (\mathcal{L}(r) \setminus \{\epsilon\})^\omega$.
$r \cup s$: Let $v \in \mathcal{I}(r \cup s) = \mathcal{I}(r) \cup \mathcal{I}(s)$. Without loss of generality, $v \in \mathcal{I}(r)$, so by induction $u \in \mathcal{I}(r) \subseteq \mathcal{I}(r \cup s)$.
$r \cap s$: Let $v \in \mathcal{I}(r \cap s) = \mathcal{I}(r) \cap \mathcal{I}(s)$. Then $v \in \mathcal{I}(r)$ and $v \in \mathcal{I}(s)$, so by induction $u \in \mathcal{I}(r)$ and $u \in \mathcal{I}(s)$, hence $u \in \mathcal{I}(r \cap s)$.

(4) $\widehat{\mathcal{L}}(\cdot)$: Follows from Definition 21 and Parts (1) and (2).

□

LEMMA D.3. *Let r be a* SERE. *If $\mathcal{L}(r) \neq \emptyset$ and $\mathcal{L}(r) \neq \{\epsilon\}$, then $\epsilon \in \mathcal{F}(r)$.*

PROOF. By induction on the structure of r.

λ: $\mathcal{L}(\lambda) = \{\epsilon\}$, so there is nothing to show.
b: $\epsilon \in \mathcal{F}(b)$ by the definition of $\mathcal{F}(b)$.
$r \cdot s$: If $\mathcal{L}(r \cdot s) \neq \emptyset$ and $\mathcal{L}(r \cdot s) \neq \{\epsilon\}$, then $\mathcal{L}(r) \neq \emptyset$ and $\mathcal{L}(s) \neq \emptyset$ and either $\mathcal{L}(r) \neq \{\epsilon\}$ or $\mathcal{L}(s) \neq \{\epsilon\}$. If $\mathcal{L}(r) \neq \{\epsilon\}$, then $\epsilon \in \mathcal{F}(r)$ by induction and $\mathcal{F}(r) \subseteq \mathcal{F}(r \cdot s)$ by the definition of $\mathcal{F}(r \cdot s)$, and thus $\epsilon \in \mathcal{F}(r \cdot s)$. If $\mathcal{L}(r) = \{\epsilon\}$ but $\mathcal{L}(s) \neq \{\epsilon\}$, then $\epsilon \in \mathcal{F}(s)$ by induction and thus $\epsilon \in \mathcal{L}(r)\mathcal{F}(s)$. $\mathcal{L}(r)\mathcal{F}(s) \subseteq \mathcal{F}(r \cdot s)$ by the definition of $\mathcal{F}(r \cdot s)$, thus $\epsilon \in \mathcal{F}(r \cdot s)$.
$r \circ s$: By the definition of fusion, $\mathcal{L}(r \circ s) \neq \{\epsilon\}$. If $\mathcal{L}(r \circ s) \neq \emptyset$, then $\mathcal{L}(r) \neq \emptyset$, $\mathcal{L}(s) \neq \emptyset$, $\mathcal{L}(r) \neq \{\epsilon\}$ and $\mathcal{L}(s) \neq \{\epsilon\}$. Then $\epsilon \in \mathcal{F}(r)$ by induction and $\mathcal{F}(r) \subseteq \mathcal{F}(r \circ s)$ by the definition of $\mathcal{F}(r \circ s)$. Thus $\epsilon \in \mathcal{F}(r \circ s)$.
r^+: If $\mathcal{L}(r^+) \neq \emptyset$ and $\mathcal{L}(r^+) \neq \{\epsilon\}$, then $\mathcal{L}(r) \neq \emptyset$ and $\mathcal{L}(r) \neq \{\epsilon\}$. Then by induction, $\epsilon \in \mathcal{F}(r)$ and thus $\epsilon \in \mathcal{L}(r)^*\mathcal{F}(r) = \mathcal{F}(r^+)$.

$r \cup s$: If $\mathcal{L}(r \cup s) \neq \emptyset$ and $\mathcal{L}(r \cup s) \neq \{\epsilon\}$, then either ($\mathcal{L}(r) \neq \emptyset$ and $\mathcal{L}(r) \neq \{\epsilon\}$) or ($\mathcal{L}(s) \neq \emptyset$ and $\mathcal{L}(s) \neq \{\epsilon\}$). Then by induction, either $\epsilon \in \mathcal{F}(r)$ or $\epsilon \in \mathcal{F}(s)$ and thus $\epsilon \in \mathcal{F}(r) \cup \mathcal{F}(s)$, which is the definition of $\mathcal{F}(r \cup s)$.
$r \cap s$: If $\mathcal{L}(r \cap s) \neq \emptyset$ and $\mathcal{L}(r \cap s) \neq \{\epsilon\}$, then $\mathcal{L}(r) \neq \emptyset$, $\mathcal{L}(s) \neq \emptyset$, $\mathcal{L}(r) \neq \{\epsilon\}$ and $\mathcal{L}(s) \neq \{\epsilon\}$. Then by induction $\epsilon \in \mathcal{F}(r)$ and $\epsilon \in \mathcal{F}(s)$ so $\epsilon \in \mathcal{F}(r) \cap \mathcal{F}(s)$, and therefore by definition, $\epsilon \in \mathcal{F}(r \cap s)$.

□

LEMMA D.4. *Let $w \in \mathcal{L}(r)$, and $u \prec w$. Then $u \in \mathcal{F}(r)$.*

PROOF. By induction on the structure of r.

λ: $w \in \mathcal{L}(\lambda) = \{\epsilon\}$, and there is no relevant u.
b: $w \in \mathcal{L}(b) = \{\ell \in \widehat{\Sigma} \mid \ell \Vdash b\}$, thus $u = \epsilon$, which is in $\mathcal{F}(b)$.
$r \cdot s$: $w \in \mathcal{L}(r \cdot s) = \mathcal{L}(r)\mathcal{L}(s)$. Thus $w = w'w''$ s.t. $w' \in \mathcal{L}(r)$ and $w'' \in \mathcal{L}(s)$. If $u \prec w'$, then by induction $u \in \mathcal{F}(r) \subseteq \mathcal{F}(r \cdot s)$. If $u \succeq w'$, then $u = w'u''$ s.t. $u'' \prec w''$. By induction, $u'' \in \mathcal{F}(s)$, thus $u \in \mathcal{L}(r)\mathcal{F}(s) \subseteq \mathcal{F}(r \cdot s)$.
$r \circ s$: $w \in \mathcal{L}(r \circ s) = \mathcal{L}(r) \circ \mathcal{L}(s)$. Thus $w = w' \circ w''$ s.t. $w' \in \mathcal{L}(r)$ and $w'' \in \mathcal{L}(s)$. If $u \prec w'$, then by induction $u \in \mathcal{F}(r) \subseteq \mathcal{F}(r \circ s)$. If $u \succeq w'$, then $u = w' \circ u''$ s.t. $u'' \prec w''$. By induction, $u'' \in \mathcal{F}(s)$, thus $u \in \mathcal{L}(r) \circ \mathcal{F}(s) \subseteq \mathcal{F}(r \circ s)$.
r^+: $w \in \mathcal{L}(r^+) = \mathcal{L}(r)^+$. Thus $w = w_1 \cdots w_n$ s.t. each $w_i \in \mathcal{L}(r)$, and there are two cases:

- $u = w_1 \cdots w_k u'$ s.t. $k < n$ and $\epsilon \prec u' \prec w_{k+1}$. By induction, $u' \in \mathcal{F}(r)$. Thus $u \in \mathcal{L}(r)^* \mathcal{F}(r) = \mathcal{F}(r^+)$.
- $u = w_1 \cdots w_k$ s.t. $k < n$. We know that $\mathcal{L}(r) \neq \emptyset$ (otherwise there is no w) and $\mathcal{L}(r) \neq \{\epsilon\}$ (otherwise there is no $u \prec w$), thus by Lemma D.3 we know that $\epsilon \in \mathcal{F}(r)$. Thus $u \in \mathcal{L}(r)^* \mathcal{F}(r) = \mathcal{F}(r^+)$.

$r \cup s$: If $w \in \mathcal{L}(r \cup s)$ then $w \in \mathcal{L}(r) \cup \mathcal{L}(s)$ and by induction $u \in \mathcal{F}(r)$ or $u \in \mathcal{F}(s)$, thus $u \in \mathcal{F}(r \cup s)$.
$r \cap s$: If $w \in \mathcal{L}(r \cap s)$ then $w \in \mathcal{L}(r) \cap \mathcal{L}(s)$ and by induction $u \in \mathcal{F}(r)$ and $u \in \mathcal{F}(s)$, thus $u \in \mathcal{F}(r \cap s)$.

□

LEMMA D.5. *Let $w \in \mathcal{I}(r)$, and $u \prec w$. Then $u \in \mathcal{F}(r)$.*

PROOF. By induction on the structure of r.

λ: $\mathcal{I}(\lambda) = \emptyset$, thus $\nexists w \in \mathcal{I}(\lambda)$.
b: $\mathcal{I}(b) = \emptyset$, thus $\nexists w \in \mathcal{I}(b)$.
$r \cdot s$: $w \in \mathcal{I}(r \cdot s) = \mathcal{I}(r) \cup (\mathcal{L}(r)\mathcal{I}(s))$. If $w \in \mathcal{I}(r)$ then by induction $u \in \mathcal{F}(r) \subseteq \mathcal{F}(r \cdot s)$. Otherwise, $w = w'w''$ s.t. $w' \in \mathcal{L}(r)$ and $w'' \in \mathcal{I}(s)$. If $u \prec w'$, then by Lemma D.4, $u \in \mathcal{F}(r)$. If $u \succeq w'$, then $u = w'u''$ s.t. $u'' \prec w''$. By induction, $u'' \in \mathcal{F}(s)$, thus $u \in \mathcal{L}(r)\mathcal{F}(s) \subseteq \mathcal{F}(r \cdot s)$.
$r \circ s$: $w \in \mathcal{I}(r \circ s) = \mathcal{I}(r) \cup (\mathcal{L}(r) \circ \mathcal{I}(s))$. If $w \in \mathcal{I}(r)$ then by induction $u \in \mathcal{F}(r) \subseteq \mathcal{F}(r \circ s)$. Otherwise, $w = w' \circ w''$ s.t. $w' \in \mathcal{L}(r)$ and $w'' \in \mathcal{I}(s)$. If $u \prec w'$, then by Lemma D.4, $u \in \mathcal{F}(r) \subseteq \mathcal{F}(r \circ s)$. If $u \succeq w'$, then $u = w' \circ u''$ s.t. $u'' \prec w''$. By induction, $u'' \in \mathcal{F}(s)$, thus $u \in \mathcal{L}(r) \circ \mathcal{F}(s) \subseteq \mathcal{F}(r \circ s)$.
r^+: $w \in \mathcal{I}(r^+) = (\mathcal{L}(r)^* \mathcal{I}(r)) \cup (\mathcal{L}(r) \setminus \{\epsilon\})^\omega$.
There are two cases:

- If $w \in \mathcal{L}(r)^* \mathcal{I}(r)$, then $w = w'w''$ s.t. $w' \in \mathcal{L}(r)^*$ and $w'' \in \mathcal{I}(r)$. If $u \prec w'$, then $w' \neq \epsilon$ and thus $w' \in \mathcal{L}(r)^+ = \mathcal{L}(r^+)$ and by Lemma D.4, $u \in \mathcal{F}(r^+)$. If $u \succeq w'$, then $u = w'u''$ s.t. $u'' \prec w''$, thus by induction $u'' \in \mathcal{F}(r)$ and so $u \in \mathcal{L}(r)^* \mathcal{F}(r) = \mathcal{F}(r^+)$.

- If $w \in (\mathcal{L}(r) \setminus \{\epsilon\})^\omega$, then $\exists w' \prec w$ s.t. $w' \in \mathcal{L}(r)^+ = \mathcal{L}(r^+)$ and $u \prec w'$. Thus by Lemma D.4, $u \in \mathcal{F}(r^+)$.

$r \cup s$: If $w \in \mathcal{I}(r \cup s)$ then $w \in \mathcal{I}(r) \cup \mathcal{I}(s)$ and by induction $u \in \mathcal{F}(r)$ or $u \in \mathcal{F}(s)$, thus $u \in \mathcal{F}(r \cup s)$.
$r \cap s$: If $w \in \mathcal{I}(r \cap s)$ then $w \in \mathcal{I}(r) \cap \mathcal{I}(s)$ and by induction $u \in \mathcal{F}(r)$ and $u \in \mathcal{F}(s)$, thus $u \in \mathcal{F}(r \cap s)$.

□

LEMMA D.6. *Let $w \in \mathcal{F}(r)$, and $u \prec w$. Then $u \in \mathcal{F}(r)$. In other words, $\mathcal{F}(r)$ is prefix closed.*

PROOF. By induction on the structure of r.

λ: $\mathcal{F}(\lambda) = \emptyset$, thus $\nexists w \in \mathcal{F}(\lambda)$.
b: $w \in \mathcal{F}(b) = \{\epsilon\}$, thus there is no relevant u.
$r \cdot s$: $w \in \mathcal{F}(r \cdot s) = \mathcal{F}(r) \cup (\mathcal{L}(r)\mathcal{F}(s))$. If $w \in \mathcal{F}(r)$ then by induction $u \in \mathcal{F}(r) \subseteq \mathcal{F}(r \cdot s)$. Otherwise, $w = w'w''$ s.t. $w' \in \mathcal{L}(r)$ and $w'' \in \mathcal{F}(s)$. If $u \prec w'$, then by Lemma D.4, $u \in \mathcal{F}(r) \subseteq \mathcal{F}(r \cdot s)$. If $u \succeq w'$, then $u = w'u''$ s.t. $u'' \prec w''$. By induction, $u'' \in \mathcal{F}(s)$, thus $u \in \mathcal{L}(r)\mathcal{F}(s) \subseteq \mathcal{F}(r \cdot s)$.
$r \circ s$: $w \in \mathcal{F}(r \circ s) = \mathcal{F}(r) \cup (\mathcal{L}(r) \circ \mathcal{F}(s))$. If $w \in \mathcal{F}(r)$ then by induction $u \in \mathcal{F}(r) \subseteq \mathcal{F}(r \circ s)$. Otherwise, $w = w' \circ w''$ s.t. $w' \in \mathcal{L}(r)$ and $w'' \in \mathcal{F}(s)$. If $u \prec w'$, then by Lemma D.4, $u \in \mathcal{F}(r) \subseteq \mathcal{F}(r \circ s)$. If $u \succeq w'$, then $u = w' \circ u''$ s.t. $u'' \prec w''$. By induction, $u'' \in \mathcal{F}(s)$, thus $u \in \mathcal{L}(r) \circ \mathcal{F}(s) \subseteq \mathcal{F}(r \circ s)$.
r^+: $w \in \mathcal{F}(r^+) = \mathcal{L}(r)^*\mathcal{F}(r)$. Thus $w = w'w''$ s.t. $w' \in \mathcal{L}(r)^*$ and $w'' \in \mathcal{F}(r)$. If $u \prec w'$, then $w' \neq \epsilon$ and so $w' \in \mathcal{L}(r)^+ = \mathcal{L}(r^+)$. By Lemma D.4, $u \in \mathcal{F}(r^+)$. Otherwise, $u \succeq w'$, thus $u = w'u''$ s.t. $u'' \prec w''$, thus by induction $u'' \in \mathcal{F}(r)$ and so $u \in \mathcal{L}(r)^*\mathcal{F}(r) = \mathcal{F}(r^+)$.
$r \cup s$: If $w \in \mathcal{F}(r \cup s)$ then $w \in \mathcal{F}(r) \cup \mathcal{F}(s)$ and by induction $u \in \mathcal{F}(r)$ or $u \in \mathcal{F}(s)$, thus $u \in \mathcal{F}(r \cup s)$.
$r \cap s$: If $w \in \mathcal{F}(r \cap s)$ then $w \in \mathcal{F}(r) \cap \mathcal{F}(s)$ and by induction $u \in \mathcal{F}(r)$ and $u \in \mathcal{F}(s)$, thus $u \in \mathcal{F}(r \cap s)$.

□

E. PROOF OF THEOREM 16

LEMMA E.1. *Let r be a* SERE.

(1) $\top^+ \subseteq \widehat{\mathcal{L}}(r)$.
(2) *No letter of any word in $\mathcal{L}(r)$, $\mathcal{F}(r)$, or $\mathcal{I}(r)$ is $\bot$.*
(3) *No letter of any word in $\widehat{\mathcal{L}}(r)$ is $\bot$.*

PROOF.

(1) Immediate from Definition 21.
(2) By straightforward induction.
(3) Follows from Part (2) by Definition 21.

□

NOTE: In proofs by induction over the structure of LTL^{sere} formulas, we may omit the cases for the derived forms $f \vee g$ and $\mathsf{X}\ f$, as justified by the following lemma. We also assume that the forms r and $[g\ \mathsf{W}\ h]$ have greater inductive weight than $r!$ and $[g\ \mathsf{U}\ h]$, respectively, so that we may make inductive appeals to the latter forms in the proofs for the cases of the former forms.

LEMMA E.2 (DERIVED OPERATORS IN LTL^{sere}). *Let f, $g \in$ LTL^{sere} and let $v \in \widehat{\Sigma}^\infty$. Then*

- $v \models f \vee g$ *iff* $v \models \neg(\neg f \wedge \neg g)$
- $v \models \mathsf{X}\, f$ *iff* $v \models \neg(\mathsf{X!}\, \neg f)$

PROOF.

- $v \models \neg(\neg f \wedge \neg g)$
  - iff $\overline{v} \not\models \neg f \wedge \neg g$
  - iff not($\overline{v} \models \neg f$ and $\overline{v} \models \neg g$)
  - iff not($v \not\models f$ and $v \not\models g$)
  - iff $v \models f$ or $v \models g$
  - iff $v \models f \vee g$
- $v \models \neg(\mathsf{X!}\, \neg f)$
  - iff $\overline{v} \not\models \mathsf{X!}\, \neg f$
  - iff not($|\overline{v}| > 1$ and $\overline{v}^{1\cdot\cdot} \models \neg f$)
  - iff either $|\overline{v}| \leq 1$ or $\overline{v}^{1\cdot\cdot} \not\models \neg f$
  - iff either $|v| \leq 1$ or $v^{1\cdot\cdot} \models f$
  - iff $v \models \mathsf{X}\, f$

□

THEOREM 16. *Let $f \in$ LTL^{sere}. Then*

(1) $\top^\omega \models f$.

(2) $\bot^\omega \not\models f$.

PROOF. By induction.

$f = r!$:
(1) By Lemma E.1, $\top^+ \subseteq \widehat{\mathcal{L}}(r)$. This proves that $\top^\omega \models r!$.
(2) $\bot^\omega \models r!$ iff $\exists k \geq 1$ s.t. $\bot^k \in \widehat{\mathcal{L}}(r)$ iff [Lemma E.1] FALSE
$f = r$:
(1) $\top^\omega \models r \Leftarrow \top^\omega \models r!$ iff [$\top^\omega \models r!$ from the case $f = r!$] TRUE
(2) $\bot^\omega \models r$
iff either $\bot^\omega \models r!$ or $\bot^\omega \in \mathcal{I}(r) \cup \mathcal{F}(r) \cup \{\epsilon\}$
iff [$\bot^\omega \not\models r!$ from the case $f = r!$]$\bot^\omega \in \mathcal{I}(r) \cup \mathcal{F}(r) \cup \{\epsilon\}$
iff [$\mathcal{F}(r), \{\epsilon\}$ consist of finite words]$\bot^\omega \in \mathcal{I}(r)$
iff [By Lemma E.1, no letter of a word in $\mathcal{I}(r)$ is $\bot$] FALSE
$f = \neg g$:
(1) $\top^\omega \models \neg g$ iff $\bot^\omega \not\models g$ iff [induction] TRUE
(2) $\bot^\omega \models \neg g$ iff $\top^\omega \not\models g$ iff [induction] FALSE
$f = g \wedge h$:
(1) $\top^\omega \models g \wedge h$ iff $\top^\omega \models g$ and $\top^\omega \models h$ iff [induction] TRUE
(2) $\bot^\omega \models g \wedge h$ iff $\bot^\omega \models g$ and $\bot^\omega \models h$ iff [induction] FALSE
$f = g \vee h$: Omitted as a derived form.
$f = \mathsf{X!}\, g$:
(1) $\top^\omega \models \mathsf{X!}\, g$ iff $\top^\omega \models g$ iff [induction] TRUE
(2) $\bot^\omega \models \mathsf{X!}\, g$ iff $\bot^\omega \models g$ iff [induction] FALSE
$f = \mathsf{X}\, g$: Omitted as a derived form.
$f = [g\ \mathsf{U}\ h]$:
(1) $\top^\omega \models [g\ \mathsf{U}\ h]$ iff $\top^\omega \models h$ iff [induction] TRUE
(2) $\bot^\omega \models [g\ \mathsf{U}\ h]$ iff $\bot^\omega \models h$ iff [induction] FALSE
$f = [g\ \mathsf{W}\ h]$:
(1) $\top^\omega \models [g\ \mathsf{W}\ h]$ iff either $\top^\omega \models [g\ \mathsf{U}\ h]$ or $\top^\omega \models g$ iff [induction] TRUE
(2) $\bot^\omega \models [g\ \mathsf{W}\ h]$ iff either $\bot^\omega \models [g\ \mathsf{U}\ h]$ or $\bot^\omega \models g$ iff [induction] FALSE

□

F. PROOF OF THEOREM 17

THEOREM 17. *$\mathcal{L}_w(r)$ is prefix closed.*

PROOF. Let $w \in \mathcal{L}_w(r)$, and $u \prec w$. Then $w \in \mathcal{L}(r) \cup \mathcal{I}(r) \cup (\mathcal{F}(r)\top^\infty) \cup \top^\infty$ and we have to show that $u \in \mathcal{L}(r) \cup \mathcal{I}(r) \cup (\mathcal{F}(r)\top^\infty) \cup \top^\infty$.

- If $w \in \mathcal{L}(r)$, then by Lemma D.4, $u \in \mathcal{F}(r)$.
- If $w \in \mathcal{I}(r)$, then by Lemma D.5, $u \in \mathcal{F}(r)$.
- If $w \in \mathcal{F}(r)\top^\infty$, then $w = w'w''$ s.t. $w' \in \mathcal{F}(r)$ and $w'' \in \top^\infty$. If $u \prec w'$, then by Lemma D.6, $u \in \mathcal{F}(r)$. If $u \succeq w'$, then $u \in (\mathcal{F}(r)\top^\infty)$.
- If $w \in \top^\infty$, then $u \in \top^\infty$.

□

G. PROOF OF THEOREM 18

THEOREM 18. *Let $w \in \widehat{\Sigma}^\omega$ and assume that w has infinitely many prefixes in $\mathcal{F}(r)$. Then $w \in \mathcal{I}(r)$.*

PROOF. By induction over the structure of r.

λ: $\mathcal{F}(\lambda) = \emptyset$, contradicting the assumption that w has infinitely many prefixes in $\mathcal{F}(\lambda)$.
b: $\mathcal{F}(b) = \{\epsilon\}$, contradicting the assumption that w has infinitely many prefixes in $\mathcal{F}(b)$.
$r \cdot s$: $\mathcal{F}(r \cdot s) = \mathcal{F}(r) \cup (\mathcal{L}(r)\mathcal{F}(s))$.
- Case: w has infinitely many prefixes in $\mathcal{F}(r) \cup \mathcal{L}(r)$. Then by Lemma D.4, there are infinitely many prefixes in $\mathcal{F}(r)$. By induction, $w \in \mathcal{I}(r) \subseteq \mathcal{I}(r \cdot s)$.
- Case: w has only finitely many prefixes in $\mathcal{F}(r) \cup \mathcal{L}(r)$. Then w has infinitely many prefixes in $\mathcal{L}(r)\mathcal{F}(s)$. In particular, w has prefixes in $\mathcal{L}(r)$, so let them be $u_1, \ldots, u_k$. Then for some i, w has infinitely many prefixes in $u_i\mathcal{F}(s)$. Let $w = u_iw'$. Then w' has infinitely many prefixes in $\mathcal{F}(s)$. By induction, $w' \in \mathcal{I}(s)$, hence $w \in \mathcal{L}(r)\mathcal{I}(s) \subseteq \mathcal{I}(r \cdot s)$.

$r \circ s$: $\mathcal{F}(r \circ s) = \mathcal{F}(r) \cup (\mathcal{L}(r) \circ \mathcal{F}(s))$.
- Case: w has infinitely many prefixes in $\mathcal{F}(r) \cup \mathcal{L}(r)$. Then by Lemma D.4, there are infinitely many prefixes in $\mathcal{F}(r)$. By induction, $w \in \mathcal{I}(r) \subseteq \mathcal{I}(r \circ s)$.
- Case: w has only finitely many prefixes in $\mathcal{F}(r) \cup \mathcal{L}(r)$. Then w has infinitely many prefixes in $\mathcal{L}(r) \circ \mathcal{F}(s)$. In particular, w has non-empty prefixes in $\mathcal{L}(r)$, so let them be $u_1, \ldots, u_k$. Then for some i, w has infinitely many prefixes in $u_i \circ \mathcal{F}(s)$. Let $w = u_i \circ w'$. Then w' has infinitely many prefixes in $\mathcal{F}(s)$. By induction, $w' \in \mathcal{I}(s)$, hence $w \in \mathcal{L}(r) \circ \mathcal{I}(s) \subseteq \mathcal{I}(r \circ s)$.

r^+: $\mathcal{F}(r^+) = \mathcal{L}(r)^*\mathcal{F}(r)$. Define an *$r$-chain* of w to be a sequence $u_1, u_2, \ldots$ (finite or infinite) such that:

(1) Each u_i is a prefix of w
(2) Each $u_i \in (\mathcal{L}(r) \setminus \{\epsilon\})^*$
(3) $u_{i+1} = u_iv$ for some $v \in \mathcal{L}(r) \setminus \{\epsilon\}$.

The prefix of w associated with the r-chain is the limit $\mathsf{lim}\, u_i$. For a finite r-chain $u_1, \ldots, u_k$, this limit is just u_k.

The finite r-chains of w can be assembled into a forest in the following way: r-chain V is a descendant of r-chain U iff U is a sequential prefix of V. To see that this defines a forest, note that two distinct successors of a node represent different choices of the next substring of w in $\mathcal{L}(r) \setminus \{\epsilon\}$ to concatenate to obtain the next element of the chain, and this difference is always preserved in descendants. Hence, paths following

different successors of a node never meet afterward.

Note that infinite r-chains correspond to infinite paths in the forest.

There are two cases:

(1) There is an infinite path in the forest. Then there is an infinite r-chain of w, hence $w \in (\mathcal{L}(r) \setminus \{\epsilon\})^\omega$, hence $w \in \mathcal{I}(r^+)$.
(2) Every path in the forest is finite. There are two cases:
   (a) Either there are infinitely many root nodes or there is a node in the forest with infinitely many successors. Then w can be written in the form uw', where $u \in (\mathcal{L}(r) \setminus \{\epsilon\})^*$ and w' has infinitely many prefixes in $\mathcal{L}(r)$. By Lemma D.4, w' has infinitely many prefixes in $\mathcal{F}(r)$, so by induction $w' \in \mathcal{I}(r)$. Then $w \in (\mathcal{L}(r) \setminus \{\epsilon\})^*\mathcal{I}(r) \subseteq \mathcal{I}(r^+)$.
   (b) There are only finitely many root nodes and each node is finitely branching. By König's Lemma, the forest itself is finite (because otherwise we would have an infinite path). Therefore w has only finitely many prefixes in $(\mathcal{L}(r) \setminus \{\epsilon\})^*$. Let them be $U_1, \ldots, U_k$. Since w has infinitely many prefixes in $\mathcal{L}(r)^*\mathcal{F}(r)$, for some i, w has infinitely many prefixes in $U_i\mathcal{F}(r)$. Let $w = U_iw'$. Then w' has infinitely many prefixes in $\mathcal{F}(r)$. By induction $w' \in \mathcal{I}(r)$, hence $w \in (\mathcal{L}(r) \setminus \{\epsilon\})^*\mathcal{I}(r) \subseteq \mathcal{I}(r^+)$.

$r \cup s$: $\mathcal{F}(r \cup s) = \mathcal{F}(r) \cup \mathcal{F}(s)$. Without loss of generality, w has infinitely many prefixes in $\mathcal{F}(r)$. By induction, $w \in \mathcal{I}(r) \subseteq \mathcal{I}(r \cup s)$.
$r \cap s$: $\mathcal{F}(r \cap s) = \mathcal{F}(r) \cap \mathcal{F}(s)$. Therefore w has infinitely many prefixes in $\mathcal{F}(r)$ and in $\mathcal{F}(s)$. By induction, $w \in \mathcal{I}(r) \cap \mathcal{I}(s) = \mathcal{I}(r \cap s)$.

□

H. PROOF OF THEOREM 19

LEMMA H.1.

- $\mathcal{L}_w(r) \cap \widehat{\Sigma}^* = \mathcal{L}(r) \cup (\mathcal{F}(r)\top^*) \cup \top^*$.
- $\mathcal{L}_w(r) \cap \widehat{\Sigma}^\omega = \mathcal{I}(r) \cup (\mathcal{F}(r)\top^\omega) \cup \top^\omega$.

PROOF. By induction, $\mathcal{L}(r)$ and $\mathcal{F}(r)$ consist only of finite words and by Lemma D.1, $\mathcal{I}(r)$ consists only of infinite words. The rest follows from the definition of $\mathcal{L}_w(r)$. □

THEOREM 19. *$\mathcal{L}_w(r)$ is limit closed.*

PROOF. Assume that $w \in \widehat{\Sigma}^\infty$ and that every finite prefix of w is in $\mathcal{L}_w(r)$. We must show that $w \in \mathcal{L}_w(r)$. If w is finite, then $w \in \mathcal{L}_w(r)$ by assumption. Therefore, we assume w is infinite in the argument below.

- Case: $w = u\top^\omega$, where u is finite. Assume without loss of generality that the last letter of u is not $\top$ (u may be empty). Now, $u \in \mathcal{L}_w(r)$ by assumption, and since u is finite and its last letter is not $\top$, we have that $u \in \mathcal{L}(r) \cup \mathcal{F}(r) \cup \{\epsilon\}$. If $u \in \mathcal{F}(r) \cup \{\epsilon\}$, then clearly $u\top^\omega = w \in \mathcal{L}_w(r)$ and we are done. Otherwise we have by assumption that $u\top \in \mathcal{L}_w(r)$ thus by Lemma H.1 we have that $u\top \in \mathcal{L}(r) \cup (\mathcal{F}(r)\top^*) \cup \top^*$. But $u \notin \mathcal{F}(r) \cup \{\epsilon\}$, thus $u\top \in \mathcal{L}(r)$ and by Lemma D.4, $u \in \mathcal{F}(r)$, thus $u\top^\omega = w \in \mathcal{L}_w(r)$.
- Case: w cannot be written in the form $u\top^\omega$, where u is finite. This means that w has infinitely many non-empty finite prefixes whose last letter is not $\top$. According to Lemma H.1, we must show that $w \in \mathcal{I}(r)$. Let u be a non-empty finite prefix of w whose last letter is not $\top$. Then $u \in \mathcal{L}_w(r)$, so by Lemma H.1, $u \in \mathcal{L}(r) \cup \mathcal{F}(r)$. Let $u \prec v$ where v is also a finite prefix of w whose last letter is not $\top$. Then also $v \in \mathcal{L}(r) \cup \mathcal{F}(r)$. By

Lemmas D.4 and D.6, it follows that $u \in \mathcal{F}(r)$. Therefore, w has infinitely many prefixes in $\mathcal{F}(r)$. By Theorem 18, $w \in \mathcal{I}(r)$, as desired.

□

I. PROOF OF THEOREM 20

LEMMA I.1.

$$\mathcal{L}_w(r) = \widehat{\mathcal{L}}(r) \cup \mathcal{I}(r) \cup (\mathcal{F}(r)(\{\epsilon\} \cup \{\top^\omega\})) \cup (\{\epsilon\} \cup \{\top^\omega\}).$$

PROOF. Note that $\top^\infty \setminus \top^+ = \{\epsilon\} \cup \{\top^\omega\}$. The result then follows from Definitions 21 and 22. □

OBSERVATION I.2. $\widehat{\mathcal{L}}(r) \subseteq \mathcal{L}_w(r)$.

LEMMA I.3. *Let r be a* SERE. *Then $\mathcal{L}_w(r)$ is closed and $cl(\widehat{\mathcal{L}}(r)) \subseteq \mathcal{L}_w(r)$.*

PROOF. Theorems 17 and 19 show that $\mathcal{L}_w(r)$ is prefix closed and limit closed. By Proposition 2, $\mathcal{L}_w(r)$ is closed. By Observation I.2, $\widehat{\mathcal{L}}(r) \subseteq \mathcal{L}_w(r)$. Since $\mathcal{L}_w(r)$ is closed, $cl(\widehat{\mathcal{L}}(r)) \subseteq \mathcal{L}_w(r)$. □

LEMMA I.4. $\mathit{fpref}(\mathcal{L}_w(r)) = \mathit{fpref}(\widehat{\mathcal{L}}(r))$.

PROOF. Since $\widehat{\mathcal{L}}(r) \subseteq \mathcal{L}_w(r)$ (Observation I.2), $\mathit{fpref}(\widehat{\mathcal{L}}(r)) \subseteq \mathit{fpref}(\mathcal{L}_w(r))$. By Theorem 17, $\mathcal{L}_w(r)$ is prefix closed, so $\mathit{fpref}(\mathcal{L}_w(r)) = \mathcal{L}_w(r) \cap \widehat{\Sigma}^* =$ [By Lemma H.1] $\mathcal{L}(r) \cup (\mathcal{F}(r)\top^*) \cup \top^* \subseteq \mathit{fpref}(\widehat{\mathcal{L}}(r))$, the last since $\widehat{\mathcal{L}}(r) = \mathcal{L}(r) \cup (\mathcal{F}(r)\top^+) \cup \top^+$. □

THEOREM 20. *Let r be a* SERE. *Then $cl(\widehat{\mathcal{L}}(r)) = \mathcal{L}_w(r)$.*

PROOF.

$$\begin{aligned}
cl(\widehat{\mathcal{L}}(r)) &= \{w \in \widehat{\Sigma}^\infty \mid \forall \text{ finite } u \preceq w : \exists v \succeq u : v \in \widehat{\mathcal{L}}(r)\} \\
&= \{w \in \widehat{\Sigma}^\infty \mid \mathit{fpref}(w) \subseteq \mathit{fpref}(\widehat{\mathcal{L}}(r))\} \\
&= [\mathit{fpref}(\widehat{\mathcal{L}}(r)) = \mathit{fpref}(\mathcal{L}_w(r)) \text{ by Lemma I.4}] \\
&\quad \{w \in \widehat{\Sigma}^\infty \mid \mathit{fpref}(w) \subseteq \mathit{fpref}(\mathcal{L}_w(r))\} \\
&= \{w \in \widehat{\Sigma}^\infty \mid \forall \text{ finite } u \preceq w : \exists v \succeq u : v \in \mathcal{L}_w(r)\} \\
&= cl(\mathcal{L}_w(r)) \\
&= [\mathcal{L}_w(r) \text{ is closed by Lemma I.3}] \\
&\quad \mathcal{L}_w(r)
\end{aligned}$$

□

J. PROOF OF THEOREM 21

LEMMA J.1.

$$((\widehat{\mathcal{L}}(r) \setminus \{\epsilon\})\widehat{\Sigma}^\infty) \cup \mathcal{I}(r) \cup \mathcal{F}(r) \cup \{\epsilon\} = ((\widehat{\mathcal{L}}(r) \setminus \{\epsilon\})\widehat{\Sigma}^\infty) \cup \mathcal{L}_w(r).$$

PROOF. Clearly, $\mathcal{I}(r) \cup \mathcal{F}(r) \cup \{\epsilon\} \subseteq \mathcal{L}_w(r)$, so the inclusion $\subseteq$ holds. Suppose $w \in \mathcal{L}_w(r)$. By Lemma I.1, $w \in \widehat{\mathcal{L}}(r) \cup \mathcal{I}(r) \cup (\mathcal{F}(r)(\{\epsilon\} \cup \{\top^\omega\})) \cup (\{\epsilon\} \cup \{\top^\omega\}) = \widehat{\mathcal{L}}(r) \cup \mathcal{I}(r) \cup \mathcal{F}(r) \cup (\mathcal{F}(r)\top^\omega) \cup \{\epsilon\} \cup \{\top^\omega\}$.

- If $w \in \widehat{\mathcal{L}}(r)$, then $w \in (\widehat{\mathcal{L}}(r) \setminus \{\epsilon\}) \cup \{\epsilon\} \subseteq ((\widehat{\mathcal{L}}(r) \setminus \{\epsilon\})\widehat{\Sigma}^\infty) \cup \{\epsilon\}$.
- The cases $w \in \mathcal{I}(r)$ and $w \in \mathcal{F}(r)$ are trivial.
- If $w \in \mathcal{F}(r)\top^\omega$, then $w = u\top^\omega$, where $u \in \mathcal{F}(r)$. Then by Definition 21, $u\top \in \widehat{\mathcal{L}}(r) \setminus \{\epsilon\}$, hence $u\top^\omega \in (\widehat{\mathcal{L}}(r) \setminus \{\epsilon\})\widehat{\Sigma}^\infty$.

- The case $w \in \{\epsilon\}$ is trivial.
- If $w \in \{\top^\omega\}$, then $w \in (\widehat{\mathcal{L}}(r) \setminus \{\epsilon\})\widehat{\Sigma}^\infty$ because $\top^+ \subseteq \widehat{\mathcal{L}}(r)$ by Definition 21.

□

THEOREM 21. *$w \models r$ iff $w \models r!$ or $w \in \mathcal{L}_w(r)$.*

PROOF.

$$\begin{array}{rll} w \models r & \text{iff [Definition 23]} & \text{either } w \models r! \text{ or } w \in \mathcal{I}(r) \cup \mathcal{F}(r) \cup \{\epsilon\} \\ & \text{iff [Definition 23]} & w \in (\widehat{\mathcal{L}}(r) \setminus \{\epsilon\})\widehat{\Sigma}^\infty \cup \mathcal{I}(r) \cup \mathcal{F}(r) \cup \{\epsilon\} \\ & \text{iff [Lemma J.1]} & w \in (\widehat{\mathcal{L}}(r) \setminus \{\epsilon\})\widehat{\Sigma}^\infty \cup \mathcal{L}_w(r) \\ & \text{iff [Definition 23]} & \text{either } w \models r! \text{ or } w \in \mathcal{L}_w(r) \end{array}$$

□

K. PROOF OF THEOREM 22

LEMMA K.1. *Let $X \subseteq \widehat{\Sigma}^\infty$ be a set of words that contains an element of the form $\top^k$, $k \geq 1$, and that is closed under switching letters to $\top$. Let $w \in \widehat{\Sigma}^\infty$. The following are equivalent*:

(1) *For every finite $u \preceq w$, there exists a nonempty finite prefix of $u\top^\omega$ that is in X.*
(2) *Either there exists a nonempty finite prefix of w that is in X or $w \in cl(X)$.*

PROOF. By Proposition 3, $cl(X)$ is the set

$$\{w \in \widehat{\Sigma}^\infty \mid \forall \text{ finite } u \preceq w : \exists v \succeq u : v \in X\}.$$

($\Rightarrow$) Assume (1). Suppose there is no nonempty finite prefix of w that is in X. Let $u \preceq w$ be finite. By (1), there exists nonempty finite $v \preceq u\top^\omega$ such that $v \in X$. We cannot have $v \preceq u$ because w has no such prefix. Therefore, $v \succeq u$. This shows that $w \in cl(X)$.

($\Leftarrow$) Assume (2). Let $u \preceq w$ be finite.

Suppose first that there exists nonempty finite $v \preceq w$ such that $v \in X$. If $u \preceq v$, then $u\top^{|v|-|u|} \in X \setminus \{\epsilon\}$, which shows that $u\top^\omega$ has a nonempty finite prefix in X. Otherwise, $v \preceq u \preceq u\top^\omega$, hence v is a nonempty finite prefix of $u\top^\omega$ that is in X.

Suppose now that $w \in cl(X)$. Then there exists $v \succeq u$ such that $v \in X$. If u is nonempty then v is nonempty. Otherwise, we may select v nonempty by choosing $v = \top^k$ for some $k \geq 1$. Then $u\top^{|v|-|u|} \in X \setminus \{\epsilon\}$, which shows that $u\top^\omega$ has a nonempty finite prefix that is in X. □

LEMMA K.2. *Let r be a* SERE *and $w \in \widehat{\Sigma}^\infty$. The following are equivalent*:

(1) *For every finite $u \preceq w : u\top^\omega \models r!$.*
(2) *$w \models r!$ or $w \in cl(\widehat{\mathcal{L}}(r))$.*

PROOF. Follows from Definition 23, Lemmas E.1 and D.2, and Lemma K.1. □

THEOREM 22. *Let r be a* SERE *and $w \in \widehat{\Sigma}^\infty$. The following are equivalent*:

(1) *For every finite $u \preceq w : u\top^\omega \models r!$.*
(2) *$w \models r$.*

PROOF. Follows from Theorems 20 and 21 and Lemma K.2. □

L. PROOF OF THEOREM 23

THEOREM 23. *Let $f \in \text{LTL}^{sere}$ and let $w \in \widehat{\Sigma}^\infty$. Then*

(1) $w \models f \Rightarrow w\top^\omega \models f$.
(2) $w\bot^\omega \models f \Rightarrow w \models f$.

PROOF. If w is infinite, then $w\top^\omega = w = w\bot^\omega$, so both (1) and (2) hold trivially. Assume therefore that w is finite. The proof is by induction over the structure of f. The cases $f = g \vee h$ and $f = \mathsf{X}\, g$ are omitted as derived forms.

$f = r!$:

$$\begin{array}{rll} w \models r! & \text{iff} & \exists u \preceq w : u \in \widehat{\mathcal{L}}(r) \setminus \{\epsilon\} \\ & \Rightarrow & \exists u \preceq w\top^\omega : u \in \widehat{\mathcal{L}}(r) \setminus \{\epsilon\} \\ & \text{iff} & w\top^\omega \models r! \\ w\bot^\omega \models r! & \text{iff} & \exists u \preceq w\bot^\omega : u \in \widehat{\mathcal{L}}(r) \setminus \{\epsilon\} \\ & \Rightarrow & [\text{Lemma E.1}]\ \exists u \preceq w : u \in \widehat{\mathcal{L}}(r) \setminus \{\epsilon\} \\ & \text{iff} & w \models r! \end{array}$$

$f = r$:

$$\begin{array}{rll} w \models r & \text{iff} & [\text{Theorem 22}]\ \forall \text{ finite } u \preceq w, u\top^\omega \models r! \\ & \Rightarrow & [w \text{ is finite, let } u = w]\ w\top^\omega \models r! \\ & \Rightarrow & w\top^\omega \models r \\ w\bot^\omega \models r & \text{iff} & [\text{Theorem 22}]\ \forall \text{ finite } u \preceq w\bot^\omega, u\top^\omega \models r! \\ & \Rightarrow & \forall \text{ finite } u \preceq w, u\top^\omega \models r! \\ & \text{iff} & [\text{Theorem 22}]\ w \models r \end{array}$$

$f = \neg g$:

$$\begin{array}{rlllll} w\top^\omega \not\models \neg g & \text{iff} & \overline{w}\bot^\omega \models g & \Rightarrow [\text{induction}] & \overline{w} \models g & \text{iff} \quad w \not\models \neg g \\ w \not\models \neg g & \text{iff} & \overline{w} \models g & \Rightarrow [\text{induction}] & \overline{w}\top^\omega \models g & \text{iff} \quad w\bot^\omega \not\models \neg g \end{array}$$

$f = g \wedge h$:

$$\begin{array}{rll} w \models g \wedge h & \text{iff} & w \models g \text{ and } w \models h \\ & \Rightarrow & [\text{induction}]\ w\top^\omega \models g \text{ and } w\top^\omega \models h \\ & \text{iff} & w\top^\omega \models g \wedge h \\ w\bot^\omega \models g \wedge h & \text{iff} & w\bot^\omega \models g \text{ and } w\bot^\omega \models h \\ & \Rightarrow & [\text{induction}]\ w \models g \text{ and } w \models h \\ & \text{iff} & w \models g \wedge h \end{array}$$

$f = \mathsf{X}!\ g$:

$$\begin{array}{rll} w \models \mathsf{X}!\ g & \text{iff} & |w| > 1 \text{ and } w^{1\cdot\cdot} \models g \\ & \Rightarrow & [\text{induction}]\ |w| > 1 \text{ and } w^{1\cdot\cdot}\top^\omega \models g \\ & \Rightarrow & [w^{1\cdot\cdot}\top^\omega = (w\top^\omega)^{1\cdot\cdot}]\ |w\top^\omega| > 1 \text{ and } (w\top^\omega)^{1\cdot\cdot} \models g \\ & \text{iff} & w\top^\omega \models \mathsf{X}!\ g \\ w\bot^\omega \models \mathsf{X}!\ g & \text{iff} & |w\bot^\omega| > 1 \text{ and } (w\bot^\omega)^{1\cdot\cdot} \models g \\ & \text{iff} & [|w\bot^\omega| = \omega,\ (w\bot^\omega)^{1\cdot\cdot} = w^{1\cdot\cdot}\bot^\omega (w^{1\cdot\cdot} = \epsilon \text{ if } |w| \leq 1)]\ w^{1\cdot\cdot}\bot^\omega \models g \\ & \text{iff} & [\bot^\omega \not\models g \text{ by Theorem 16}]\ |w| > 1 \text{ and } w^{1\cdot\cdot}\bot^\omega \models g \\ & \Rightarrow & [\text{induction}]\ |w| > 1 \text{ and } w^{1\cdot\cdot} \models g \\ & \text{iff} & w \models \mathsf{X}!\ g \end{array}$$

$f = [g\ \mathsf{U}\ h]$:
$w \models [g\ \mathsf{U}\ h]$

iff $\exists k,\ 0 \leq k < |w|$ such that $w^{k\cdot\cdot} \models h$ and $\forall j,\ 0 \leq j < k, w^{j\cdot\cdot} \models g$
$\Rightarrow$ [induction] $\exists k,\ 0 \leq k < |w|$ such that $w^{k\cdot\cdot}\top^\omega \models h$ and $\forall j,\ 0 \leq j < k, w^{j\cdot\cdot}\top^\omega \models g$
iff $[v^{i\cdot\cdot}\top^\omega = (v\top^\omega)^{i\cdot\cdot}$ (if $i \geq |v|$ then $v^{i\cdot\cdot} = \epsilon)]$ $\exists k,\ 0 \leq k < |w|$ such that $(w\top^\omega)^{k\cdot\cdot} \models h$ and $\forall j,\ 0 \leq j < k, (w\top^\omega)^{j\cdot\cdot} \models g$
$\Rightarrow$ [$\exists k < |w|$ implies that $\exists k < |w\top^\omega|$] $w\top^\omega \models [g \mathsf{U} h]$

$w\bot^\omega \models [g \mathsf{U} h]$
iff $\exists k \geq 0$ such that $(w\bot^\omega)^{k\cdot\cdot} \models h$ and $\forall j,\ 0 \leq j < k, (w\bot^\omega)^{j\cdot\cdot} \models g$
iff [$\bot^\omega \not\models h$ by Theorem 16] $\exists k,\ 0 \leq k < |w|$ such that $(w\bot^\omega)^{k\cdot\cdot} \models h$ and $\forall j,\ 0 \leq j < k, (w\bot^\omega)^{j\cdot\cdot} \models g$
iff $[v^{i\cdot\cdot}\bot^\omega = (v\bot^\omega)^{i\cdot\cdot}$ (if $i \geq |v|$ then $v^{i\cdot\cdot} = \epsilon)]$ $\exists k,\ 0 \leq k < |w|$ such that $w^{k\cdot\cdot}\bot^\omega \models h$ and $\forall j,\ 0 \leq j < k, w^{j\cdot\cdot}\bot^\omega \models g$
$\Rightarrow$ [induction] $\exists k,\ 0 \leq k < |w|$ such that $w^{k\cdot\cdot} \models h$ and $\forall j,\ 0 \leq j < k, w^{j\cdot\cdot} \models g$
iff $w \models [g \mathsf{U} h]$

$f = [g \mathsf{W} h]$:
$w \models [g \mathsf{W} h]$
iff either $w \models [g \mathsf{U} h]$ or $\forall k,\ 0 \leq k < |w| : w^{k\cdot\cdot} \models g$
$\Rightarrow$ [induction] either $w\top^\omega \models [g \mathsf{U} h]$ or $\forall k,\ 0 \leq k < |w| : w^{k\cdot\cdot}\top^\omega \models g$
iff $[v^{i\cdot\cdot}\top^\omega = (v\top^\omega)^{i\cdot\cdot}$ (if $i \geq |v|$ then $v^{i\cdot\cdot} = \epsilon)]$ either $w\top^\omega \models [g \mathsf{U} h]$ or $\forall k,\ 0 \leq k < |w| : (w\top^\omega)^{k\cdot\cdot} \models g$
iff [$\top^\omega \models g$ by Theorem 16] either $w\top^\omega \models [g \mathsf{U} h]$ or $\forall k,\ 0 \leq k < |w\top^\omega| : (w\top^\omega)^{k\cdot\cdot} \models g$
iff $w\top^\omega \models [g \mathsf{W} h]$

$w\bot^\omega \models [g \mathsf{W} h]$
iff either $w\bot^\omega \models [g \mathsf{U} h]$ or $\forall k \geq 0 : (w\bot^\omega)^{k\cdot\cdot} \models g$
iff [$\bot^\omega \not\models g$ by Theorem 16] $w\bot^\omega \models [g \mathsf{U} h]$
$\Rightarrow$ [induction] $w \models [g \mathsf{U} h]$
$\Rightarrow w \models [g \mathsf{W} h]$

□

M. PROOF OF THEOREM 24

LEMMA M.1. *Let $f \in \text{LTL}^{sere}$, and let $w \in \widehat{\Sigma}^\infty$ be such that $w \models f$.*

(1) *If u results from w by switching (0 or more) letters to $\top$, then $u \models f$.*
(2) *If v results from w by switching (0 or more) letters away from $\bot$, then $v \models f$.*

PROOF. By induction. The cases $f = g \vee h$ and $f = \mathsf{X}\, g$ are omitted as derived forms.

$f = r!$: Since $w \models r!$, there exists non-empty $x \preceq w : x \in \widehat{\mathcal{L}}(r)$. Let y be the prefix of u of the same length as x. By Lemma D.2, $y \in \widehat{\mathcal{L}}(r)$, and so $u \models r!$. Let z be the prefix of v of the same length as x. By Lemma E.1, no letter of x is $\bot$, so $z = x \in \widehat{\mathcal{L}}(r)$, and so $v \models r!$.

$f = r$: Since $w \models r$, by Theorem 22, $\forall$ finite $x \preceq w : x\top^\omega \models r!$.
Let y be a finite prefix of u. Let x_y be the finite prefix of w of the same length as y. Then y results from x_y by switching letters to $\top$, so by induction $y\top^\omega \models r!$. Since y is arbitrary, Theorem 22 implies that $u \models r$.
Let z be a finite prefix of v. Let x_z be the finite prefix of w of the same length as z. Then z results from x_z by switching letters away from $\bot$, so by induction $z\top^\omega \models r!$.

Since z is arbitrary, Theorem 22 implies that $v \models r$.

$f = \neg g$:
$w \models \neg g$
iff $\overline{w} \not\models g$
$\Rightarrow$ [induction; $\overline{w}$ results from $\overline{u}$ by switching letters away from $\bot$; $\overline{w}$ results from $\overline{v}$ by switching letters to $\top$] both $\overline{u} \not\models g$ and $\overline{v} \not\models g$
iff both $u \models \neg g$ and $v \models \neg g$

$f = g \wedge h$:
$w \models g \wedge h$
iff $w \models g$ and $w \models h$
$\Rightarrow$ [induction] both $u \models g$ and $u \models h$ and $v \models g$ and $v \models h$
iff both $u \models g \wedge h$ and $v \models g \wedge h$

$f = \mathsf{X!}\ g$:
$w \models \mathsf{X!}\ g$
iff $|w| > 1$ and $w^{1\cdot\cdot} \models g$
$\Rightarrow$ [induction] both $|u| > 1$ and $u^{1\cdot\cdot} \models g$ and $|v| > 1$ and $v^{1\cdot\cdot} \models g$
iff both $u \models \mathsf{X!}\ g$ and $v \models \mathsf{X!}\ g$

$f = [g\ \mathsf{U}\ h]$:
$w \models [g\ \mathsf{U}\ h]$
iff $\exists k,\ 0 \leq k < |w| : w^{k\cdot\cdot} \models h$ and $\forall j,\ 0 \leq j < k : w^{j\cdot\cdot} \models g$
$\Rightarrow$ [induction] both $\exists k,\ 0 \leq k < |u| : u^{k\cdot\cdot} \models h$ and $\forall j,\ 0 \leq j < k : u^{j\cdot\cdot} \models g$ and $\exists k,\ 0 \leq k < |v| : v^{k\cdot\cdot} \models h$ and $\forall j,\ 0 \leq j < k : v^{j\cdot\cdot} \models g$
iff both $u \models [g\ \mathsf{U}\ h]$ and $v \models [g\ \mathsf{U}\ h]$

$f = [g\ \mathsf{W}\ h]$:
$w \models [g\ \mathsf{W}\ h]$
iff either $w \models [g\ \mathsf{U}\ h]$ or $\forall k,\ 0 \leq k < |w| : w^{k\cdot\cdot} \models g$
$\Rightarrow$ [induction] both (either $u \models [g\ \mathsf{U}\ h]$ or $\forall k,\ 0 \leq k < |u| : u^{k\cdot\cdot} \models g$) and (either $v \models [g\ \mathsf{U}\ h]$ or $\forall k,\ 0 \leq k < |v| : v^{k\cdot\cdot} \models g$)
iff both $u \models [g\ \mathsf{W}\ h]$ and $v \models [g\ \mathsf{W}\ h]$

□

Corollary M.2 (Prefix/Extension). *Let $f \in \text{LTL}^{sere}$, and let $u, v, w \in \widehat{\Sigma}^\infty$.*

(1) $w\top^\omega \models f$ *iff* $\forall u \preceq w : u\top^\omega \models f$.
(2) $w\bot^\omega \models f$ *iff* $\forall v \succeq w : v\bot^\omega \models f$.

Proof. The ($\Leftarrow$) directions are trivial. The ($\Rightarrow$) directions follow from Lemma M.1. □

Lemma M.3. *Let $f \in \text{LTL}^{sere}$, and let $u, v, w \in \widehat{\Sigma}^\infty$. Then*

(1) $w\top^\omega \models f$ *iff* $\exists v \succeq w : v \models f$.
(2) $w\bot^\omega \models f$ *iff* $\forall v \succeq w : v \models f$.

Proof.

(1) ($\Rightarrow$) Trivial.
($\Leftarrow$) Let $v \succeq w$ be such that $v \models f$. By Strength Relation (Theorem 23), $v\top^\omega \models f$. By Prefix/Extension (Corollary M.2) or by Lemma M.1 directly, $w\top^\omega \models f$.

(2) ($\Rightarrow$) Assume that $w\bot^\omega \models f$. By Prefix/Extension (Corollary M.2), $\forall v \succeq w : v\bot^\omega \models f$. Then by Strength Relation (Theorem 23), $\forall v \succeq w : v \models f$.
($\Leftarrow$) Trivial.

□

THEOREM 24. *Let $f \in \text{LTL}^{sere}$. Then*

(1) $cl(\llbracket f \rrbracket) = \{w \in \widehat{\Sigma}^\infty \mid \forall \textit{ finite } u \preceq w : u\top^\omega \in \llbracket f \rrbracket\}$.
(2) $int(\llbracket f \rrbracket) = \{w \in \widehat{\Sigma}^\infty \mid \exists \textit{ finite } u \preceq w : u\bot^\omega \in \llbracket f \rrbracket\}$.

PROOF.

$$\begin{aligned} cl(\llbracket f \rrbracket) &= \text{[Proposition 3]} \ \{w \in \widehat{\Sigma}^\infty \mid \forall \text{ finite } u \preceq w : \exists v \succeq u \text{ s.t. } v \in \llbracket f \rrbracket\} \\ &= \text{[Lemma M.3]} \quad \{w \in \widehat{\Sigma}^\infty \mid \forall \text{ finite } u \preceq w : u\top^\omega \in \llbracket f \rrbracket\} \end{aligned}$$

This proves (1). A dual argument proves (2). □

N. PROOF OF THEOREM 25

THEOREM 25. *Let r be a* SERE. *Then*

- $\llbracket r \rrbracket = cl(\llbracket r! \rrbracket) = cl(\llbracket r \rrbracket)$.
- $\llbracket r! \rrbracket = int(\llbracket r \rrbracket) = int(\llbracket r! \rrbracket)$.

In particular, r is semantically weak and $r!$ is semantically strong.

PROOF. The *cl* and *int* operators are idempotent, so in each case the second equality follows from the first. From Definitions 24 and 26, the outer equality $\llbracket r \rrbracket = cl(\llbracket r \rrbracket)$ implies r is semantically weak and the outer equality $\llbracket r! \rrbracket = int(\llbracket r! \rrbracket)$ implies $r!$ is semantically strong. It remains to prove the first equalities.

- $w \in cl(\llbracket r! \rrbracket)$
  iff [Theorem 24] $\forall$ finite $u \preceq w : u\top^\omega \models r!$
  iff [Theorem 22] $w \in \llbracket r \rrbracket$

- $w \in int(\llbracket r \rrbracket)$
  iff [Theorem 24] $\exists$ finite $u \preceq w : u\bot^\omega \models r$
  iff [Definition 23] $\exists$ finite $u \preceq w : u\bot^\omega \in ((\widehat{\mathcal{L}}(r) \setminus \{\epsilon\})\widehat{\Sigma}^\infty) \cup \mathcal{I}(r) \cup \mathcal{F}(r) \cup \{\epsilon\}$
  iff [$\mathcal{F}(r) \cup \{\epsilon\} \subseteq \widehat{\Sigma}^*$; no letter of any word in $\mathcal{I}(r)$ is $\bot$ by Lemma E.1]
  $\exists$ finite $u \preceq w : u\bot^\omega \in (\widehat{\mathcal{L}}(r) \setminus \{\epsilon\})\widehat{\Sigma}^\infty$
  iff [no letter of any word in $\widehat{\mathcal{L}}(r)$ is $\bot$ by Lemma E.1]
  $\exists$ finite $u \preceq w : \exists v \preceq u : v \in \widehat{\mathcal{L}}(r) \setminus \{\epsilon\}$
  iff $\exists$ finite $v \preceq w : v \in \widehat{\mathcal{L}}(r) \setminus \{\epsilon\}$
  iff $w \in \llbracket r! \rrbracket$

□

O. PROOF OF THEOREM 26

LEMMA O.1. *Let g, $h \in \text{LTL}^{sere}$.*

- $cl([\![g \wedge h]\!]) = cl([\![g]\!]) \cap cl([\![h]\!])$
- $cl([\![g \vee h]\!]) = cl([\![g]\!]) \cup cl([\![h]\!])$
- $cl([\![X\ g]\!]) = cl([\![X!\ g]\!])$
- $cl([\![[g\ W\ h]]\!]) = cl([\![[g\ U\ h]]\!])$
- $int([\![g \wedge h]\!]) = int([\![g]\!]) \cap int([\![h]\!])$
- $int([\![g \vee h]\!]) = int([\![g]\!]) \cup int([\![h]\!])$
- $int([\![X\ g]\!]) = int([\![X!\ g]\!])$
- $int([\![[g\ W\ h]]\!]) = int([\![[g\ U\ h]]\!])$

PROOF.

- $w \in cl([\![g \wedge h]\!])$
  iff [Theorem 24] $\forall$ finite $u \preceq w : u\top^\omega \models g \wedge h$
  iff $\forall$ finite $u \preceq w : u\top^\omega \models g$ and $u\top^\omega \models h$
  iff ($\forall$ finite $u \preceq w : u\top^\omega \models g$) and ($\forall$ finite $u \preceq w : u\top^\omega \models h$)
  iff [Theorem 24] $w \in cl([\![g]\!]) \cap cl([\![h]\!])$
  A dual argument shows that $int([\![g \vee h]\!]) = int([\![g]\!]) \cup int([\![h]\!])$.

- $int([\![g \wedge h]\!]) = int([\![g]\!]) \cap int([\![h]\!])$ because $int(X \cap Y) = int(X) \cap int(Y)$ in any topological space. A dual argument shows that $cl([\![g \vee h]\!]) = cl([\![g]\!]) \cup cl([\![h]\!])$.

- $w \in cl([\![\mathsf{X}\ g]\!])$
  iff [Theorem 24] $\forall$ finite $u \preceq w : u\top^\omega \models \mathsf{X}\ g$
  iff [$|u\top^\omega| > 1$] $\forall$ finite $u \preceq w : u\top^\omega \models \mathsf{X!}\ g$
  iff [Theorem 24] $w \in cl([\![\ \mathsf{X!}\ g]\!])$

- $w \in int([\![\mathsf{X}\ g]\!])$
  iff [Theorem 24] $\exists$ finite $u \preceq w : u\bot^\omega \models \mathsf{X}\ g$
  iff [$|u\bot^\omega| > 1$] $\exists$ finite $u \preceq w : u\bot^\omega \models \mathsf{X!}\ g$
  iff [Theorem 24] $w \in int([\![\mathsf{X!}\ g]\!])$

- $w \in cl([\![[g\ \mathsf{W}\ h]]\!])$
  iff [Theorem 24] $\forall$ finite $u \preceq w : u\top^\omega \models [g\ \mathsf{W}\ h]$
  iff $\forall$ finite $u \preceq w :$ either $u\top^\omega \models [g\ \mathsf{U}\ h]$ or $\forall i \geq 0 : u^{i\cdot\cdot}\top^\omega \models g$
  iff $\forall$ finite $u \preceq w :$ either ($\exists k \geq 0 : u^{k\cdot\cdot}\top^\omega \models h$ and $\forall j,\ 0 \leq j < k : u^{j\cdot\cdot}\top^\omega \models g$) or $\forall i \geq 0 : u^{i\cdot\cdot}\top^\omega \models g$
  iff [($\Rightarrow$) Let $u \preceq w$ be finite. Suppose that $\forall i \geq 0 : u^{i\cdot\cdot}\top^\omega \models g$. Pick $k = |u|$. Then $u^{k\cdot\cdot} = \epsilon$, so $u^{k\cdot\cdot}\top^\omega = \top^\omega$, which satisfies h by Theorem 16.]
  $\forall$ finite $u \preceq w : \exists k \geq 0 : u^{k\cdot\cdot}\top^\omega \models h$ and $\forall j,\ 0 \leq j < k : u^{j\cdot\cdot}\top^\omega \models g$
  iff $\forall$ finite $u \preceq w : u\top^\omega \models [g\ \mathsf{U}\ h]$
  iff [Theorem 24] $w \in cl([\![[g\ \mathsf{U}\ h]]\!])$

- $w \in int([\![[g\ \mathsf{W}\ h]]\!])$
  iff [Theorem 24] $\exists$ finite $u \preceq w : u\bot^\omega \models [g\ \mathsf{W}\ h]$
  iff $\exists$ finite $u \preceq w :$ either $u\bot^\omega \models [g\ \mathsf{U}\ h]$ or $\forall i \geq 0 : u^{i\cdot\cdot}\bot^\omega \models g$
  iff $\exists$ finite $u \preceq w :$ either ($\exists k \geq 0 : u^{k\cdot\cdot}\bot^\omega \models h$ and $\forall j,\ 0 \leq j < k : u^{j\cdot\cdot}\bot^\omega \models g$) or ($\forall i \geq 0 : u^{i\cdot\cdot}\bot^\omega \models g$)
  iff [($\Rightarrow$) Let $u \preceq w$ be finite. Pick $i = |u|$. Then $u^{i\cdot\cdot} = \epsilon$ so $u^{i\cdot\cdot}\bot^\omega = \bot^\omega$, which does not satisfy g by Theorem 16.] $\exists$ finite $u \preceq w : \exists k \geq 0 : u^{k\cdot\cdot}\bot^\omega \models h$ and $\forall j,\ 0 \leq j < k : u^{j\cdot\cdot}\bot^\omega \models g$
  iff $\exists$ finite $u \preceq w : u\bot^\omega \models [g\ \mathsf{U}\ h]$
  iff [Theorem 24] $w \in int([\![[g\ \mathsf{U}\ h]]\!])$

□

LEMMA O.2. *Let $g, h \in \text{LTL}^{sere}$. Let $u \in \widehat{\Sigma}^*$, $w \in \widehat{\Sigma}^\infty$.*

- $u\top^\omega \models [g\ \mathsf{U}\ h]$ *iff* $\forall i \geq 0:$ *either* $u^{i\cdot\cdot}\top^\omega \models g$ *or* $\exists j,\ 0 \leq j \leq i : u^{j\cdot\cdot}\top^\omega \models h$.
- $w \models [g\ \mathsf{W}\ h]$ *iff* $\forall i,\ 0 \leq i < |w|:$ *either* $w^{i\cdot\cdot} \models g$ *or* $\exists j,\ 0 \leq j \leq i : w^{j\cdot\cdot} \models h$.

PROOF.

- $u\top^\omega \models [g\ \mathsf{U}\ h]$
  iff $\exists k \geq 0 : u^{k\cdot\cdot}\top^\omega \models h$ and $\forall j,\ 0 \leq j < k : u^{j\cdot\cdot}\top^\omega \models g$
  iff [($\Rightarrow$) Let $0 \leq i$. If $k \leq i$, then $\exists j,\ 0 \leq j \leq i$ (namely $j = k$): $u^{j\cdot\cdot}\top^\omega \models h$. Otherwise, $i < k$, so $u^{i\cdot\cdot}\top^\omega \models g$.
  ($\Leftarrow$) $u^{|u|\cdot\cdot}\top^\omega = \top^\omega \models h$, so pick k minimal such that $u^{k\cdot\cdot}\top^\omega \models h$. Let $0 \leq j < k$. There is no $0 \leq j' \leq j$ such that $u^{j'\cdot\cdot}\top^\omega \models h$, so we must have $u^{j\cdot\cdot}\top^\omega \models g$.] $\forall i \geq 0$: either $u^{i\cdot\cdot}\top^\omega \models g$ or $\exists j,\ 0 \leq j \leq i : u^{j\cdot\cdot}\top^\omega \models h$
- $w \models [g\ \mathsf{W}\ h]$
  iff either
  (A): $\exists k \geq 0 : w^{k\cdot\cdot} \models h$ and $\forall j,\ 0 \leq j < k : w^{j\cdot\cdot} \models g$
  or
  (B): $\forall i,\ 0 \leq i < |w| : w^{i\cdot\cdot} \models g$
  iff [($\Rightarrow$) Let $0 \leq i < |w|$. Suppose (A) holds. If $i < k$, then $w^{i\cdot\cdot} \models g$. Otherwise $k \leq i$, so, taking $j = k$, $\exists j,\ 0 \leq j \leq i : w^{j\cdot\cdot} \models h$. Suppose now that (B) holds. Then $w^{i\cdot\cdot} \models g$.
  ($\Leftarrow$) Suppose that (B) fails. Choose k minimal such that $w^{k\cdot\cdot} \not\models g$. Then $\forall j,\ 0 \leq j < k : w^{j\cdot\cdot} \models g$. From below, $\exists k', 0 \leq k' \leq k : w^{k'\cdot\cdot} \models h$. Therefore, (A) is satisfied.] $\forall i,\ 0 \leq i < |w|:$ either $w^{i\cdot\cdot} \models g$ or $\exists j,\ 0 \leq j \leq i : w^{j\cdot\cdot} \models h$

□

LEMMA O.3. *Let $g,\ h \in \text{LTL}^{sere}$ be semantically weak and let $g',\ h' \in \text{LTL}^{sere}$ be semantically strong.*

- $[\![\mathsf{X}\ g]\!] = cl([\![\mathsf{X}\ g]\!]) = cl([\![\mathsf{X!}\ g]\!])$.
- $[\![\mathsf{X!}\ g']\!] = int([\![\mathsf{X}\ g']\!]) = int([\![\mathsf{X!}\ g']\!])$.

- $[\![[g\ \mathsf{W}\ h]]\!] = cl([\![[g\ \mathsf{W}\ h]]\!]) = cl([\![[g\ \mathsf{U}\ h]]\!])$.
- $[\![[g'\ \mathsf{U}\ h']]\!] = int([\![[g'\ \mathsf{W}\ h']]\!]) = int([\![[g'\ \mathsf{U}\ h']]\!])$.

In particular, $\mathsf{X}\ g$ and $[g\ \mathsf{W}\ h]$ are semantically weak, and $\mathsf{X!}\ g'$ and $[g'\ \mathsf{U}\ h']$ are semantically strong.

PROOF.

- From Lemma O.1 we have that $cl([\![\mathsf{X}\ g]\!]) = cl([\![\mathsf{X!}\ g]\!])$. Clearly, $w \in [\![\mathsf{X}\ g]\!] \Rightarrow w \in cl([\![\mathsf{X}\ g]\!])$. It remains to show the other direction:

  $w \in cl([\![\mathsf{X}\ g]\!])$
  iff $w \in cl([\![\mathsf{X!}\ g]\!])$
  iff [Theorem 24] $\forall$ finite $u \preceq w : u\top^\omega \models \mathsf{X!}\ g$
  iff [$|u\top^\omega| > 1$] $\forall$ finite $u \preceq w : u^{1\cdot\cdot}\top^\omega \models g$
  iff $\forall$ finite $v \preceq w^{1\cdot\cdot} : v\top^\omega \models g$
  iff [Theorem 24] $w^{1\cdot\cdot} \in cl([\![g]\!])$
  iff [g is semantically weak] $w^{1\cdot\cdot} \in [\![g]\!]$
  $\Rightarrow w \in [\![\mathsf{X}\ g]\!]$

- From Lemma O.1 we have that $int(\llbracket \mathsf{X}\ g' \rrbracket) = int(\llbracket \mathsf{X}!\ g' \rrbracket)$. Clearly, $w \in int(\llbracket \mathsf{X}!\ g' \rrbracket) \Rightarrow w \in \llbracket \mathsf{X}!\ g' \rrbracket$. It remains to show the other direction:

  $w \in int(\llbracket \mathsf{X}!\ g' \rrbracket)$
  iff [Theorem 24] $\exists$ finite $u \preceq w : u\bot^\omega \models \mathsf{X}!\ g'$
  iff [$|u\bot^\omega| > 1$] $\exists$ finite $u \preceq w : u^{1\cdot\cdot}\bot^\omega \models g'$
  iff $\exists$ finite $v \preceq w^{1\cdot\cdot} : v\bot^\omega \models g'$
  iff [Theorem 24] $w^{1\cdot\cdot} \in int(\llbracket g' \rrbracket)$
  iff [g' is semantically strong] $w^{1\cdot\cdot} \in \llbracket g' \rrbracket$
  $\Leftarrow w \in \llbracket \mathsf{X}!\ g' \rrbracket$

- From Lemma O.1 we have that $cl(\llbracket [g\ \mathsf{W}\ h] \rrbracket) = cl(\llbracket [g\ \mathsf{U}\ h] \rrbracket)$. Clearly, $w \in \llbracket [g\ \mathsf{W}\ h] \rrbracket \Rightarrow w \in cl(\llbracket [g\ \mathsf{W}\ h] \rrbracket)$. It remains to show the other direction:

  $w \in cl(\llbracket [g\ \mathsf{W}\ h] \rrbracket)$
  iff $w \in cl(\llbracket [g\ \mathsf{U}\ h] \rrbracket)$
  iff [Theorem 24] $\forall$ finite $u \preceq w : u\top^\omega \models [g\ \mathsf{U}\ h]$
  iff [Lemma O.2]
  $\forall$ finite $u \preceq w : \forall i \geq 0 :$
  either $u^{i\cdot\cdot}\top^\omega \models g$
  or $\exists j,\ 0 \leq j \leq i : u^{j\cdot\cdot}\top^\omega \models h$

  $\Rightarrow$ [Let $0 \leq i < |w|$. Assume that (A) below fails, so that $\forall j,\ 0 \leq j \leq i :$ $\exists$ finite $v_j \preceq w^{j\cdot\cdot} : v_j\top^\omega \not\models h$. Let $u_i \preceq w^{i\cdot\cdot}$ be finite. Then $u_i = w^{i..i+|u_i|-1}$ and $v_j = w^{j..j+|v_j|-1}$. Let m be the maximum of $i + |u_i| - 1$ and the $j + |v_j| - 1$, $0 \leq j \leq i$. Let $u = w^{0..m}$. Then $u_i \preceq u^{i\cdot\cdot}$ and for all $j,\ 0 \leq j \leq i$, $v_j \preceq u^{j\cdot\cdot}$. By Lemma M.1, $u^{j\cdot\cdot}\top^\omega \not\models h$. From above, $u^{i\cdot\cdot}\top^\omega \models g$, and so by Lemma M.1, $u_i\top^\omega \models g$.]

  $\forall i,\ 0 \leq i < |w| :$
  either $\forall$ finite $u \preceq w^{i\cdot\cdot} : u\top^\omega \models g$
  or (A) : $\exists j,\ 0 \leq j \leq i : \forall$ finite $u \leq w^{j\cdot\cdot} : u\top^\omega \models h$
  iff [Theorem 24]
  $\forall i,\ 0 \leq i < |w| :$
  either $w^{i\cdot\cdot} \in cl(\llbracket g \rrbracket)$
  or $\exists j,\ 0 \leq j \leq i : w^{j\cdot\cdot} \in cl(\llbracket h \rrbracket)$
  iff [g, h are semantically weak]
  $\forall i,\ 0 \leq i < |w| :$
  either $w^{i\cdot\cdot} \models g$
  or $\exists j,\ 0 \leq j \leq i : w^{j\cdot\cdot} \models h$
  iff [Lemma O.2] $w \in \llbracket [g\ \mathsf{W}\ h] \rrbracket$

- From Lemma O.1 we have that $int(\llbracket [g'\ \mathsf{W}\ h'] \rrbracket) = int(\llbracket [g'\ \mathsf{U}\ h'] \rrbracket)$.
  Clearly, $w \in int(\llbracket g'\ \mathsf{U}\ h' \rrbracket) \Rightarrow w \in \llbracket g'\ \mathsf{U}\ h' \rrbracket$. It remains to show the other direction:

  $w \in int(\llbracket [g'\ \mathsf{U}\ h'] \rrbracket)$
  iff [Theorem 24] $\exists$ finite $u \preceq w : u\bot^\omega \models [g'\ \mathsf{U}\ h']$
  iff
  $\exists$ finite $u \preceq w : \exists k \geq 0 :$
  both $u^{k\cdot\cdot}\bot^\omega \models h'$
  and $\forall j,\ 0 \leq j < k : u^{j\cdot\cdot}\bot^\omega \models g'$

$\Leftarrow$ [Let k be given from below, so that $\exists$ finite $u_k \preceq w^{k\cdot\cdot} : u_k\bot^\omega \models h'$ and $\forall j,\ 0 \le j < k : \exists$ finite $v_j \preceq w^{j\cdot\cdot} : v_j\bot^\omega \models g'$. Then $u_k = w^{k..k+|u_k|-1}$ and $v_j = w^{j..j+|v_j|-1}$. Let m be the maximum of $k+|u_k|-1$ and the $j+|v_j|-1$, $0 \le j < k$. Let $u = w^{0..m}$. Then $u_k \preceq u^{k\cdot\cdot}$ and for all $j,\ 0 \le j < k$, $v_j \preceq u^{j\cdot\cdot}$. By Lemma M.1, $u^{k\cdot\cdot}\bot^\omega \models h'$ and $u^{j\cdot\cdot}\bot^\omega \models g'$ for all $j,\ 0 \le j < k$.]

$\exists k,\ 0 \le k < |w|$:
both $\exists$ finite $u \preceq w^{k\cdot\cdot} : u\bot^\omega \models h'$
and $\forall j,\ 0 \le j < k : \exists$ finite $u \preceq w^{j\cdot\cdot} : u\bot^\omega \models g'$
iff [Theorem 24]
$\exists k,\ 0 \le k < |w|$:
both $w^{k\cdot\cdot} \in int(\llbracket h' \rrbracket)$
and $\forall j,\ 0 \le j < k : w^{j\cdot\cdot} \in int(\llbracket g' \rrbracket)$
iff [g', h' are semantically strong]
$\exists k,\ 0 \le k < |w|$:
both $w^{k\cdot\cdot} \models h'$
and $\forall j,\ 0 \le j < k : w^{j\cdot\cdot} \models g'$
iff $w \in \llbracket [g' \mathsf{U} h'] \rrbracket$

□

Lemma O.4 (Top. Characterization of Syntactically Weak/Strong Formulas). *Let $f \in \text{LTL}^{sere}$.*

(1) *If f is syntactically weak, then f is semantically weak.*
(2) *If f is syntactically strong, then f is semantically strong.*

Proof.

(1) By induction (assume that f is syntactically weak):
$f = r$: By Theorem 25, r is semantically weak.
$f = g \wedge h$: $cl(\llbracket g \wedge h \rrbracket) =$ [Lemma O.1] $cl(\llbracket g \rrbracket) \cap cl(\llbracket h \rrbracket) =$ [induction] $\llbracket g \rrbracket \cap \llbracket h \rrbracket = \llbracket g \wedge h \rrbracket$.
$f = g \vee h$: $cl(\llbracket g \vee h \rrbracket) = cl(\llbracket g \rrbracket \cup \llbracket h \rrbracket) = cl(\llbracket g \rrbracket) \cup cl(\llbracket h \rrbracket) =$ [induction] $\llbracket g \rrbracket \cup \llbracket h \rrbracket = \llbracket g \vee h \rrbracket$.
$f = \mathsf{X}\, g$, $f = [g\ \mathsf{W}\ h]$: g, h are syntactically weak, hence semantically weak by induction. By Lemma O.3, f is semantically weak.
(2) The arguments are dual to those for Part (1).

□

Lemma O.5 (Weakening/Strengthening a Formula). *Let $f \in \text{LTL}^{sere}$ be a formula in positive normal form. Let f_w be the formula obtained from f by weakening all operators (replacing $\mathsf{X}!$ with X, U with W, and $r!$ with r for a regular expression r). Let f_s be the formula obtained from f by strengthening all operators. Then*

- $\llbracket f_w \rrbracket = cl(\llbracket f \rrbracket)$.
- $\llbracket f_s \rrbracket = int(\llbracket f \rrbracket)$.

Proof. Using Theorem 25, Lemma O.1, and induction, $cl(\llbracket f \rrbracket) = cl(\llbracket f_w \rrbracket)$. f_w is syntactically weak, so by Lemma O.4, $cl(\llbracket f_w \rrbracket) = \llbracket f_w \rrbracket$. This proves the first item. The proof of the second item is similar. □

THEOREM 26. *Let φ be an* LTL^{sere} *formula in positive normal form. Let φ_w be the formula obtained by weakening all operators (replacing* X! *with* X, U *with* W, *and* r! *with* r.*). Let φ_s be the formula obtained by strengthening all operators. Then*

- $[\![\varphi_w]\!] = weak(\varphi)$
- $[\![\varphi_s]\!] = strong(\varphi)$

PROOF. Follows from Lemma O.5 and Definition 24. □

P. PROOF OF THEOREM 27

THEOREM 27. *Let φ be an* LTL^{sere} *formula.*
- $weak(\varphi) = \{w \in \widehat{\Sigma}^\infty \mid \forall \textit{ finite } u \preceq w : u\top^\omega \in [\![\varphi]\!]\}$.
- $strong(\varphi) = \{w \in \widehat{\Sigma}^\infty \mid \exists \textit{ finite } u \preceq w : u\bot^\omega \in [\![\varphi]\!]\}$.

PROOF. Follows from Theorem 24 and Definition 24. □

Q. PROOF OF THEOREM 28

THEOREM 28. *Let φ be an* LTL^{sere} *formula. Then*

$$\mathit{fpref}(\ (\widehat{\Sigma}^\infty \setminus strong(\varphi)) \setminus (\widehat{\Sigma}^\infty \setminus weak(\varphi))\ )\ \cap\ (\widehat{\Sigma}^\infty \setminus weak(\varphi)) = \emptyset.$$

PROOF. $(\widehat{\Sigma}^\infty \setminus strong(\varphi)) \setminus (\widehat{\Sigma}^\infty \setminus weak(\varphi)) = weak(\varphi) \setminus strong(\varphi) = bd([\![\varphi]\!])$. Since $weak(\varphi)$ is closed, it follows that $bd([\![\varphi]\!]) \subseteq weak(\varphi)$ and hence also that $\mathit{fpref}(bd([\![\varphi]\!])) \subseteq weak(\varphi)$. Therefore, $\mathit{fpref}(bd([\![\varphi]\!])) \cap (\widehat{\Sigma}^\infty \setminus weak(\varphi)) = \emptyset$. □

R. PROOF OF THEOREM 29

LEMMA R.1. *$\overline{\cdot} : \widehat{\Sigma}^\infty \to \widehat{\Sigma}^\infty$ is a homeomorphism.*

PROOF. Clearly, $\overline{\cdot}$ is bijective, and $\overline{\cdot}$ is its own inverse. Therefore, it remains to show that $\overline{\cdot}$ is continuous.

$V \subseteq \widehat{\Sigma}^\infty$ is topologically closed
iff [Proposition 2] V is both prefix closed and limit closed
iff $\overline{V}$ is both prefix closed and limit closed
iff [Proposition 2] $\overline{V}$ is topologically closed

□

REMARK R.2. Note that by definition, for $\varphi \in$ LTL^{sere}, $[\![\neg\varphi]\!] = [\![\varphi]\!]^c$. Note also that $W^{cc} = \widehat{\Sigma}^\infty \setminus \overline{W^c} = \widehat{\Sigma}^\infty \setminus \overline{(\widehat{\Sigma}^\infty \setminus \overline{W})} = \widehat{\Sigma}^\infty \setminus (\widehat{\Sigma}^\infty \setminus \overline{\overline{W}}) = \overline{\overline{W}} = W$.

LEMMA R.3. *Let $f \in$ LTL^{sere}. Then*

(1) $cl([\![\neg f]\!]) = (int([\![f]\!]))^c$.
(2) $int([\![\neg f]\!]) = (cl([\![f]\!]))^c$.

PROOF.

(1)

$$\begin{array}{ll} & cl(\llbracket \neg f \rrbracket) \\ = & cl(\llbracket f \rrbracket^c) \\ = & cl(\widehat{\Sigma}^\infty \setminus \overline{\llbracket f \rrbracket}) \\ = & [\text{if } X \text{ is a space and } A \subseteq X, \text{ then } cl(X \setminus A) = X \setminus int(A)\ ] \\ & \widehat{\Sigma}^\infty \setminus int(\overline{\llbracket f \rrbracket}) \\ = & [\overline{\cdot} \text{ is a homeomorphism by Lemma R.1}] \\ & \widehat{\Sigma}^\infty \setminus \overline{int(\llbracket f \rrbracket)} \\ = & (int(\llbracket f \rrbracket))^c \end{array}$$

(2)

$$\begin{array}{ll} & int(\llbracket \neg f \rrbracket) \\ = & (int(\llbracket \neg f \rrbracket))^{cc} \\ = & [\text{Part (1)}]\ (cl(\llbracket \neg\neg f \rrbracket))^c \\ = & (cl(\llbracket f \rrbracket))^c \end{array}$$

□

THEOREM 29. *Let φ be an* LTL^{sere} *formula. Then*

- $weak(\neg\varphi) = (strong(\varphi))^c$
- $strong(\neg\varphi) = (weak(\varphi))^c$

PROOF. Follows from Lemma R.3 and Definition 24. □

S. PROOF OF THEOREM 30

THEOREM 30. *Let φ be an* LTL^{sere} *formula. Then φ is semantically strong iff $\neg\varphi$ is semantically weak.*

PROOF.

$$\begin{array}{ll} & \neg\varphi \text{ is semantically weak} \\ \text{iff} & cl(\llbracket \neg\varphi \rrbracket) = \llbracket \neg\varphi \rrbracket \\ \text{iff} & cl(\llbracket \varphi \rrbracket^c) = \llbracket \varphi \rrbracket^c \\ \text{iff} & [\overline{\cdot} \text{ is bijective}] \\ & \overline{cl(\llbracket \varphi \rrbracket^c)} = \overline{\llbracket \varphi \rrbracket^c} \\ \text{iff} & [\overline{\cdot} \text{ is a homeomorphism by Lemma R.1}] \\ & cl(\overline{\llbracket \varphi \rrbracket^c}) = \overline{\llbracket \varphi \rrbracket^c} \\ \text{iff} & cl(\widehat{\Sigma}^\infty \setminus \llbracket \varphi \rrbracket) = \widehat{\Sigma}^\infty \setminus \llbracket \varphi \rrbracket \\ \text{iff} & \widehat{\Sigma}^\infty \setminus cl(\widehat{\Sigma}^\infty \setminus \llbracket \varphi \rrbracket) = \llbracket \varphi \rrbracket \\ \text{iff} & [\text{if } X \text{ is a space and } A \subseteq X, \text{ then } int(A) = X \setminus cl(X \setminus A)] \\ & int(\llbracket \varphi \rrbracket) = \llbracket \varphi \rrbracket \\ \text{iff} & \varphi \text{ is semantically strong} \end{array}$$

□

T. PROOF OF THEOREM 31

THEOREM 31. *Let φ be an* LTL^{sere} *formula.*

(1) *φ is semantically weak iff φ is non-pathologically safe.*
(2) *φ is semantically strong iff $\neg\varphi$ is non-pathologically safe.*

PROOF.

(1) Follows from Theorem 30 and the second part.

(2) φ is semantically strong
$\Leftrightarrow w \models \varphi$ iff $\exists$ finite $u \preceq w$ such that $u\bot^\omega \in [\![\varphi]\!]$
$\Leftrightarrow w \models \varphi$ iff $\exists$ an informative bad prefix of w for $\neg\varphi$
$\Leftrightarrow \neg\varphi$ is non-pathologically safe

□

U. PROOF OF THEOREM 32

THEOREM 32. *Let φ be an LTL^{sere} formula, and let $M \subseteq \Sigma^\infty$. Searching for words in M that have informative bad prefixes for φ is equivalent to searching for words in M that are not in $weak(\varphi)$.*

PROOF.

$\exists w \in M$ s.t. $w \notin weak(\varphi)$
iff $\exists w \in M$ s.t. $w \notin \{v \in \widehat{\Sigma}^\infty \mid \forall u \preceq v : u\top^\omega \models \varphi\}$
iff $\exists w \in M$ s.t. not$(\forall u \preceq w : u\top^\omega \models \varphi)$
iff $\exists w \in M$ s.t. $\exists u \preceq w : u\top^\omega \not\models \varphi$
iff $[\bar{u} = u$ since $M \subseteq \Sigma^\infty]$
$\exists w \in M$ s.t. $\exists u \preceq w : u\bot^\omega \models \neg\varphi$
iff $\exists w \in M$ s.t. w has an informative bad prefix of φ

□

V. PROOFS OF THEOREMS 33 AND 34

THEOREM 33. *Let $f \in$ LTL^{sere}. Then*

(1) $[\![f]\!] \subseteq [\![f]\!]^- \subseteq cl([\![f]\!])$.
(2) $[\![f]\!] \supseteq [\![f]\!]^+ \supseteq int([\![f]\!])$.

PROOF.

(1) The first containment is by Strength Relation (Theorem 23). Suppose that $w \in [\![f]\!]^-$. Then $w\top^\omega \models f$. By Prefix/Extension (Corollary M.2), $\forall u \preceq w : u\top^\omega \models f$, so by Theorem 24, $w \in cl([\![f]\!])$.
(2) The first containment is by Strength Relation (Theorem 23). Suppose $w \in int([\![f]\!])$. By Theorem 24, $\exists$ finite $u \preceq w : u\bot^\omega \models f$. By Prefix/Extension (Corollary M.2), $\forall v \succeq u : v\bot^\omega \models f$. In particular, $w\bot^\omega \models f$, so $w \in [\![f]\!]^+$.

□

THEOREM 34. *Let φ in LTL^{sere}. The sets of finite words in $[\![f]\!]^-$ and $cl([\![f]\!])$ are the same, i.e.,*

$$[\![f]\!]^- \cap \widehat{\Sigma}^* = cl([\![f]\!]) \cap \widehat{\Sigma}^*.$$

PROOF. From Theorem 33 we have that $[\![f]\!]^- \subseteq cl([\![f]\!])$. Let $w \in cl([\![f]\!])$ be finite. Then there exists $v \succeq w$ s.t. $v \in [\![f]\!]$, and by Theorem 23, $v \in [\![f]\!]^-$, so $v\top^\omega \models f$. Now, by Lemma M.1, it follows that $w\top^\omega \models f$, hence $w \in [\![f]\!]^-$. □