Brief paper

Preservation of exponential stability for linear non-autonomous functional differential systems

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\textbf{A B S T R A C T}

We consider preservation of exponential stability for a system of linear equations with a distributed delay under the addition of new terms and a delay perturbation. As particular cases, the system includes models with concentrated delays and systems of integrodifferential equations. Our method is based on Bohl–Perron type theorems.

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1. Introduction

Equations with a distributed delay provide a more realistic description for real world delay models in mathematical biology, mechanical engineering and other applications (Kuang, 1993). Here we consider a type of delay which involves, as special cases, equations with concentrated delays and integrodifferential equations. To the best of our knowledge the first systematic study of equations with a distributed delay can be found in the monograph of Myshkis (1972), the results obtained by 1993 are summarized in the book by Kuang (1993).

In most publications on distributed delays (see, for example, Bernard, Bélair, & Mackey, 2001) integrodifferential equations are studied, however sometimes models incorporate both integral terms and concentrated delays; such equations are considered in the present paper. Here we study the following question: assuming that the original equation is exponentially stable, when can we state that the perturbed equation (generally, the perturbation has a distributed delay) preserves this property? As corollaries, we obtain some known results (see, for example, Fu, Olbrot, & Polis, 1989; Halanay, 1966; Louisell, 1992; Stokes, 1974; Su & Huang, 1992; Tsykin & Fu, 1993) for equations with concentrated delays

\begin{equation}
\dot{x}(t) + \sum_{k=1}^{m} A_k(t)x(h_k(t)) = 0, \quad (1)
\end{equation}

where the equation is perturbed by adding new delay terms:

\begin{equation}
\dot{x}(t) + \sum_{k=1}^{m} A_k(t)x(h_k(t)) + \sum_{l=1}^{j} B_l(t)x(g_l(t)) = 0. \quad (2)
\end{equation}

Delay perturbations were studied in Driver (1977): the equation

\begin{equation}
\dot{x}(t) = A_0(t)x(t) + \sum_{k=1}^{m} A_k(t)x(t - \tau_k) \quad (3)
\end{equation}

was considered as a perturbation of the equation

\begin{equation}
\dot{x}(t) = \sum_{k=0}^{m} A_k(t)x(t) \quad (4)
\end{equation}

without delays. Suppose that the fundamental matrix of (4) has exponential estimation \( \|X(t, s)\| \leq Ke^{-\lambda(t-s)} \) for \( K > 0, \lambda > 0 \). Then (Driver, 1977) Eq. (3) is asymptotically stable as far as

\begin{equation}
\max_{k} \tau_k \left( \sup_{t \geq 0} \sum_{j=1}^{m} \|A_k(t)\| \left( \sum_{k=0}^{m} \sup_{t \geq 0} \|A_k(t)\| \right) \right) < \frac{\lambda}{K}. \quad (5)
\end{equation}

In Györi, Hartung, and Turi (1998) the authors considered a scalar equation with variable delays and constant coefficients as a
perturbation of an autonomous delay equation. Some publications (see, for example, Gil’, 1998) deal with the generalizations of the classical Wazewski and Lozinskii inequalities for linear ordinary differential equations to vector functional differential equations with small delays; see also recent papers (Burton, 2005; Chen & Guan, 2004; Corduneanu & Ignatieve, 2005; Desch, Göyör, & Pongor, 1997; Fridman, 2006; Goubet-Bartholoméus, Dambrine, & Richard, 1997; Göyör & Hartung, 2001, 2002, 2006; Hou, Gao, & Qian, 1999; Kharitonov & Niculescu, 2003; Zevin & Pinsky, 2006) for results on stability preservation under perturbations and applications of these results to control theory.

In this paper we consider differential equations with distributed delays. Our main method is based on the Bohl–Perron theorem. Previously it was applied to perturbation problems for impulsive delay differential equations in Berezansky and Braverman (1995). We consider here several types of perturbed equations. In Section 3 some additional terms constitute the perturbation of the original equation. We obtain standard stability results which extend well-known stability conditions for ordinary differential equations, including equations in Banach spaces. In Section 4 we consider perturbations of delays and generalize some results of Driver (1977). Finally, Section 5 contains discussion and states some open problems.

2. Preliminaries

We consider a system of linear differential equations with a distributed delay

\[ \dot{x}(t) + \int_{h(t)}^{t} \sum_{i=1}^{m} \int_{g_{i}(t)}^{t} d_{i}R(t, s) x(s) \, ds + \int_{h(t)}^{t} d_{Q}(t, s) x(s) = 0 \]  \tag{6}

for \( t \geq t_{0} \geq 0 \), which will be treated as a perturbation of the equation with one integral term

\[ \dot{x}(t) + \int_{h(t)}^{t} d_{R}(t, s) x(s) = 0, \quad t \geq t_{0}. \]  \tag{7}

Here \( x(t) \) is a column \( n \)-vector, \( R \) and \( Q \) are \( n \times n \) matrix functions, \( \|x\| \) is the Euclidean norm of the vector \( x \in \mathbb{R}^{n} \),

\[ \|A\| = \sup_{\|x\|=1} \|Ax\|. \]

We consider Eqs. (6) and (7) for a fixed \( t_{0} \geq 0 \) with the initial condition

\[ x(t) = \psi(t), \quad t \leq t_{0} \]  \tag{8}

under the following assumptions:

(a1) \( R(t, \cdot) \) and \( Q(t, \cdot) \) are left continuous matrix functions of bounded variation for any \( t \); \( R(\cdot, s) \) and \( Q(\cdot, s) \) are locally integrable for any \( s \),

\[ R(t, h(t)) = Q(t, g(t)) = 0, \]

\( R(t, s) \), \( Q(t, s) \) are constant for \( s > t \) and coincide with the right limits \( R(t, t+) \), \( Q(t, t+) \) of the functions

\[ \alpha(t) = \int_{h(t)}^{t} \|d_{R}(t, s)\|, \quad \beta(t) = \int_{h(t)}^{t} \|d_{Q}(t, s)\| \]  \tag{9}

are Lebesgue measurable and bounded on \([0, \infty)\);

(a2) \( h, g : [0, \infty) \to \mathbb{R} \) are Lebesgue measurable functions, \( h(t) \leq t, g(t) \leq t, \lim \sup_{t \to \infty}[t - h(t)] < \infty, \lim \sup_{t \to \infty}[t - g(t)] < \infty \).

The Lebesgue–Stiltjes integral \( \int_{h(t)}^{t} d_{R}(t, s) x(s) \, ds \) is understood as

\[ \int_{h(t)}^{t} d_{R}(t, s) x(s) = \int_{h(t)}^{t+\varepsilon} d_{R}(t, s) x(s) \]

for any \( \varepsilon > 0 \), where the Lebesgue–Stiltjes matrix measure produced by \( d_{R}(t, s) \) is multiplied by the column vector \( x(\cdot) \) for any \( t \) (\( x \) is not under the sign of the differential). Thus, \( \alpha(t) \) defined in (9) has the form

\[ \alpha(t) = \int_{h(t)}^{t} \|d_{R}(t, s)\| = \sum_{k=1}^{m} \|A_{k}(t)\| \]

for (1) at points \( t \) where \( h_{k}(t) \neq h_{j}(t) \), \( k \neq j \), and obviously satisfies

\[ \alpha(t) \leq \int_{h(t)}^{t} \|K(t, s)\| \, ds \]

for the integro-differential equation

\[ \dot{x}(t) + \int_{h(t)}^{t} K(t, s)x(s)ds = 0, \quad t \geq t_{0}. \]  \tag{10}

Definition. An absolutely continuous on \([t_{0}, \infty)\) function \( x : \mathbb{R} \to \mathbb{R}^{n} \) will be called a solution of the problem (7), (8) if it satisfies Eq. (7) for almost all \( t \in [t_{0}, \infty) \) and conditions (8) for \( t \leq t_{0} \).

In addition to (7) we consider the non-homogeneous equation

\[ \dot{x}(t) + \int_{h(t)}^{t} d_{R}(t, s)x(s) = f(t), \]  \tag{11}

where \( f(t) \) is a Lebesgue measurable locally essentially bounded vector function.

Definition. For each \( s \geq t_{0} \) and \( t \geq s \) the solution \( X(t, s) \) of the problem

\[ \dot{x}(t) + \int_{h(t)}^{t} d_{R}(t, \tau)x(\tau) = 0, \quad t \geq s. \]

\[ x(t) = 0, \quad t < s, \quad x(s) = I, \]  \tag{12}

where \( I \) is the identity matrix, is called the fundamental matrix of Eq. (7). Here \( X(t, s) = 0, 0 \leq t \leq s < \infty \).

Lemma 1 (Azebilev, Berezansky, & Rahmatullina, 1977; Hale & Verduyn Lunel, 1993). The solution of the initial value problem (11), (8) has the following representation

\[ x(t) = X(t, t_{0})\psi(t_{0}) - \int_{t_{0}}^{t} X(t, s) \left[ \int_{h(s)}^{s} d_{R}(s, \tau)x(\tau) d\tau \right] ds + \int_{t_{0}}^{t} X(t, s)f(s)ds, \quad \text{where} \quad \psi(t) = 0, \quad t > t_{0}. \]  \tag{13}

Definition. Eq. (7) is (uniformly) exponentially stable, if there exist \( K > 0, \lambda > 0 \), such that the fundamental matrix \( X(t, s) \) defined by (12) has the estimate

\[ \|X(t, s)\| \leq Ke^{-\lambda(t-s)}, \quad t \geq s \geq 0. \]  \tag{14}

Let us introduce some functional spaces on a halfline. Denote by \( L_{\infty}[t_{0}, \infty) \) the space of all essentially bounded functions \( y : [t_{0}, \infty) \to \mathbb{R}^{n} \) with the essential supremum norm \( \|y\|_{\infty} = \text{ess sup}_{t \geq t_{0}} \|y(t)\| \), by \( C[t_{0}, \infty) \) the space of all continuous bounded functions on \([t_{0}, \infty) \) with the sup-norm.

We will use the following Bohl–Perron type result presented in Azebilev and Simonov (2003) and Halanay (1966).

Lemma 2. Suppose that for any \( f \in L_{\infty}[t_{0}, \infty) \) the solution of (11) with the zero initial conditions belongs to \( C[t_{0}, \infty) \). Then Eq. (7) is exponentially stable.
3. Perturbation by adding new terms

**Theorem 1.** Suppose that (7) is exponentially stable, its fundamental matrix satisfies the exponential estimate (14) and there exist \( t_0 \geq 0 \) and \( \mu \in (0, 1) \) such that

\[
\sup_{t \geq t_0 \Delta} \left\| e^{\lambda(t-s)} \int_{g(s)}^{t} dQ(s, \tau) \right\| ds \leq \mu. \tag{15}
\]

Then Eq. (6) is also exponentially stable.

**Proof.** Let us demonstrate that the absolutely continuous solution of the non-homogeneous equation

\[
\dot{x}(t) + \int_{h(t)}^{t} dR(t, s) x(s) + \int_{h(t)}^{t} dQ(t, s) x(s) = f(t), \tag{16}
\]

for \( t \geq t_0 \) with the zero initial conditions is bounded on \([t_0, \infty)\) for any \( f \in L_{\infty}[t_0, \infty)\). This solution also satisfies the following integral equation

\[
x(t) + \int_{t_0}^{t} X(t, s) \int_{g(s)}^{t} dQ(s, \tau) x(\tau)ds = \int_{t_0}^{t} X(t, s)f(s)ds,
\]

where \( X(t, s) \) is the fundamental matrix of Eq. (7) which has the estimate (14), thus the right hand side is \( r(t) := \int_{t_0}^{t} X(t, s)f(s)ds \in L_{\infty}[t_0, \infty) \). Eq. (16) has the form

\[
x + Hx = r,
\]

where

\[
\| (Hx)(t) \| = \left\| \int_{t_0}^{t} X(t, s) \int_{g(s)}^{t} dQ(s, \tau) x(\tau)ds \right\|
\]

\[
\leq \left( \int_{t_0}^{t} Ke^{-\lambda(t-s)}ds \right) \| X(t, s) \| \left\| dQ(s, \tau) \right\| \| x \|_{L_{\infty}[t_0, \infty)}
\]

thus the norm of \( H \) in \( L_{\infty}[t_0, \infty) \) does not exceed \( \mu < 1 \), consequently, the inverse \((I + H)^{-1} = I - H + H^2 - H^3 + H^4 \ldots \) is a bounded operator and the function \( x = (I + H)^{-1}r \) is bounded. By Lemma 2 Eq. (6) is exponentially stable. \( \square \)

**Corollary 1.** Suppose that Eq. (7) is exponentially stable. Then for every \( h > 0 \) there exists \( q_0 > 0 \) such that the inequality

\[
\sup_{t \to \infty} \int_{t}^{t+h} \left\| dQ(s, \tau) \right\| ds < q_0 \tag{17}
\]

implies that Eq. (6) is exponentially stable.

**Proof.** Suppose that for the Eq. (7) the fundamental matrix has the exponential estimate (14). Let us prove that the statement of the corollary holds for

\[
q_0 = \frac{e^{kh} - 1}{K e^{\lambda h}}.
\]

For simpler notations, we assume \( h = 1 \), the proof for an arbitrary \( h \) is similar. Denote \( p(s) = \int_{g(s)}^{t} dQ(s, \tau) \). We have for \( t \geq t_0 \geq 0 \)

\[
K \int_{t_0}^{t} e^{-\lambda(t-s)} \| p(s) \| ds
\]

\[
\leq K \int_{t_0}^{t+1} e^{-\lambda(t-s)} \| p(s) \| ds
\]

\[
= Ke^{-\lambda(t)} \left( \int_{t_0}^{t} e^{\lambda s} \| p(s) \| ds + \cdots + \int_{t}^{t+1} e^{\lambda s} \| p(s) \| ds \right)
\]

\[
\leq Ke^{-\lambda(t)} \left( \sup_{t \geq t_0} \int_{t}^{t+1} \| p(s) \| ds \right) \left( e^{\lambda t_0} + \cdots + e^{\lambda(t+1)} \right)
\]

\[
\leq Ke^{-\lambda(t)} \left( \sup_{t \geq t_0} \int_{t}^{t+1} \| p(s) \| ds \right) \frac{e^{\lambda t_0} (e^{\lambda(t+1)} - 1)}{e^{\lambda(t+1)} - 1}
\]

\[
\leq K \sup_{t \geq t_0} \int_{t}^{t+1} \| p(s) \| ds \frac{e^{\lambda(t+1)} - 1}{e^{\lambda(t+1)} - 1}
\]

\[
\leq K \sup_{t \geq t_0} \int_{t}^{t+1} \| p(s) \| ds \frac{e^{\lambda(t+1)}}{e^{\lambda(t+1)} - 1} < 1.
\]

By Theorem 1 Eq. (6) is exponentially stable. \( \square \)

**Corollary 2.** Suppose that (7) is exponentially stable and at least one of the following conditions holds:

1. \( \lim_{t \to \infty} \int_{t}^{t+h} \left\| dQ(s, \tau) \right\| ds = 0 \) for some \( h > 0 \);
2. \( \lim_{t \to \infty} \int_{t}^{t+h} \left\| dQ(s, \tau) \right\| ds = 0 \),
3. \( \int_{0}^{\infty} \left\| dQ(s, \tau) \right\| ds < \infty \).

Then Eq. (6) is also exponentially stable.

4. Delays perturbation

Consider the following equation as a perturbation of Eq. (1)

\[
\dot{x}(t) + \sum_{k=1}^{m} A_k(t)x(g_k(t)) = 0. \tag{18}
\]

**Theorem 2.** Suppose that Eq. (1) is exponentially stable, its fundamental matrix has the exponential estimate (14), and in addition there exist \( t_0 \geq 0 \) and \( \mu \in (0, 1) \) such that for any \( t \geq t_0 \)

\[
\int_{t_0}^{t} Ke^{-\lambda(t-s)} \sum_{k=1}^{m} \| A_k(s) \| \int_{h_k(s)}^{g_k(s)} \sum_{k=1}^{m} \| A_k(\tau) \| d\tau ds \leq \mu.
\]

Then Eq. (18) is also exponentially stable.

**Proof.** Similarly to the proof of Theorem 1, we demonstrate that the solution of the equation

\[
\dot{x}(t) + \sum_{k=1}^{m} A_k(t)x(g_k(t)) = f(t), \quad t \geq t_0,
\]

with the zero initial conditions is bounded on \([t_0, \infty)\) for any \( f \in L_{\infty}[t_0, \infty) \). Eq. (19) can be rewritten as

\[
\dot{x}(t) + \sum_{k=1}^{m} A_k(t)x(h_k(t)) + \sum_{k=1}^{m} A_k(t) \int_{h_k(t)}^{g_k(t)} \dot{x}(s) ds = f(t),
\]

hence after substituting \( \dot{x} \) from (19) we have

\[
\dot{x}(t) + \sum_{k=1}^{m} A_k(t)x(h_k(t))
\]

\[
= \sum_{k=1}^{m} A_k(t) \int_{h_k(t)}^{g_k(t)} \sum_{k=1}^{m} A_k(\tau)x(g_k(\tau)) d\tau ds
\]

\[
= F(t),
\]

where \( F(t) = f(t) - \sum_{k=1}^{m} A_k(t) \int_{h_k(t)}^{g_k(t)} f(s) ds \). Evidently \( F \in L_{\infty}[t_0, \infty) \). Hence a solution of Eq. (19) is also a solution of the operator equation \( x + Hx = r \), where \( r(t) := \int_{t_0}^{t} X(t, s)f(s)ds \in L_{\infty}[t_0, \infty) \) and

\[
\| (Hx)(t) \|
\]

\[
= \left\| \int_{t_0}^{t} X(t, s) \sum_{k=1}^{m} A_k(s) \int_{h_k(s)}^{g_k(s)} \sum_{k=1}^{m} A_k(\tau)x(g_k(\tau)) d\tau ds \right\|
\]

\[
\leq \int_{t_0}^{t} Ke^{-\lambda(t-s)} \sum_{k=1}^{m} \| A_k(s) \| \int_{h_k(s)}^{g_k(s)} \sum_{k=1}^{m} \| A_k(\tau) \| d\tau ds \| x \|_{L_{\infty}[t_0, \infty)} \leq \mu \| x \|_{L_{\infty}[t_0, \infty)}.
\]
Thus the norm of $H$ in $L_\infty[\tau_0, \infty)$ does not exceed $\mu < 1$. Consequently, the inverse $(I + H)^{-1} = I - H + H^2 - H^3 + H^4 \ldots$ is a bounded operator and the function $x = (I + H)^{-1}r$ is bounded. By Lemma 2 Eq. (18) is exponentially stable. □

Next, consider the equation with a distributed delay
\[ \dot{x}(t) + \int_0^t x(s) R(t, s) ds = 0, \quad t \geq t_0, \] (20)
as a perturbation of Eq. (7). Denote
\[ H(t) = \min\{h(t), g(t)\}, \quad G(t) = \max\{h(t), g(t)\}. \]

**Theorem 3.** Suppose that (7) is exponentially stable, its fundamental matrix satisfies (14) and there exist $t_0 \geq 0$ and $\mu \in (0, 1)$ such that
\[ \sup_{t \geq t_0} \int_0^t K e^{-\alpha(t-s)} \|d(t, R(t), s)\| ds \leq \mu. \]

Then Eq. (20) is also exponentially stable.

The proof is similar to the proof of Theorem 2.

**Corollary 3.** Suppose that Eq. (7) is exponentially stable. For every $\delta > 0$ there exists $q_0 > 0$ such that the inequality
\[ \lim_{t \to \infty} \sup_{t \geq t_0} \left[ \int_{t_0}^t \|d(t, R(s), s)\| ds \right] \leq q_0 \]
implies exponential stability of Eq. (20).

**Corollary 4.** Suppose that Eq. (7) is exponentially stable, and at least one of the following conditions holds:

1. \[ \lim_{t \to \infty} \sup_{t \geq t_0} \int_{t_0}^t \|d(t, R(s), s)\| ds = 0 \quad \text{for some } \delta > 0; \]
2. \[ \int_{t_0}^\infty \|d(t, R(s), s)\| ds < \infty; \]
3. \[ \lim_{t \to \infty} \sup_{t \geq t_0} \int_{t_0}^t \|d(t, R(s), s)\| ds = 0. \]

Then Eq. (20) is exponentially stable.

5. Discussion, example and open problems

For equations with a distributed delay, exponential stability of (7) and the condition $\lim_{t \to \infty} \sup_{t \geq t_0} |h(t) - g(t)| = 0$ in general do not imply exponential stability of (20). A perturbation of the value of a pointwise delay may be large even for small perturbations of the lower bound.

**Theorem 2** improves and extends stability condition (5) obtained in Driver (1977) for Eq. (3) with constant delays. Indeed, as a corollary of **Theorem 2** the following inequality implies asymptotic stability of (3) with variable delays $\tau_k(t)$:
\[ \left( \sup_{t \geq t_0} \sum_{k=1}^m \|A_k(t)\| \tau_k(t) \right) \left( \sum_{k=1}^m \sup_{t \geq t_0} \|A_k(t)\| \right) < \frac{\lambda}{N}. \]

**Example.** Consider the scalar equation with a nondelay term
\[ \dot{x}(t) + x(t) + \sum_{k=1}^m a_k(t)x(t - \tau_k(t)) = 0. \]

Then $K = \lambda = 1$ and (21) implies
\[ \left( 1 + \sup_{t \geq t_0} \sum_{k=1}^m \|a_k(t)\| \tau_k(t) \right) \left( \sup_{t \geq t_0} \sum_{k=1}^m \|a_k(t)\| \right) < 1. \]

**Theorem 1** yields that (22) is exponentially stable if for some $\mu \in (0, 1)$, $a_k(t) \geq 0$, $k = 1, \ldots, m$, $\sum_{k=1}^m a_k(t) < \mu < 1$, which is a delay-independent stability condition similar to Zevin and Pinsky (2006). In particular, for the equation
\[ \dot{x}(t) + x(t) + \alpha(1 + \cos \tau)(t - \tau(t)) = 0. \]

**Theorem 1** implies exponential stability of (24) for any $\alpha < 2/(2 + \sqrt{2}) \approx 0.8284$ since
\[ \int_0^t e^{-\alpha(t-s)}(1 + \cos s) ds \leq \int_0^t e^{-\alpha(t-s)}(1 + \cos s) ds \leq \alpha \left[ 1 + \frac{\cos t + \sin t}{2} - \frac{1 - e^{-1}}{2} \right] \leq \frac{\alpha(2 + \sqrt{2})}{2} = \mu < 1. \]

Hence **Theorem 1** implies that for $\alpha < 0.8284$, while (23) yields $\alpha(1 + 2k) \sup_{t \geq t_0} \|R(t)(1 + \cos t)\| < 1$, thus **Theorem 2** outperforms **Theorem 1** (and thus Zevin & Pinsky, 2006) for small $\tau$, where $\tau(t) \leq \tau$, but is obviously worse than the $2/(2 + \sqrt{2})$ bound for $\tau > 0.23$.

We also mention that the equation considered in the present paper is more general than most of the previously considered models.

Finally, let us formulate some open problems.

(1) Formulate sufficient conditions for (6) which for the case (2) generalizes inequality (5).
(2) Most results of the present paper are based on the explicit estimates for the fundamental functions of equations with a distributed delay. Develop techniques to obtain such estimates.
(3) Based on the results of the present paper, deduce explicit exponential stability results for Eq. (7), as well as for its particular cases (1) and (10).

6. Conclusion

The present paper investigated preservation of exponential stability under perturbations for linear systems with a distributed delay. The most general setting is considered where the systems with concentrated delays can be perturbed by adding similar or integral terms, or delays can be perturbed. In real world problems delays are not, generally, concentrated, but may be distributed around the expectancy value. We obtained sufficient conditions under which exponential stability is preserved under perturbations. The superiority of the results of the paper, compared to some previously known, is demonstrated in Section 5, and some features of equations with a distributed delay which distinguish them from both equations with concentrated delays and integrodifferential equations are outlined.

**References**


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