Fall 2017-2018 Compiler Principles
Lecture 5: Intermediate Representation

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Previously

• Becoming parsing ninjas
  – Going from text to an Abstract Syntax Tree

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From scanning to parsing

program text

59 + (1257 * xPosition)

token stream

| num | + | ( | num | * | id | ) |

Grammar:
E → id
E → num
E → E + E
E → E * E
E → ( E )

Abstract Syntax Tree

Parse tree
AGENDA

• The role of intermediate representations

• Two example languages
  – A high-level language
  – An intermediate language

• Lowering

• Correctness
  – Formal meaning of programs
Role of intermediate representation

• Bridge between front-end and back-end

- High-level Language (scheme)
- Lexical Analysis
- Syntax Analysis
- Parsing
- AST
- Symbol Table etc.
- Inter. Rep. (IR)
- Code Generation
- Executable Code

• Allow implementing optimizations independent of source language and executable (target) language
MOTIVATION FOR INTERMEDIATE REPRESENTATION
Intermediate representation

- A language that is between the source language and the target language
  - Not specific to any source language or machine language
- Goal 1: retargeting compiler components for different source languages/target machines
Intermediate representation

- A language that is between the source language and the target language
  - Not specific to any source language or machine language
- Goal 1: retargeting compiler components for different source languages/target machines
- Goal 2: machine-independent optimizer
  - Narrow interface: small number of node types (instructions)
Multiple IRs

• Some optimizations require high-level constructs while others more appropriate on low-level code

• Solution: use multiple IR stages

![Diagram showing the process from AST to LIR with optimize steps and connections to Pentium, Sparc, and Java bytecode]
Multiple IRs example

Elixir – a language for parallel graph algorithms
# AST vs. LIR for imperative languages

<table>
<thead>
<tr>
<th>AST</th>
<th>LIR</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Rich set of language constructs</td>
<td>• An abstract machine language</td>
</tr>
<tr>
<td>• Rich type system</td>
<td>• Very limited type system</td>
</tr>
<tr>
<td>• Declarations: types (classes, interfaces), functions, variables</td>
<td>• Only computation-related code</td>
</tr>
<tr>
<td>• Control flow statements: if-then-else, while-do, break-continue,</td>
<td>• Labels and conditional/unconditional jumps, no looping</td>
</tr>
<tr>
<td>switch, exceptions</td>
<td>• Data movements, generic memory access statements</td>
</tr>
<tr>
<td>• Data statements: assignments, array access, field access</td>
<td>• No sub-expressions, logical as numeric, temporaries, constants,</td>
</tr>
<tr>
<td>• Expressions: variables, constants, arithmetic operators, logical</td>
<td>function calls – explicit argument passing</td>
</tr>
<tr>
<td>operators, function calls</td>
<td></td>
</tr>
</tbody>
</table>
THREE ADDRESS CODE
Three-Address Code IR

• A popular form of IR
• High-level assembly where instructions have at most three operands

• There exist other types of IR
  – For example, IR based on acyclic graphs – more amenable for analysis and optimizations
BASE LANGUAGE: While
Syntax

\[ n \in \text{Num} \quad \text{numerals} \]
\[ x \in \text{Var} \quad \text{program variables} \]

\[
A \rightarrow n \mid x \mid A \text{ArithOp} A \mid (A)
\]
\[
\text{ArithOp} \rightarrow - \mid + \mid \ast \mid / \]
\[
B \rightarrow \text{true} \mid \text{false}
\]
\[
\quad \mid A = A \mid A \leq A \mid \neg B \mid B \land B \mid (B)
\]
\[
S \rightarrow x := A \mid \text{skip} \mid S; S \mid \{S\}
\]
\[
\quad \mid \text{if } B \text{ then } S \text{ else } S
\]
\[
\quad \mid \text{while } B \ S
\]
Example program

while x < y {
  x := x + 1
}

y := x;
INTERMEDIATE LANGUAGE:

IL
Syntax

\[ n \in \textbf{Num} \quad \text{Numerals} \]
\[ l \in \textbf{Num} \quad \text{Labels} \]
\[ x \in \textbf{Temp} \cup \textbf{Var} \quad \text{Temporaries and variables} \]

\[
V \rightarrow n \mid x \\
R \rightarrow V \, \text{Op} \, V \\
\text{Op} \rightarrow - \mid + \mid * \mid / \mid = \mid \leq \mid \ll \mid \gg \mid \ldots \\
C \rightarrow l: \text{skip} \\
\mid l: x := R \\
\mid l: \text{Goto } l' \\
\mid l: \text{IfZ } x \text{ Goto } l' \\
\mid l: \text{IfNZ } x \text{ Goto } l' \\
IR \rightarrow C^+ \]
Intermediate language programs

• An intermediate program $P$ has the form
  
  1: $c_1$

  ...  

  n: $c_n$

• We can view it as a map from labels to individual commands and write $P(j) = c_j$

```plaintext
1: t0 := 137
2: y := t0 + 3
3: IfZ x Goto 7
4: t1 := y
5: z := t1
6: Goto 9
7: t2 := y
8: x := t2
9: skip
```
LOWERING
TAC generation

• At this stage in compilation, we have
  – an AST
  – annotated with scope information
  – and annotated with type information

• We generate TAC recursively (bottom-up)
  – First for expressions
  – Then for statements
TAC generation for expressions

- Define a function $cgen(expr)$ that generates TAC that:
  1. Computes an expression,
  2. Stores it in a temporary variable,
  3. Returns the name of that temporary

- Define $cgen$ directly for atomic expressions (constants, this, identifiers, etc.)

- Define $cgen$ recursively for compound expressions (binary operators, function calls, etc.)
Translation rules for expressions

\[
cgen(n) = (l: t:=n, t) \quad \text{where } l \text{ and } t \text{ are fresh}
\]

\[
cgen(x) = (l: t:=x, t) \quad \text{where } l \text{ and } t \text{ are fresh}
\]

\[
cgen(e_1) = (P_1, t_1) \quad cgen(e_2) = (P_2, t_2)
\]

\[
cgen(e_1 \ op \ e_2) = (P_1 \cdot P_2 \cdot l: t:=t_1 \ op \ t_2, t)
\]

where \( l \) and \( t \) are fresh
cgen for basic expressions

- Maintain a counter for temporaries in $c$, and a counter for labels in $l$
- Initially: $c = 0$, $l = 0$

\[
cgen(k) = \{ \begin{array}{l}
// k is a constant \\
    c = c + 1, l = l + 1 \\
    \text{emit}(l; tc := k) \\
    \text{return } tc
\end{array}\}
\]

\[
cgen(id) = \{ \begin{array}{l}
// id is an identifier \\
    c = c + 1, l = l + 1 \\
    \text{emit}(l; tc := id) \\
    \text{return } tc
\end{array}\}
\]
Naive `cgen` for binary expressions

- \[ \text{cgen}(e_1 \ op \ e_2) = \{
\text{let } A = \text{cgen}(e_1) \]
  \text{let } B = \text{cgen}(e_2) \]
  \text{c} = c + 1, \text{l} = l + 1 \]
  \text{emit}( l: tc := A \ op \ B; ) \]
  \text{return } tc \]
\}
Example: **cgen** for binary expressions

\[ cgen( (a*b)-d) \]
Example: \textbf{cgen} for binary expressions

\begin{align*}
\text{c} &= 0, \text{l} = 0 \\
\text{cgen}( (a*b)-d) &
\end{align*}
Example: `cgen` for binary expressions

\[
c = 0, \ l = 0
\]

\[
cgen( (a*b)-d) = \{
  \text{let } A = cgen(a*b)
  \text{let } B = cgen(d)
  c = c + 1, \ l = l +1
  \text{emit}(l: \text{tc := A - B}; )
  \text{return } \text{tc}
\}
\]
Example: \texttt{cgen} for binary expressions

\begin{verbatim}
c = 0, l = 0
cgen( (a*b)-d) = {
  let A = {
    let A = cgen(a)
    let B = cgen(b)
    c = c + 1, l = l + 1
    emit(l: tc := A * B; )
    return tc
  }
  let B = cgen(d)
  c = c + 1, l = l + 1
  emit(l: tc := A - B; )
  return tc
}
\end{verbatim}
Example: \texttt{cgen} for binary expressions

\begin{verbatim}
c = 0, l = 0
cgen( \ (a*b)-d) = {
  let A = {
    let A = {c=c+1, l=l+1, emit(l: tc := a;), return tc }
    let B = {c=c+1, l=l+1, emit(l: tc := b;), return tc }
    c = c + 1, l = l +1
    emit(l: tc := A * B; )
    return tc
  }
  let B = {c=c+1, l=l+1, emit(l: tc := d;), return tc }
  c = c + 1, l = l +1
  emit(l: tc := A - B; )
  return tc
}
\end{verbatim}
Example: `cgen` for binary expressions

\[
c = 0, \ l = 0
\]
\[
cgen( (a*b)-d) = \{
\]
\[
\text{let } A = \{
\]
\[
\text{let } A = \{c=c+1, \ l=l+1, \ \text{emit}(l: tc := a;), \ \text{return } tc \} \\
\text{let } B = \{c=c+1, \ l=l+1, \ \text{emit}(l: tc := b;), \ \text{return } tc \} \\
c = c + 1, \ l = l + 1 \\
\text{emit}(l: tc := A \ast B; ) \\
\text{return } tc
\}
\]
\[
\text{let } B = \{c=c+1, \ l=l+1, \ \text{emit}(l: tc := d;), \ \text{return } tc \} \\
c = c + 1, \ l = l + 1 \\
\text{emit}(l: tc := A - B; ) \\
\text{return } tc
\}
\]
Example: \texttt{cgen} for binary expressions

\begin{verbatim}
c = 0, l = 0
cgen( (a*b)-d) = {
  let A = {
    let A = {c=c+1, l=l+1, emit(l: tc := a;), return tc }
    let B = {c=c+1, l=l+1, emit(l: tc := b;), return tc }
    c = c + 1, l = l + 1
    emit(l: tc := A * B; )
    return tc
  }
  let B = {c=c+1, l=l+1, emit(l: tc := d;), return tc }
  c = c + 1, l = l + 1
  emit(l: tc := A - B; )
  return tc
}
\end{verbatim}

Code
1: t1:=a;
2: t2:=b;
Example: cgen for binary expressions

c = 0, l = 0

cgen( (a*b)-d) = {
  let A = {
    let A = {c=c+1, l=l+1, emit(l: tc := a;), return tc }
    let B = {c=c+1, l=l+1, emit(l: tc := b;), return tc }
    c = c + 1, l = l +1
    emit(l: tc := A * B; )
    return tc
  }
  let B = {c=c+1, l=l+1, emit(l: tc := d;), return tc }
  c = c + 1, l = l +1
  emit(l: tc := A - B; )
  return tc
}

Code
1: t1:=a;
2: t2:=b;
3: t3:=t1*t2
Example: \texttt{cgen} for binary expressions

\begin{verbatim}
c = 0, l = 0
cgen( (a*b) - d) = {
  let A = {
    let A = {c=c+1, l=l+1, emit(l: tc := a;), return tc }
    let B = {c=c+1, l=l+1, emit(l: tc := b;), return tc }
    c = c + 1, l = l + 1
    emit(l: tc := A * B; )
    return tc
  }
  let B = {c=c+1, l=l+1, emit(l: tc := d;), return tc }
  c = c + 1, l = l + 1
  emit(l: tc := A - B; )
  return tc
}
\end{verbatim}

Code
1: t1:=a;
2: t2:=b;
3: t3:=t1*t2

\begin{center}
\begin{tabular}{ll}
\texttt{Code} & \texttt{t1:=a;} \\
\texttt{t2:=b;} & \texttt{t3:=t1*t2} \\
\end{tabular}
\end{center}
Example: \texttt{cgen} for binary expressions

\begin{aligned}
c = 0, \  l = 0 \\
c \text{gen}( (a * b) - c) &= \\
\text{let} \ A = \\
\text{let} \ A = \{ c=c+1, \ l=l+1, \ \text{emit}(l: tc := a;) , \ \text{return} \ tc \} \\
\text{let} \ B = \{ c=c+1, \ l=l+1, \ \text{emit}(l: tc := b;) , \ \text{return} \ tc \} \\
c = c + 1, \ l = l + 1 \\
\text{emit}(l: tc := A * B; ) \\
\text{return} \ tc \\
\} \\
\text{let} \ B = \{ c=c+1, \ l=l+1, \ \text{emit}(l: tc := d;) , \ \text{return} \ tc \} \\
c = c + 1, \ l = l + 1 \\
\text{emit}(l: tc := A - B; ) \\
\text{return} \ tc \\
\}
\end{aligned}

\begin{verbatim}
Code
1: t1:=a;
2: t2:=b;
3: t3:=t1*t2
4: t4:=d
\end{verbatim}
Example: \texttt{cgen} for binary expressions

\begin{verbatim}
c = 0, l = 0
cgen( (a*b) - d) = {
  let A = {
    let A = {c=c+1, l=l+1, emit(l: tc := a;), return tc }
    let B = {c=c+1, l=l+1, emit(l: tc := b;), return tc }
    c = c + 1, l = l + 1
    emit(l: tc := A * B; )
    return tc
  }
  let B = {c=c+1, l=l+1, emit(l: tc := d;), return tc }
  c = c + 1, l = l + 1
  emit(l: tc := A - B; )
  return tc
}
\end{verbatim}

Code
1: t1:=a;
2: t2:=b;
3: t3:=t1*t2
4: t4:=d
5: t5:=t3-t4
cgen for statements

• We can extend the cgen function to operate over statements as well

• Unlike cgen for expressions, cgen for statements does not return the name of a temporary holding a value
  — (Why?)
Syntax

\[ n \in \textbf{Num} \quad \text{numerals} \]
\[ x \in \textbf{Var} \quad \text{program variables} \]

\[
A \rightarrow n \mid x \mid A \; \text{ArithOp} \; A \mid (A) \\
\text{ArithOp} \rightarrow - \mid + \mid * \mid / \\
B \rightarrow \text{true} \mid \text{false} \\
\mid A = A \mid A \leq A \mid \neg B \mid B \land B \mid (B) \\
S \rightarrow x := A \mid \text{skip} \mid S; S \mid \{ S \} \\
\mid \text{if} \; B \; \text{then} \; S \; \text{else} \; S \\
\mid \text{while} \; B \; \text{S} 
\]
## Translation rules for statements

<table>
<thead>
<tr>
<th>Rule</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>cgen(\textbf{skip}) = l: \text{skip}</td>
<td>where ( l ) is fresh</td>
</tr>
<tr>
<td>cgen((e) = (P, t))</td>
<td>(\frac{cgen(x := e) = P \cdot l: x := t}{\text{where } l \text{ is fresh}})</td>
</tr>
<tr>
<td>cgen((S_1) = P_1, \ cgen(S_2) = P_2)</td>
<td>(cgen(S_1; S_2) = P_1 \cdot P_2)</td>
</tr>
</tbody>
</table>
### Translation rules for conditions

\[
\begin{align*}
c\text{gen}(b) &= (Pb, t), \quad c\text{gen}(S_1) = P_1, \quad c\text{gen}(S_2) = P_2 \\
c\text{gen}(\textbf{if } b \textbf{ then } S_1 \textbf{ else } S_2) &= \\
&\quad Pb \\
&\quad \text{IfZ } t \text{ Goto } l_{\text{false}} \\
&\quad P_1 \\
&\quad l_{\text{finish}} : \text{ Goto } L_{\text{after}} \\
&\quad l_{\text{false}} : \text{ skip} \\
&\quad P_2 \\
&\quad l_{\text{after}} : \text{ skip}
\end{align*}
\]

where \(l_{\text{finish}}, l_{\text{false}}, l_{\text{after}}\) are fresh
Translation rule for loops

\[\text{cgen}(b) = (Pb, t), \quad \text{cgen}(S) = P\]

\[\text{cgen(while } b \text{ } S) =\]
\[
\text{l}_{\text{before}} : \text{skip} \\
Pb \\
\text{IfZ } t \text{ Goto } l_{\text{after}} \\
P \\
l_{\text{loop}} : \text{Goto } l_{\text{before}} \\
l_{\text{after}} : \text{skip}
\]

where \(l_{\text{after}}, l_{\text{before}}, l_{\text{loop}}\) are fresh
y := 137+3;
if x=0
  z := y;
else
  x := y;

1: t1 := 137
2: t2 := 3
3: t3 := t1 + t2
4: y := t3
5: t4 := x
6: t5 := 0
7: t6 := t4=t5
8: IfZ t6 Goto 12
9: t7 := y
10: z := t7
11: Goto 14
12: t8 := y
13: x := t8
14: skip
TRANSLATION CORRECTNESS
Compiler correctness

• Compilers translate programs in one language (usually high) to another language (usually lower) such that they are both equivalent

• Our goal is to formally define the meaning of this equivalence
  – First, we must formally define the meaning of programs
FORMAL SEMANTICS
What is formal semantics?

“Formal semantics is concerned with rigorously specifying the meaning, or behavior, of programs, pieces of hardware, etc.”
Why formal semantics?

• Implementation-independent definition of a programming language

• Automatically generating interpreters (and some day maybe full fledged compilers)

• Optimization, verification, and debugging
  – If you don’t know what it does, how do you know its correct/incorrect?
  – How do you know whether a given optimization is correct?
Operational semantics

• Elements of the semantics

• **States/configurations**: the (aggregate) values that a program computes during execution

• **Transition rules**: how the program advances from one configuration to another
OPERATIONAL SEMANTICS OF WHILE
While syntax reminder

\[ n \in \textbf{Num} \quad \text{numerals} \]
\[ x \in \textbf{Var} \quad \text{program variables} \]

\[ A \rightarrow n \mid x \mid A \, \text{ArithOp} \, A \mid (A) \]
\[ \text{ArithOp} \rightarrow - \mid + \mid * \mid / \]
\[ B \rightarrow \text{true} \mid \text{false} \]
\[ \quad \mid A = A \mid A \leq A \mid \neg B \mid B \wedge B \mid (B) \]
\[ S \rightarrow x := A \mid \text{skip} \mid S; S \mid \{ S \} \]
\[ \quad \mid \text{if } B \text{ then } S \text{ else } S \]
\[ \quad \mid \text{while } B \text{ do } S \]
Semantic categories

\( Z \)  
Integers \{0, 1, -1, 2, -2, \ldots\}

\( T \)  
Truth values \{ff, tt\}

State  
\( \text{Var} \rightarrow Z \)

Example state:  \( \sigma = [x\mapsto 5, y\mapsto 7, z\mapsto 0] \)

Lookup:  \( \sigma(x) = 5 \)

Update:  \( \sigma[x\mapsto 6] = [x\mapsto 6, y\mapsto 7, z\mapsto 0] \)
SEMANTICS OF EXPRESSIONS
Semantics of arithmetic expressions

- Semantic function $\llbracket A \rrbracket : \mathbf{State} \rightarrow \mathbb{Z}$

- Defined by induction on the syntax tree

  $\llbracket n \rrbracket \sigma = n$

  $\llbracket x \rrbracket \sigma = \sigma(x)$

  $\llbracket a_1 + a_2 \rrbracket \sigma = \llbracket a_1 \rrbracket \sigma + \llbracket a_2 \rrbracket \sigma$

  $\llbracket a_1 - a_2 \rrbracket \sigma = \llbracket a_1 \rrbracket \sigma - \llbracket a_2 \rrbracket \sigma$

  $\llbracket a_1 \times a_2 \rrbracket \sigma = \llbracket a_1 \rrbracket \sigma \times \llbracket a_2 \rrbracket \sigma$

  $\llbracket (a_1) \rrbracket \sigma = \llbracket a_1 \rrbracket \sigma$ --- not needed

  $\llbracket - a \rrbracket \sigma = 0 - \llbracket a_1 \rrbracket \sigma$

- Compositional

- Expressions in While are side-effect free
Arithmetic expression exercise

Suppose $\sigma x = 3$

Evaluate $[(\mathit{x+1})] \sigma$
Semantics of boolean expressions

- Semantic function $\mathcal{[} B \mathcal{]} : \text{State} \rightarrow \text{T}$
- Defined by induction on the syntax tree
  - $\mathcal{[} \text{true} \mathcal{]} \sigma = \text{tt}$
  - $\mathcal{[} \text{false} \mathcal{]} \sigma = \text{ff}$
  - $\mathcal{[} a_1 = a_2 \mathcal{]} \sigma =$
  - $\mathcal{[} a_1 \leq a_2 \mathcal{]} \sigma =$
  - $\mathcal{[} b_1 \land b_2 \mathcal{]} \sigma =$
  - $\mathcal{[} \neg b \mathcal{]} \sigma =$

- Compositional
- Expressions in While are side-effect free
SEMANTICS OF STATEMENTS

\[ \langle S, \sigma \rangle \rightarrow \gamma \]

first step
Operational semantics

• Developed by Gordon Plotkin

• **Configurations**: \( \gamma \) has one of two forms:
  \[ \langle S, \sigma \rangle \] Statement \( S \) is about to execute on state \( \sigma \)
  \[ \sigma \] Terminal (final) state

• **Transitions** \( \langle S, \sigma \rangle \rightarrow \gamma \)
  \( \gamma = \langle S', \sigma' \rangle \) Execution of \( S \) from \( \sigma \) is not completed and remaining computation proceeds from intermediate configuration \( \gamma \)
  \( \gamma = \sigma' \) Execution of \( S \) from \( \sigma \) has terminated and the final state is \( \sigma' \)

• \( \langle S, \sigma \rangle \) is **stuck** if there is no \( \gamma \) such that \( \langle S, \sigma \rangle \rightarrow \gamma \)
Form of semantic rules

• → defined by rules of the form

\[
\langle S_1, \sigma_1 \rangle \rightarrow \gamma_1', \ldots, \langle S_n, \sigma_n \rangle \rightarrow \gamma_n' \quad \text{if...}
\]

\[
\langle S, \sigma \rangle \rightarrow \gamma'
\]

• The meaning of compound statements is defined using the meaning immediate constituent statements
Operational semantics for **While**

**[ass]** \[x:=a, \sigma] \rightarrow \sigma[x\mapsto[a]]\sigma

**[skip]** \langle \text{skip}, \sigma \rangle \rightarrow \sigma

**[comp\textsuperscript{1}]** \[
\dfrac{\langle S_1, \sigma \rangle \rightarrow \langle S_1', \sigma' \rangle}{\langle S_1; S_2, \sigma \rangle \rightarrow \langle S_1'; S_2, \sigma' \rangle}
\]

**[comp\textsuperscript{2}]** \[
\dfrac{\langle S_1, \sigma \rangle \rightarrow \sigma'}{\langle S_1; S_2, \sigma \rangle \rightarrow \langle S_2, \sigma' \rangle}
\]

**[if\textsuperscript{tt}]** \[\langle \text{if } b \text{ then } S_1 \text{ else } S_2, \sigma \rangle \rightarrow \langle S_1, \sigma \rangle \text{ if } \llbracket b \rrbracket \sigma = \text{tt} \]

**[if\textsuperscript{ff}]** \[\langle \text{if } b \text{ then } S_1 \text{ else } S_2, \sigma \rangle \rightarrow \langle S_2, \sigma \rangle \text{ if } \llbracket b \rrbracket \sigma = \text{ff} \]

**[while\textsuperscript{tt}]** \[\langle \text{while } b \text{ do } S, \sigma \rangle \rightarrow \langle S; \text{while } b \text{ do } S, \sigma \rangle \text{ if } \llbracket b \rrbracket \sigma = \text{tt} \]

**[while\textsuperscript{ff}]** \[\langle \text{while } b \text{ do } S, \sigma \rangle \rightarrow \sigma \text{ if } \llbracket b \rrbracket \sigma = \text{ff} \]
Factorial ($n!$) example

- **Input state** $\sigma$ such that $\sigma x = 3$

  $y := 1; \text{ while } \neg(x=1) \text{ do } (y := y \times x; x := x - 1)$

  \[
  \langle y := 1; W, \sigma \rangle \\
  \rightarrow \langle W, \sigma[y \leftarrow 1] \rangle \\
  \rightarrow \langle ((y := y \times x; x := x - 1); W), \sigma[y \leftarrow 1] \rangle \\
  \rightarrow \langle (x := x - 1; W), \sigma[y \leftarrow 3] \rangle \\
  \rightarrow \langle W, \sigma[y \leftarrow 3][x \leftarrow 2] \rangle \\
  \rightarrow \langle ((y := y \times x; x := x - 1); W), \sigma[y \leftarrow 3][x \leftarrow 2] \rangle \\
  \rightarrow \langle (x := x - 1; W), \sigma[y \leftarrow 6][x \leftarrow 2] \rangle \\
  \rightarrow \langle W, \sigma[y \leftarrow 6][x \leftarrow 1] \rangle \\
  \rightarrow \sigma[y \leftarrow 6][x \leftarrow 1]
  \]
Derivation sequences

• An interpreter
  – Given a statement $S$ and an input state $\sigma$
    start with the initial configuration $\langle S, \sigma \rangle$
  – Repeatedly apply rules until either a terminal state
    is reached or an infinite sequence

• We will write $\langle S, \sigma \rangle \rightarrow^k \sigma'$ if $\sigma'$ can be obtained
  from $\langle S, \sigma \rangle$ by $k$ derivation steps

• We will write $\langle S, \sigma \rangle \rightarrow^* \sigma'$ if $\langle S, \sigma \rangle \rightarrow^k \sigma'$ for
  some $k \geq 0$
The semantics of statements

• The meaning of a statement $S$ is defined as

\[
[S] \sigma = \begin{cases} 
\sigma' & \text{if } \langle S, \sigma \rangle \rightarrow^* \sigma' \\
\bot & \text{else}
\end{cases}
\]

• Examples:

\[
[Skip] \sigma = \sigma
\]
\[
[x := 1] \sigma = \sigma[x \mapsto 1]
\]
\[
[\text{while true do skip}] \sigma = \bot
\]
PROPERTIES OF OPERATIONAL SEMANTICS
Deterministic semantics for **While**

- **Theorem**: for all statements $S$ and states $\sigma_1, \sigma_2$
  
  if $\langle S, \sigma \rangle \rightarrow^* \sigma_1$ and $\langle S, \sigma \rangle \rightarrow^* \sigma_2$ then $\sigma_1 = \sigma_2$
Program termination

• Given a statement $S$ and input $\sigma$
  – $S$ terminates on $\sigma$ if there exists a state $\sigma'$ such that $\langle S, \sigma \rangle \rightarrow^* \sigma'$
  – $S$ loops on $\sigma$ if there is no state $\sigma'$ such that $\langle S, \sigma \rangle \rightarrow^* \sigma$

• Given a statement $S$
  – $S$ always terminates if for every input state $\sigma$, $S$ terminates on $\sigma$
  – $S$ always loops if for every input state $\sigma$, $S$ loops on $\sigma$
Semantic equivalence

• $S_1$ and $S_2$ are **semantically equivalent** if for all $\sigma$:

  $\langle S_1, \sigma \rangle \rightarrow^* \gamma$ if and only if $\langle S_2, \sigma \rangle \rightarrow^* \gamma$

  $\gamma$ is either stuck or terminal

  and there is an infinite derivation sequence for $\langle S_1, \sigma \rangle$ if and only if there is one for $\langle S_2, \sigma \rangle$

• Examples

  - $S; \, \text{skip}$ and $S$
  - while $b$ do $S$ and if $b$ then ($S; \, \text{while} \, b \, S$) else skip
  - $(S_1; \, S_2; \, S_3)$ and $(S_1; \, (S_2; \, S_3))$
  - $(x:=5; \, y:=x*8)$ and $(x:=5; \, y:=40)$
Exercise

Why do we need this?

\[(x, y) := (a, b) \quad x, y, a, b \in \text{Var} \cup \text{Temp}\]


\[
\begin{align*}
\text{par} & \quad m(p_c) = l \quad P(l) = (x, y) := (a, b) \\
& \quad m \triangleright m[p_c \mapsto m(p_c) + 1, x \mapsto m(a), y \mapsto m(b)] \quad x \neq y
\end{align*}
\]

\[
\begin{align*}
\text{ser} & \quad m(p_c) = l \quad P(l) = (x, y) := (a, b) \\
& \quad m' = m[x \mapsto m(a)] \\
& \quad m \triangleright m'[p_c \mapsto m(p_c) + 1, y \mapsto m'(b)]
\end{align*}
\]

נתון התכניות הבאות, ביעול פקודת בודדת:

<table>
<thead>
<tr>
<th>פקודת</th>
<th>שורש</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P_1)</td>
<td>((x, y) := (a, b))</td>
</tr>
<tr>
<td>(P_2)</td>
<td>((x, y) := (y, b))</td>
</tr>
<tr>
<td>(P_3)</td>
<td>((x, y) := (a, x))</td>
</tr>
<tr>
<td>(P_4)</td>
<td>((x, y) := (y, x))</td>
</tr>
</tbody>
</table>

אם ה_OPCODE של התכניות השכולות סמוכית תחת של הפורים של

(4) \(\text{אמ\(\)י)}

השכולת בודדת.
OPERATIONAL SEMANTICS OF IL
IL syntax reminder

\[ n \in \text{Num} \quad \text{Numerals} \]
\[ l \in \text{Num} \quad \text{Labels} \]
\[ x \in \text{Var} \cup \text{Temp} \quad \text{Variables and temporaries} \]

\[ V \rightarrow n \mid x \]
\[ R \rightarrow V \text{ Op } V \mid V \]
\[ \text{Op} \rightarrow - \mid + \mid \ast \mid / \mid = \mid \leq \mid \ll \mid \gg \mid \ldots \]
\[ C \rightarrow l: \text{skip} \]
\[ \mid l: x := R \]
\[ \mid l: \text{Goto } l' \]
\[ \mid l: \text{IfZ } x \text{ Goto } l' \]
\[ IR \rightarrow C^+ \]
Intermediate program states

\[ \mathbb{Z} \quad \text{Integers \{0, 1, -1, 2, -2, \ldots\}} \]

\text{IState} \quad (\text{Var} \cup \text{Temp} \cup \{pc\}) \rightarrow \mathbb{Z}

\begin{itemize}
  \item Var, Temp, and \{pc\} are all disjoint
  \item For every state \( m \) and program \( P=1:c_1,\ldots,n:c_n \)
    we have that \( 1 \leq m(pc) \leq n+1 \)
    \begin{itemize}
      \item We can check that the labels used in \( P \) are legal
    \end{itemize}
\end{itemize}
Rules for executing commands

• We will use rules of the following form

\[
\frac{m(pc) = \ell \quad P(\ell) = C}{m \triangleright m'}
\]

• Here \( m \) is the **pre-state**, which is scheduled to be executed as the program counter indicates, and \( m' \) is the **post-state**

• The rules specialize for the particular type of command \( C \) and possibly other conditions
Evaluating values

\[ [n]m = n \]

\[ [x]m = m(x) \]

\[ [v_1 \text{ op } v_2]m = [v_1]m \text{ op } [v_2]m \]
Rules for executing commands

\begin{align*}
m(pc) &= l & P(l) &= \text{skip} \\
m \triangleright m[pc\mapsto l+1]
\end{align*}

\begin{align*}
m(pc) &= l & P(l) &= x := v \\
m \triangleright m[pc\mapsto l+1, x\mapsto [v]m]
\end{align*}

\begin{align*}
m(pc) &= l & P(l) &= x := v_1 \ op \ v_2 \\
m \triangleright m[pc\mapsto l+1, x\mapsto [v_1]m \ op \ [v_2]m]
\end{align*}

\begin{align*}
m(pc) &= l & P(l) &= \text{Goto} \ l' \\
m \triangleright m[pc\mapsto l']
\end{align*}
Rules for executing commands

\[
\begin{align*}
    m(pc) &= l & P(l) &= \textbf{IfZ } x \textbf{ Goto } l' & m(x) &= 0 \\
    m &\succ m[pc\mapsto l']
\end{align*}
\]

\[
\begin{align*}
    m(pc) &= l & P(l) &= \textbf{IfZ } x \textbf{ Goto } l' & m(x) &\neq 0 \\
    m &\succ m[pc\mapsto l+1]
\end{align*}
\]

\[
\begin{align*}
    m(pc) &= l & P(l) &= \textbf{IfNZ } x \textbf{ Goto } l' & m(x) &\neq 0 \\
    m &\succ m[pc\mapsto l']
\end{align*}
\]

\[
\begin{align*}
    m(pc) &= l & P(l) &= \textbf{IfNZ } x \textbf{ Goto } l' & m(x) &= 0 \\
    m &\succ m[pc\mapsto l+1]
\end{align*}
\]
Executing programs

• For a program $P=1:c_1,\ldots,n:c_n$ we define executions as finite or infinite sequences $m_1 \triangleright m_2 \triangleright \ldots \triangleright m_n \triangleright \ldots$

• We write $m \triangleright^* m'$ if there is a finite execution starting at $m$ and ending at $m'$:

$$m = m_1 \triangleright m_2 \triangleright \ldots \triangleright m_n = m'$$
Semantics of a program

• For a program $P=1:c_1,...,n:c_n$ and a state $m$ s.t. $m(pc)=1$ we define the result of executing $P$ on $m$ as

$$\llbracket P \rrbracket m = \begin{cases} m' & \text{if } m \searrow^* m' \text{ and } m'(pc)=n+1 \\ \bot & \text{else} \end{cases}$$

• **Lemma:** the function is well-defined (i.e., at most one output state)
Execution example

• Execute the following intermediate language program on a state where all variables evaluate to 0

```c
1: t1 := 137
2: y := t1 + 3
3: IfZ x Goto 7
4: t2 := y
5: z := t2
6: Goto 9
7: t3 := y
8: x := t3
9: skip
```

\[ m = [pc\rightarrow 1, t1\rightarrow 0, t2\rightarrow 0, t3\rightarrow 0, x\rightarrow 0, y\rightarrow 0] \]
Execution example

• Execute the following intermediate language program on a state where all variables evaluate to 0

```plaintext
1: t1 := 137
2: y := t1 + 3
3: IfZ x Goto 7
4: t2 := y
5: z := t2
6: Goto 9
7: t3 := y
8: x := t3
9: skip
```

\[ m = \left[ pc\rightarrow 1, t1\rightarrow 0, t2\rightarrow 0, t3\rightarrow 0, x\rightarrow 0, y\rightarrow 0 \right] \triangleright \]
\[ m[pc\rightarrow 2, t1\rightarrow 137] \triangleright \]
\[ m[pc\rightarrow 3, t1\rightarrow 137, y\rightarrow 140] \triangleright \]
\[ m[pc\rightarrow 7, t1\rightarrow 137, y\rightarrow 140] \triangleright \]
\[ m[pc\rightarrow 8, t1\rightarrow 137, t3\rightarrow 140, y\rightarrow 140] \triangleright \]
\[ m[pc\rightarrow 9, t1\rightarrow 137, t3\rightarrow 140, x\rightarrow 140, y\rightarrow 140] \triangleright \]
\[ m[pc\rightarrow 10, t1\rightarrow 137, t3\rightarrow 140, x\rightarrow 140, y\rightarrow 140] \]
Semantic equivalence

• $P_1$ and $P_2$ are semantically equivalent if for all $m$ the following holds:

$$((\llbracket P_1 \rrbracket m) |_{\text{Var}} = ((\llbracket P_2 \rrbracket m) |_{\text{Var}}$$
ADDING PROCEDURES
While+procedures syntax

\[ n \in \text{Num} \quad \text{Numerals} \]
\[ x \in \text{Var} \quad \text{Program variables} \]
\[ f \quad \text{Function identifiers, including main} \]

\[ A \rightarrow n \mid x \mid A \text{ ArithOp } A \mid (A) \mid f(A^*) \]
\[ \text{ArithOp} \rightarrow - \mid + \mid * \mid / \]
\[ B \rightarrow \text{true} \mid \text{false} \mid A = A \mid A \leq A \mid \neg B \mid B \land B \mid (B) \]
\[ S \rightarrow x := A \mid \text{skip} \mid S; S \mid \{ S \} \]
\[ \mid \text{if } B \text{ then } S \text{ else } S \]
\[ \mid \text{while } B \ S \]
\[ \mid f(A^*) \]
\[ F \rightarrow f(x^*) \{ S \} \]
\[ P ::= F^+ \]

Can be used to call side-effecting functions and to return values from functions.
IL+ procedures syntax

\[ n \in \text{Num} \quad \text{Numerals} \]
\[ l \in \text{Num} \quad \text{Labels and function identifiers} \]
\[ x \in \text{Var} \cup \text{Temp} \cup \text{Params} \quad \text{Variables, temporaries, and parameters} \]
\[ f \quad \text{Function identifiers} \]

\[
V \rightarrow n \mid x \\
R \rightarrow V \ Op \ V \mid V \mid \text{Call } f(V^*) \mid pn \\
Op \rightarrow - \mid + \mid * \mid / \mid = \mid \leq \mid <\mid >\mid >\mid \ldots \\
C \rightarrow l: \text{skip} \\
\quad l: x := R \\
\quad l: \text{Goto } l' \\
\quad l: \text{IfZ } x \text{ Goto } l' \\
\quad l: \text{IfNZ } x \text{ Goto } l' \\
\quad l: \text{Call } f(x^*) \\
\quad l: \text{Ret } x \\
\]
\[
IR \rightarrow C^+ 
\]
Exercise

Given the set of instructions to be inserted into a program to handle an instruction, we need to determine the effect of these instructions. The program consists of a sequence of instructions that are to be executed in order. The goal is to ensure that the program behaves correctly in all possible scenarios.

The problem involves analyzing the effect of each instruction on the variables in the program. Specifically, we need to determine the value of the variable `x` after the execution of the given sequence of instructions.

The instructions are as follows:

1. `x = y = z`
2. `x = y = z` (for different values of `x`, `y`, and `z`)
3. `x = y = z` (for different values of `x`, `y`, and `z`)
4. `x = y = z` (for different values of `x`, `y`, and `z`)

The variables `x`, `y`, and `z` are assigned values in the given sequence. The task is to determine the final value of `x` after the execution of these instructions.

The solution involves analyzing the effect of each instruction on the variables and determining the final value of `x`.

The solution to the exercise is as follows:

- After the first instruction, `x` is assigned the value of `y`, which is then assigned the value of `z`.
- After the second instruction, `x` is assigned the value of `y`, which is then assigned the value of `z`.
- After the third instruction, `x` is assigned the value of `y`, which is then assigned the value of `z`.
- After the fourth instruction, `x` is assigned the value of `y`, which is then assigned the value of `z`.

The final value of `x` is determined by analyzing the effect of each instruction on the variables and ensuring that the program behaves correctly in all possible scenarios.

The solution to the exercise is as follows:

The final value of `x` is determined by analyzing the effect of each instruction on the variables and ensuring that the program behaves correctly in all possible scenarios.

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DEFINING TRANSLATION CORRECTNESS
Exercise 1

• Are the following equivalent in your opinion?

**WHILE**

\[
x := 137
\]

**IL**

1: t1 := 137
2: x := t1
Exercise 2

• Are the following equivalent in your opinion?

**WHILE**

\[
\begin{align*}
x & := 137 \\
\end{align*}
\]

**IL**

\[
\begin{align*}
1: & \ y := 137 \\
2: & \ x := y \\
\end{align*}
\]
Exercise 3

• Are the following equivalent in your opinion?

\begin{align*}
\textbf{WHILE} & \quad \textbf{IL} \\
x & := 137 \\
2: & \quad t_2 := 138 \\
3: & \quad t_1 := 137 \\
4: & \quad x := t_1
\end{align*}
Exercise 4

• Are the following equivalent in your opinion?

\texttt{WHILE}

\begin{align*}
\text{x := 137}
\end{align*}

\texttt{IL}

\begin{align*}
2: & \ t2 := 138 \\
3: & \ t1 := 137 \\
4: & \ x := t1 \\
5: & \ t1 := 138
\end{align*}
Equivalence of arithmetic expressions

- Define \( T_{\text{Var}} = \text{Var} \cup \text{Temp} \)
- While state: \( \sigma \in \text{Var} \rightarrow \mathbb{Z} \)
- IL state: \( m \in (\text{Var} \cup \text{Temp} \cup \{pc\}) \rightarrow \mathbb{Z} \)
- Define \( \text{label}(P) \) to be first label of IL program \( P \)
- An arithmetic While expression \( a \) is equivalent to an IL program \( P \) and \( x \in T_{\text{Var}} \) iff for every input state \( m \) such that \( m(\text{pc}) = \text{label}(P) \):
  \[
  \left[ [a] \right] m |_{\text{Var}} = (\left[ [P] \right] m) \ x
  \]
Statement equivalence

• We define state equivalence by

\[ \sigma \approx m \iff \sigma = m \mid_{\text{Var}} \]

– That is, for each \( x \in \text{Var} \) \( \sigma(x) = m(x) \)

• A While statement \( S \) is equivalent to an IL program \( P \) iff for every input state \( m \) such that \( m(\text{pc}) = \text{label}(P) \):

\[ \llbracket S \rrbracket m \mid_{\text{Var}} \approx \llbracket P \rrbracket m \]
Next lecture:
Dataflow-based Optimizations