Oscillation criteria for a linear neutral differential equation

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Abstract

For a neutral differential equation

$$\dot{x}(t) - a(t)\dot{x}\left(g(t)\right) + b(t)x\left(h(t)\right) = 0,$$

$$0 \leq a(t) < 1, \quad b(t) \geq 0, \quad g(t) \leq t, \quad h(t) \leq t,$$

a connection between oscillation properties of the differential equation and differential inequalities is established. Explicit nonoscillation and oscillation conditions and a comparison theorem are presented.

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1. Introduction

This paper deals with oscillation properties of a scalar neutral differential equation. Linear neutral type equation can be written in any one of the following two forms:

$$\left(x(t) - a(t)x\left(g(t)\right)\right)' + \sum_{k=1}^{m} b_k(t)x\left(h_k(t)\right) = 0$$

(1)

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or
\[
\dot{x}(t) - a(t)\dot{x}(g(t)) + \sum_{k=1}^{m} b_k(t)x(h_k(t)) = 0, \quad t \geq t_0,
\]
(2)

where \( g(t) \leq t, \ h_k(t) \leq t \).

Equations (1) and (2) are similar; however, there are differences between them. For example, unlike (2), solution \( x \) of (1) is an arbitrary continuous function, such that \( x(t) - a(t)x(g(t)) \) is differentiable. Thus (1) in general cannot be rewritten in form (2), and vice versa.

Concerning the connection of (1) with (2), we mention here paper [6] where the oscillation of (1) was studied by applying an adjoint equation which has form (2). For the autonomous case in the “neutral part” when \( a(t) \equiv a, \ g(t) \equiv t - \sigma \) (1) and (2) are the same equation, once we consider only differentiable solutions \( x(t) \). In this case the results of this paper coincide with the known ones.

It is to be emphasized that Eq. (1) is much better studied than Eq. (2). Extensive literature on (1) is concerned with existence and uniqueness theorems and especially stability and oscillation theories (see monographs [8,9,11,12] and references therein).

Equation (2) is a natural representative of neutral type equations. There exist applied problems which can be written in form (2) [14]. Monograph [2] contains solvability and uniqueness results, the solution representation for (2) and elements of stability theory. Recent monograph [15] is devoted to the stability of Eq. (2). We also mention here paper [1] where a new method based on Bohl–Perron theorem was applied to stability investigation of (2).

Though there exists a developed stability theory for (2) surprisingly there are only few publications on its oscillation. We mention here paper [10] where comparison results for (2) were obtained and two papers [4,5] where the positiveness for the fundamental function of Eq. (2) is studied. The purpose of the present paper is to fill up this gap and to investigate the oscillation of (2). For simplicity we consider an equation with a single delay.

The paper is organized as follows. Section 2 contains relevant definitions and notations and auxiliary lemmas. Section 3 includes the main result of the paper which is the equivalence of the nonoscillation of (2), the existence of a positive solution for a differential inequality and the existence of a nonnegative solution of some explicitly constructed by (2) nonlinear integral inequality. This section also contains a comparison theorem and nonoscillation results for Eq. (2). Section 4 presents conditions when all solutions of (2) are oscillatory. These results are obtained by applying nonoscillation criteria and comparison with a differential equation containing an infinite number of delays.

It is to be noted that in cases when the neutral equation turns into a delay equation (either \( a(t) \equiv 0 \) or \( g(t) \equiv t \)) the oscillation results for (2) coincide with the known ones for delay equations.

2. Preliminaries

We consider a scalar delay differential equation
\[
\dot{x}(t) - a(t)\dot{x}(g(t)) + b(t)x(h(t)) = 0, \quad t \geq t_0,
\]
(3)
under the following conditions:

(a1) \(a(t), b(t), g(t), h(t)\) are Lebesgue measurable locally essentially bounded functions;
(a2) \(a(t) \geq 0, \limsup_{t \to \infty} a(t) < 1, b(t) \geq 0;\)
(a3) \(g(t) \leq t, \text{mes } E = 0 \Rightarrow \text{mes } g^{-1}(E) = 0,\) where \(\text{mes } E\) is Lebesgue measure of the set \(E;\)
(a4) \(h(t) \leq t, \lim_{t \to \infty} h(t) = \infty.\)

Together with (3) we consider for each \(t_1 \geq t_0\) an initial value problem

\[
\dot{x}(t) - a(t)\dot{x}(g(t)) + b(t)x(h(t)) = f(t), \quad t \geq t_1, \tag{4}
\]
\[
x(t) = \varphi(t), \quad \dot{x}(t) = \psi(t), \quad t < t_1, \quad x(t_1) = x_0, \quad \dot{x}(t_1) = x_1. \tag{5}
\]

We also assume that the following hypothesis holds:

(a5) \(f: [t_1, \infty) \to \mathbb{R}\) is a Lebesgue measurable locally essentially bounded function, \(\varphi, \psi: (-\infty, t_1) \to \mathbb{R}\) are Borel measurable bounded functions.

**Definition.** An absolutely continuous on each interval \([t_1, b]\) function \(x: \mathbb{R} \to \mathbb{R}\) is called a solution of problem (4)–(5), if it satisfies Eq. (4) for almost all \(t \in [t_1, \infty)\) and also satisfies (5).

**Lemma 2.1** [2]. Let (a1)–(a5) hold. Then there exists one and only one solution of problem (4)–(5).

Denote by \(L_\infty[t_0, b]\) the space of all Lebesgue measurable essentially bounded in the interval \([t_0, b]\) functions with the usual sup-norm. Define in this space a linear operator

\[
(Sy)(t) = \begin{cases} 
 a(t)y(g(t)), & g(t) \geq t_0, \\
 0, & g(t) < t_0.
\end{cases}
\]

**Lemma 2.2** [2]. Suppose \(a, g\) are Lebesgue measurable locally essentially bounded functions,

\[
\limsup_{t \to \infty} \left|a(t)\right| < 1
\]

and condition (a3) holds. Then for every \(b > t_0\) operator \(S\) acts in the space \(L_\infty[t_0, b]\), its operator norm \(\|S\| < 1,\) and for the inverse operator we have a representation

\[
(I - S)^{-1} = I + S + S^2 + \cdots,
\]

where \(I\) is an identical operator. Operator \((I - S)^{-1}\) is positive if \(a(t) \geq 0.\)

Consider now a differential equation with infinite number of delays

\[
\dot{x}(t) + \sum_{k=0}^{\infty} b_k(t)x(h_k(t)) = 0, \quad t \geq t_0, \tag{6}
\]
where
\[ b_0(t) = b(t), \quad b_{k+1}(t) = (Sb_k)(t), \]
\[ h_0(t) = h(t), \quad h_{k+1}(t) = h_k(g(t)). \]

(7)

By induction it is easy to see that
\[ \sup_{t \in [t_0, b]} |b_k(t)| \leq \sup_{t \in [t_0, b]} |a(t)|^k \sup_{t \in [t_0, b]} |b(t)|. \]

Then
\[ B(t) = \sum_{k=0}^{\infty} b_k(t) \]
is an essentially locally bounded function. Equation (6) with this condition was considered in [3].

**Definition.** We will say that Eq. (3) has a nonoscillatory solution if there exists a solution of (3), (5), which is eventually positive or eventually negative. Otherwise all solutions of (3) are oscillatory.

The same definition we will use for Eq. (6).

**Lemma 2.3.** Let conditions (a1)-(a4) hold. Equation (6) has a nonoscillatory solution if and only if Eq. (3) has a nonoscillatory solution.

**Proof.** Equation (3) can be rewritten in the form
\[ \dot{x}(t) + (I - S)^{-1}[b(t)x(h(t))] = 0. \]

This equation is the same to (6). Then oscillation properties of (3) and (6) coincide. \( \square \)

We will need some properties of Eq. (6). Consider together with (6) the following equation:
\[ \dot{x}(t) + \sum_{k=0}^{\infty} c_k(t)x(p_k(t)) = 0, \]

(8)

where functions \( c_k \) are essentially locally bounded and for \( p_k \) conditions (a4) hold.

**Lemma 2.4** [3]. (1) Suppose Eq. (6) has a nonoscillatory solution. Then there exists \( t_1 \) such that the solution of (6) satisfying \( x(t_1) = 1, x(t) = 0, t < t_1, \) is positive for \( t \geq t_1. \)

(2) Suppose all solutions of (6) are oscillatory and \( c_k(t) \geq b_k(t), p_k(t) \leq h_k(t). \) Then all solutions of (8) are oscillatory.
3. Nonoscillation criteria

The following theorem establishes nonoscillation criteria.

**Theorem 3.1.** Suppose (a1)–(a4) hold. Then the following hypotheses are equivalent:

1. Differential inequality
   \[
   \dot{y}(t) - a(t)\dot{y}(g(t)) + b(t)y(h(t)) \leq 0, \quad t \geq t_0, \tag{9}
   \]
   has an eventually positive solution.

2. For some \( t_1 \geq t_0 \) and for \( t \geq t_1 \) an integral inequality
   \[
   u(t) \geq a(t)u(g(t)) \exp \left\{ \int_{g(t)}^{t} u(s) \, ds \right\} + b(t) \exp \left\{ \int_{h(t)}^{t} u(s) \, ds \right\}, \tag{10}
   \]
   has a nonnegative locally integrable solution, where the first (the second) of two terms in the right-hand side is added only if \( g(t) \geq t_1 \) or \( h(t) \geq t_1 \), respectively.

3. Equation (3) has a nonoscillatory solution.

**Proof.** (1) \( \Rightarrow \) (2) Let \( y(t) \) be a positive solution of inequality (9) for \( t \geq t_1 \). There exists a point \( t_2 \geq t_1 \) such that \( g(t) \geq t_1, h(t) \geq t_1 \) if \( t \geq t_2 \). Then (see the proof of Lemma 2.3) \( y \) is also a solution of the inequality
   \[
   \dot{y}(t) + (I - S)^{-1} \left[ b(t)y(h(t)) \right] \leq 0, \quad t > t_2,
   \]
   i.e., \( \dot{y}(t) \leq - (I - S)^{-1} [b(t)y(h(t))] < 0 \) for each \( t > t_2 \), since by Lemma 2.2 the operator \( (I - S)^{-1} \) is positive. Hence \( y \) is nonincreasing and the function \( u(t) \) defined by the equality
   \[
   u(t) = - \frac{d}{dt} \ln \frac{y(t)}{y(t_2)}
   \]
   is positive for \( t \geq t_2 \).

   Then
   \[
   y(t) = y(t_2) \exp \left\{ - \int_{t_2}^{t} u(s) \, ds, \quad t \geq t_2 \right\}. \tag{11}
   \]
   After substituting (11) into (9) and carrying the exponent out of the brackets we obtain
   \[
   - \exp \left\{ - \int_{t_2}^{t} u(s) \, ds \right\} y(t_2) \times \left[ u(t) - a(t)u(g(t)) \exp \left\{ \int_{g(t)}^{t} u(s) \, ds \right\} - b(t) \exp \left\{ \int_{h(t)}^{t} u(s) \, ds \right\} \right] \leq 0,
   \]
   which implies (10).
(2) ⇒ (3) Suppose \( u_0(t), \ t \geq t_1 \), is a nonnegative solution of (10). Denote

\[
u_{n+1}(t) = a(t)u_n(g(t)) \exp \left\{ \int_{g(t)}^{t} u_n(s) \, ds \right\} + b(t) \exp \left\{ \int_{h(t)}^{t} u_n(s) \, ds \right\},
\]

(12)

Since \( a, b \) are nonnegative and (10) holds for \( u = u_0 \), then

\[0 \leq u_{n+1}(t) \leq u_n(t) \leq \cdots \leq u_0(t)\]

Hence there exists a pointwise limit \( u(t) = \lim_{n \to \infty} u_n(t) \). Lebesgue convergence theorem and (12) imply

\[ u(t) = a(t)u(g(t)) \exp \left\{ \int_{g(t)}^{t} u(s) \, ds \right\} + b(t) \exp \left\{ \int_{h(t)}^{t} u(s) \, ds \right\}.
\]

Obviously

\[ x(t) = \exp \left\{ - \int_{t_1}^{t} u(s) \, ds \right\}, \quad t \geq t_1,
\]

is a nonoscillatory solution of (3), which completes the proof. \( \square \)

**Remark.** The equivalence of oscillation properties for Eq. (1) and the corresponding differential inequality was demonstrated in [16,17].

As a corollary of Theorem 3.1 we obtain a comparison result. Consider a neutral differential equation

\[
\dot{x}(t) - a_1(t)\dot{x}(g_1(t)) + b_1(t)x(h_1(t)) = 0, \quad t \geq t_0,
\]

(13)

where for parameters of (13) hypotheses (a1)–(a4) hold.

**Theorem 3.2.** (1) Suppose

\[ a_1(t) \leq a(t), \quad b_1(t) \leq b(t), \quad g(t) \leq g_1(t), \quad h(t) \leq h_1(t)
\]

and Eq. (3) has a nonoscillatory solution. Then Eq. (13) also has a nonoscillatory solution.

(2) Suppose

\[ a(t) \leq a_1(t), \quad b(t) \leq b_1(t), \quad g_1(t) \leq g(t), \quad h_1(t) \leq h(t)
\]

and all solutions of (3) are oscillatory. Then all solutions of (13) are oscillatory.

**Proof.** (1) Theorem 3.1 yields that there exists a nonnegative solution \( u \) of inequality (10). Then \( u \) is also a solution of this inequality, where \( a, b, g, h \) are replaced by \( a_1, b_1, g_1, h_1 \).

By Theorem 3.1 Eq. (13) has a nonoscillatory solution.

Statement (2) is a consequence of (1). \( \square \)

**Remark.** Another comparison theorem for Eq. (3) was obtained in [10].
Corollary 3.1. Let \( a, b, \sigma, \tau > 0 \).

(1) Suppose \( a(t) \leq a, b(t) \leq b, g(t) \geq t - \sigma, h(t) \geq t - \tau \) and equation

\[
\dot{x}(t) - a\dot{x}(t - \sigma) + bx(t - \tau) = 0
\]

has a nonoscillatory solution. Then Eq. (3) also has a nonoscillatory solution.

(2) Suppose \( a(t) \geq a, b(t) \geq b, g(t) \leq t - \sigma, h(t) \leq t - \tau \) and all solutions of (14) are oscillatory. Then all solution of (3) are oscillatory.

Using Theorem 3.1 we will obtain now explicit nonoscillation conditions.

Theorem 3.3. Suppose (a1)–(a4) hold, \( b(t) \neq 0 \) almost everywhere and at least one of the following conditions holds:

(1) \( 0 < \lambda < \lim_{t \to \infty} \left( \frac{1}{e} - \frac{a(t)b(g(t))}{b(t)} \right) \exp \left\{ \frac{1}{\lambda} \int_{g(t)}^{h(t)} b(s) \, ds \right\} \),

where \( \lambda = \lim sup_{t \to \infty} \int_{g(t)}^{h(t)} b(s) \, ds \);

(2) \( 0 < \lambda < \lim_{t \to \infty} \left( \frac{1}{e} - \frac{a(t)b(g(t))}{b(t)} \right) \exp \left\{ \frac{1}{\lambda} \int_{g(t)}^{h(t)} b(s) \, ds \right\} \),

where \( \lambda = \lim sup_{t \to \infty} \int_{h(t)}^{g(t)} b(s) \, ds \);

(3) \( 0 < \lambda < \lim_{t \to \infty} \left( \frac{1}{e} \left( 1 - \frac{a(t)b(g(t))}{b(t)} \right) \right) \exp \left\{ \frac{1}{\lambda} \int_{g(t)}^{h(t)} b(s) \, ds \right\} \),

where \( \lambda = \lim sup_{t \to \infty} \int_{h(t)}^{g(t)} b(s) \, ds \).

Then Eq. (3) has a nonoscillatory solution.

Proof. (1) We will show that \( u(t) = \frac{b(t)}{\lambda} \) is a solution of inequality (10).

The definition of \( \lambda \) and inequality (15) yield that there exists \( t_1 > t_0 \) and \( \epsilon > 0 \) such that

\[
\lambda e^\epsilon \leq \left( \frac{1}{e} - \frac{a(t)b(g(t))}{b(t)} \right) \exp \left\{ \frac{1}{\lambda} \int_{g(t)}^{h(t)} b(s) \, ds \right\}, \quad t \geq t_1,
\]

and

\[
\frac{1}{\lambda} \int_{g(t)}^{h(t)} b(s) \, ds \leq 1 + \epsilon, \quad t > t_1.
\]

Inequality (17) implies

\[
\frac{b(t)}{\lambda e^1 + \epsilon} \geq \frac{1}{\lambda} a(t)b(g(t)) + b(t) \exp \left\{ \frac{1}{\lambda} \int_{g(t)}^{h(t)} b(s) \, ds \right\}.
\]
Then by (18) we have
\[ \frac{b(t)}{\lambda} \exp\left\{ -\frac{1}{\lambda} \int_{g(t)}^{t} b(s) \, ds \right\} \geq \frac{1}{\lambda} a(t)b(g(t)) + b(t) \exp\left\{ -\frac{1}{\lambda} \int_{h(t)}^{t} b(s) \, ds \right\} \]
for \( t \geq t_1 \). Hence
\[ \frac{b(t)}{\lambda} \geq \frac{1}{\lambda} a(t)b(g(t)) \exp\left\{ \frac{1}{\lambda} \int_{g(t)}^{t} b(s) \, ds \right\} + b(t) \exp\left\{ \frac{1}{\lambda} \int_{h(t)}^{t} b(s) \, ds \right\}, \quad (19) \]
which implies \( u(t) = b(t)/\lambda \) is a nonnegative solution of inequality (10); consequently, Eq. (3) has a nonoscillatory solution.

The proof of (2) is similar.

(3) We will show that \( u(t) = b(t)/\lambda \) is a solution of inequality (10).

The definition of \( \lambda \) and inequality (16) imply for some \( t_1 > t_0 \) and \( \varepsilon > 0 \),
\[ \frac{1}{\lambda} \int_{g(t)}^{t} b(s) \, ds \leq 1 + \varepsilon, \quad t > t_1, \]
and
\[ \lambda b(t)e^{1+\varepsilon} \leq b(t) - a(t)b(g(t)) \exp\left\{ \frac{1}{\lambda} \int_{g(t)}^{t} b(s) \, ds \right\}. \quad t > t_1. \]
Then
\[ \lambda b(t) \exp\left\{ \frac{1}{\lambda} \int_{h(t)}^{t} b(s) \, ds \right\} \leq b(t) - a(t)b(g(t)) \exp\left\{ \frac{1}{\lambda} \int_{g(t)}^{t} b(s) \, ds \right\} \]
for \( t > t_1 \). The latter inequality is equivalent to (19), therefore by Theorem 3.1, Eq. (3) has a nonoscillatory solution.

Corollary 3.2. Suppose (a1)–(a4) hold, \( b(t) \neq 0 \) almost everywhere, \( h(t) - g(t) \) is eventually positive or eventually negative, and
\[ 0 < \limsup_{t \to \infty} \int_{h(t)}^{t} b(s) \, ds < \liminf_{t \to \infty} \left( \frac{1}{e} - \frac{a(t)b(g(t))}{b(t)} \right). \]
Then Eq. (3) has a nonoscillatory solution.

The proof follows from conditions (2) and (3) of Theorem 3.3.

Corollary 3.3. Suppose \( 0 < a < 1, b, \tau, \sigma > 0 \) and at least one of the following conditions holds:
Then Eq. (14) has a nonoscillatory solution.

Remark. (1) The same results as in Corollary 3.3 by another method were obtained in [9].
(2) Corollaries 3.1 and 3.3 can be employed to obtain explicit nonoscillation conditions for Eq. (3).

Another set of explicit nonoscillation conditions for Eq. (3) can be obtained by applying the following result.

Theorem 3.4. Suppose (a1)–(a4) hold, \( b(t) \neq 0 \) almost everywhere and at least one of the following conditions holds:

1. \( 0 < \lambda < \liminf_{t \to \infty} \left( \frac{1}{e(1 - a(t))} - \frac{a(t)b(g(t))}{b(t)[1 - a(g(t))]} \right) \exp \left\{ \frac{1}{\lambda} \int_{g(t)}^{h(t)} \frac{b(s)}{1 - a(s)} \, ds \right\} \),

where
\[
\lambda = \limsup_{t \to \infty} \int_{g(t)}^{t} \frac{b(s)}{1 - a(s)} \, ds;
\]

2. \( 0 < \lambda < \liminf_{t \to \infty} \left( \frac{1}{e(1 - a(t))} - \frac{a(t)b(g(t))}{b(t)[1 - a(g(t))]} \right) \exp \left\{ \frac{1}{\lambda} \int_{h(t)}^{g(t)} \frac{b(s)}{1 - a(s)} \, ds \right\} \),

where
\[
\lambda = \limsup_{t \to \infty} \int_{h(t)}^{t} \frac{b(s)}{1 - a(s)} \, ds;
\]

3. \( 0 < \lambda < \liminf_{t \to \infty} \frac{1}{e} \left( \frac{1}{1 - a(t)} - \frac{a(t)b(g(t))}{b(t)[1 - a(g(t))]} \right) \exp \left\{ \frac{1}{\lambda} \int_{g(t)}^{h(t)} \frac{b(s)}{1 - a(s)} \, ds \right\} \),

where
\[
\lambda = \limsup_{t \to \infty} \int_{h(t)}^{t} \frac{b(s)}{1 - a(s)} \, ds.
\]

Then Eq. (3) has a nonoscillatory solution.
The proof is similar to the proof of Theorem 3.3 if we assume
\[ u(t) = \frac{b(t)}{\lambda(1 - a(t))}. \]

**Corollary 3.4.** Suppose (a1)–(a4) hold, \( b(t) \neq 0 \) almost everywhere, \( h(t) - g(t) \) is eventually positive or eventually negative, and
\[
0 < \limsup_{t \to \infty} \int_{h(t)}^{t} \frac{b(s)}{1 - a(s)} \, ds < \liminf_{t \to \infty} \left( \frac{1}{e(1 - a(t))} - \frac{a(t)b(g(t))}{b(t)[1 - a(g(t))]} \right).
\]

Then Eq. (3) has a nonoscillatory solution.

**Remark.** If in Eq. (3) \( g(t) \equiv t \) then from Theorem 3.4 we obtain the best possible nonoscillation condition for this delay equation,
\[
\limsup_{t \to \infty} \int_{h(t)}^{t} \frac{b(s)}{1 - a(s)} \, ds < \frac{1}{e}.
\]

The following theorem is a generalization of the well-known nonoscillation condition to neutral equations.

**Theorem 3.5.** Suppose (a1)–(a4) hold,
\[
\int_{t_0}^{\infty} b(s) \, ds < \infty, \tag{20}
\]
there exist \( \lambda > 0 \) and \( t_1 \geq t_0 \) such that \( \lambda a < 1 \), where
\[ a = \limsup_{t \to \infty} a(t) \]
and \( b(g(t)) \leq \lambda b(t) \), \( t \geq t_1 \). Then Eq. (3) has a nonoscillatory solution.

**Proof.** Let us define two positive numbers \( \beta \) and \( \alpha \) in the following way:
\[ \beta > \frac{1}{1 - \lambda a}, \quad 1 < \alpha \leq \frac{\beta}{\lambda a \beta + 1}. \]

There exist \( T \geq t_1 \) and \( T_1 > T \) such that \( \exp\{\beta \int_{T}^{\infty} b(s) \, ds\} < \alpha \) and \( h(t) \geq T, \ g(t) \geq T, \ t \geq T_1 \).

We will show that \( u(t) = \beta b(t) \) is a nonnegative solution of inequality (10). We have for \( t \geq T_1 \),
\[
\begin{align*}
& a(t)u(g(t)) \exp \left\{ \int_{g(t)}^{t} u(s) \, ds \right\} + b(t) \exp \left\{ \int_{h(t)}^{t} u(s) \, ds \right\} \\
& \leq a \lambda \beta b(t) \alpha + b(t) \alpha = b(t)(\lambda a \beta + 1) \alpha \leq \beta b(t) = u(t).
\end{align*}
\]

Theorem 3.1 implies that Eq. (3) has a nonoscillatory solution.  \(\square\)
Corollary 3.5. Suppose (20) holds and
\[
\lim_{t \to \infty} \frac{b(g(t))}{b(t)} = 1.
\]
Then Eq. (3) has a nonoscillatory solution.

Example 1. Consider an equation
\[
\dot{x}(t) - a(t)\dot{x}(t - \sigma) + \frac{b}{t^\alpha}x(h(t)) = 0, \quad t \geq t_0 > 0,
\]
where \(0 < a(t) < 1\), \(b > 0\), \(\sigma > 0\), \(\alpha > 1\), \(h(t) \leq t\). By Corollary 3.5, Eq. (21) has a nonoscillatory solution.

Theorem 3.6. Let \(\int_{t_0}^{\infty} b(s) \, ds = \infty\). Then for every nonoscillatory solution of (3) we have
\[
\lim_{t \to \infty} x(t) = 0.
\]

Proof. If \(x(t) > 0\), \(t \geq t_1\), then for some \(t_2 \geq t_1\) function \(u(t) = -\dot{x}(t)/x(t)\) is a nonnegative solution of (10) for \(t \geq t_2\) (see the proof of Theorem 3.1). Inequality (10) implies that \(u(t) \geq b(t)\), hence \(\int_{t_0}^{\infty} u(s) \, ds = \infty\). For solution \(x\) of (3) we have
\[
x(t) = x(t_2) \exp\left\{-\int_{t_2}^{t} u(s) \, ds\right\} \quad \text{for} \quad t \geq t_2.
\]
Then \(\lim_{t \to \infty} x(t) = 0\).

4. Explicit oscillation conditions

Denote \(p(t) = \max\{g(t), h(t)\}\).

Theorem 4.1. Suppose (a1)–(a4) hold and
\[
\lim_{t \to \infty} \inf_{p(t)} \int_{t_0}^{t} \left( a(s) b\left(g(s)\right) \exp\left\{ \int_{g(s)}^{p(s)} b(\tau) \, d\tau \right\} + b(s) \exp\left\{ \int_{h(s)}^{p(s)} b(\tau) \, d\tau \right\} \right) \, ds > \frac{1}{e}.
\]
(22)

Then all solutions of (3) are oscillatory.

Proof. Suppose there exists a nonoscillatory solution of (3). Then there exists a nonnegative solution \(u\) of inequality (10) for \(t \geq t_1 \geq t_0\). Rewrite inequality (10) in the form
\[
u(t) \exp\left\{-\int_{p(t)}^{t} u(s) \, ds\right\}
\]
\[ \geq a(t)u\left(g(t)\right) \exp \left\{ \int_{g(t)}^{p(t)} u(s) \, ds \right\} + b(t) \exp \left\{ \int_{h(t)}^{p(t)} u(s) \, ds \right\}, \quad t \geq t_1. \]

Then
\[ \int_{p(t)}^{t} u(s) \exp \left\{ - \int_{p(t)}^{s} u(\tau) \, d\tau \right\} \, ds \]
\[ \geq \int_{p(t)}^{t} \left( a(s)u\left(g(s)\right) \exp \left\{ \int_{g(s)}^{p(s)} u(\tau) \, d\tau \right\} + b(s) \exp \left\{ \int_{h(s)}^{p(s)} u(\tau) \, d\tau \right\} \right) \, ds \]
for \( t \geq t_1 \). Inequality (10) yields that \( u(t) \geq b(t) \), therefore
\[ \int_{p(t)}^{t} u(s) \exp \left\{ - \int_{p(t)}^{s} u(\tau) \, d\tau \right\} \, ds \]
\[ \geq \int_{p(t)}^{t} \left( a(s)b\left(g(s)\right) \exp \left\{ \int_{g(s)}^{p(s)} b(\tau) \, d\tau \right\} + b(s) \exp \left\{ \int_{h(s)}^{p(s)} b(\tau) \, d\tau \right\} \right) \, ds \]
for \( t \geq t_1 \). Since
\[ \int_{p(t)}^{t} u(s) \exp \left\{ - \int_{p(t)}^{s} u(\tau) \, d\tau \right\} \, ds \leq \int_{p(t)}^{t} u(s) \exp \left\{ - \inf_{t \geq t_1} \int_{p(t)}^{t} u(\tau) \, d\tau \right\} \, ds \]
\[ = \exp \left\{ - \inf_{t \geq t_1} \int_{p(t)}^{t} u(\tau) \, d\tau \right\} \int_{p(t)}^{t} u(s) \, ds, \]
then
\[ \exp \left\{ - \inf_{t \geq t_1} \int_{p(t)}^{t} u(\tau) \, d\tau \right\} \inf_{t \geq t_1} \int_{p(t)}^{t} u(s) \, ds \]
\[ \geq \inf_{t \geq t_1} \int_{p(t)}^{t} \left( a(s)b\left(g(s)\right) \exp \left\{ \int_{g(s)}^{p(s)} b(\tau) \, d\tau \right\} + b(s) \exp \left\{ \int_{h(s)}^{p(s)} b(\tau) \, d\tau \right\} \right) \, ds. \]
We have \( \sup_{t \geq 0} te^{-t} = 1/e \), which implies
\[ \liminf_{t \to \infty} \int_{p(t)}^{t} \left( a(s)b\left(g(s)\right) \exp \left\{ \int_{g(s)}^{p(s)} b(\tau) \, d\tau \right\} + b(s) \exp \left\{ \int_{h(s)}^{p(s)} b(\tau) \, d\tau \right\} \right) \, ds \leq \frac{1}{e}. \]
This is a contradiction with (22), which proves the theorem. \( \square \)
Corollary 4.1. Suppose at least one of the following conditions holds:

1. \( \sigma \leq \tau, \sigma b(a + e^{\tau - \sigma}) > 1/e. \)
2. \( \sigma \geq \tau, \tau b(1 + ae^{\sigma - \tau}) > 1/e. \)

Then all solutions of (14) are oscillatory.

Lemma 2.3 yields that oscillation properties of Eqs. (3) and (6) are equivalent. As a corollary of this statement we obtain new explicit oscillation conditions.

Theorem 4.2. Suppose (a1)–(a4) hold and all solutions of delay differential equation

\[
\dot{x}(t) + b(t)x(h(t)) = 0 \tag{23}
\]

are oscillatory. Then all solutions of (3) are oscillatory.

Proof. Suppose (3) has a nonoscillatory solution. Lemma 2.3 yields that (6) has a nonoscillatory solution. Lemma 2.4 implies that for some \( t_1 \geq t_0 \) solution \( x \) of (6) with \( x(t) = 0, t \leq t_1, x(t_1) = 1 \) is positive. Then

\[
\dot{x}(t) + b(t)x(h(t)) \leq 0, \quad t \geq t_1.
\]

Hence [11] Eq. (23) has a nonoscillatory solution. We have a contradiction with our assumption. \( \square \)

Corollary 4.2. Suppose (a1)–(a4) hold and

\[
\lim_{t \to \infty} \inf_{h(t)} \int_{h(t)}^{t} b(s) \, ds > \frac{1}{e}.
\]

Then all solutions of (3) are oscillatory.

Remark. The same result as in Corollary 4.2 for Eq. (1) was obtained in [7].

Corollary 4.3. Suppose (a1)–(a3) hold, \( h(t) \equiv t \), and

\[
\lim_{t \to \infty} \inf_{g(t)} \int_{g(t)}^{t} a(s)b(g(s)) \exp \left\{ \int_{g(s)}^{t} b(\tau) \, d\tau \right\} \, ds > \frac{1}{e}. \tag{24}
\]

Then all solutions of (3) are oscillatory.

Proof. If \( h(t) \equiv t \) then (6) has a form

\[
\dot{x}(t) + b(t)x(t) + a(t)b(g(t))x(g(t)) + \cdots = 0. \tag{25}
\]

After substituting

\[
x(t) = y(t) \exp \left\{ - \int_{t_0}^{t} b(s) \, ds \right\}
\]
in Eq. (25) and multiplying both sides by \( \exp\left[\int_{t_0}^{t} b(s) \, ds\right] \) we have
\[
\dot{y}(t) + a(t)b\left(g(t)\right) \exp\left\{\int_{g(t)}^{t} b(s) \, ds\right\} y\left(g(t)\right) + \cdots = 0.
\]
Condition (24) and the proof of Theorem 4.2 imply that all solutions of this equation and therefore all solutions of (3) are oscillatory. \( \square \)

**Theorem 4.3.** Suppose (a1)–(a4) hold, \( h \) is a nondecreasing function and all solutions of equation
\[
\dot{x}(t) + \left(\left(I - S\right)^{-1}b\right)(t)x\left(h(t)\right) = 0
\]
are oscillatory. Then all solutions of (3) are oscillatory.

**Proof.** Equation (26) can be rewritten in the form
\[
\dot{x}(t) + \sum_{k=0}^{\infty} b_k(t)x\left(h(t)\right) = 0, \tag{27}
\]
where \( b_k(t) \) are defined by (7). We have \( h(g(t)) \leq h(t) \) and hence \( h_k(t) \leq h(t) \), where \( h_k(t) \) were also defined in (7). Lemma 2.4 implies that all solutions of (6) are oscillatory. By Lemma 2.3 all solutions of (3) are also oscillatory. \( \square \)

**Corollary 4.4.** Suppose (a1)–(a4) hold, \( h \) is a nondecreasing function, and
\[
\liminf_{t \to \infty} \int_{h(t)}^{t} \left(\left(I - S\right)^{-1}b\right)(s) \, ds > \frac{1}{e},
\]
Then all solutions of (3) are oscillatory.

**Corollary 4.5.** Suppose (a1)–(a4) hold, \( h \) is a nondecreasing function and for some \( n \geq 0 \) all solutions of equation
\[
\dot{x}(t) + \sum_{k=0}^{n} b_k(t)x\left(h(t)\right) = 0
\]
are oscillatory, where \( b_k \) are defined by (7). Then all solutions of (3) are oscillatory.

**Remark.** A similar result to Corollary 4.5 for Eq. (1) was obtained in [13].

**Corollary 4.6.** Let \( a, b, \sigma, \tau > 0 \). If \( b\tau e \geq 1 - a > 0 \), then all solutions of (14) are oscillatory.

This can be obtained by applying Corollary 4.4 and equality \( \left(I - S\right)^{-1}b = b/(1 - a) \), where \( a, b \) are constants.
Remark. (1) A more general result was obtained in [9] by another method. (2) By Corollaries 3.1, 4.1 and 4.6 one can obtain explicit oscillation conditions for (3).

**Corollary 4.7.** Suppose (a1)–(a4) hold, h is a nondecreasing function, a(t), b(t) are nonincreasing functions, and all solutions of equation

\[ \dot{x}(t) + \frac{b(t)}{1 - a(t)} x(h(t)) = 0 \]  

are oscillatory. Then all solutions of (3) are oscillatory.

**Proof.** Let us rewrite Eq. (28) in the form

\[ \dot{x}(t) + \sum_{k=0}^{\infty} b(t) a^k(t) x(h(t)) = 0. \]

We have \( b(t) a^k(t) \leq b_k(t) \), where \( b_k(t) \) are defined by (7). Theorem 4.3 and Lemma 2.4 imply that all solutions of (3) are oscillatory. \( \square \)

**Corollary 4.8.** Suppose (a1)–(a4) hold, h is a nondecreasing function, a and b are nonincreasing functions, and

\[ \lim\inf_{t \to \infty} \int_{h(t)}^{t} \frac{b(s)}{1 - a(s)} \, ds > \frac{1}{e}. \]

Then all solutions of (3) are oscillatory.

**Remark.** If \( g(t) \equiv t \) then (29) is the best possible oscillation condition for this delay differential equation.

**Example 2.** Consider the following equation:

\[ \dot{x}(t) - a \dot{x}(g(t)) + \frac{b}{t} x(t^{\mu}) = 0, \quad t \geq t_0 > 0, \]  

where 0 < a < 1, b > 0, \( \mu > 1 \). We have

\[ \int_{h(t)}^{t} \frac{b(s)}{1 - a(s)} \, ds = \int_{t^{\mu}}^{t} \frac{b}{(1 - a)s} \, ds = \frac{b}{1 - a} \ln \mu. \]

Hence if \( b/(1 - a) \ln \mu > 1/e \) then all solutions of (30) are oscillatory.

We will obtain now explicit oscillation conditions without the assumption that parameters of (3) are monotone functions. Denote

\[ \tilde{a}(t) = \sup_{t \geq t_1} a(t), \quad \underline{a}(t) = \inf_{t \geq t_1} a(t). \]
Theorem 4.4. Suppose (a1)–(a4) hold and there exists $t_1 \geq t_0$ such that all solutions of equation

$$\dot{x}(t) + \frac{b(t)}{1-a(t)} x\left(h(t)\right) = 0 \quad (31)$$

are oscillatory. Then all solutions of (3) are oscillatory.

Proof. Let us rewrite Eq. (31) in the form

$$\dot{x}(t) + \sum_{k=0}^{\infty} \frac{b(t)}{1-a(t)} a(t)^k x\left(h(t)\right) = 0.$$ 

We have

$$h_1(t) = h(g(t)) \leq \hat{h}(g(t)) \leq \bar{h}(t),$$

$$b_1(t) = a(t)b\left(g(t)\right) \geq a(t)b\left(g(t)\right) \geq a(t)^k b(t).$$

By induction we obtain that $h_k(t) \leq \hat{h}(t)$, $b_k(t) \geq a(t)^k b(t)$. Lemma 2.4 implies that all solutions of (6) are oscillatory. Then all solutions of (3) are oscillatory.  

Corollary 4.9. Suppose

$$\liminf_{t \to \infty} \int_{\hat{h}(t)}^{t} \frac{b(s)}{1-a(s)} \, ds > \frac{1}{e}.$$ 

Then all solutions of (3) are oscillatory.

References

[3] L. Berezansky, E. Braverman, On oscillation of a differential equation with infinite number of delays,