Elicitation of the parameters of a general ordinal sorting model in a conjoint measurement setting

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Catania Workshop COST IC0602
Algorithmic Decision Theory   April 16, 2008

Motivation

D. Bouyssou and myself (M.P.) (together with others like S. Greco, B. Matarazzo, R. Slowinski, Th. Marchant, . . .) have been trying to develop axiomatic foundations for outranking methods for many years.

And we have done so in a conjoint measurement framework.

Why? To imitate what had been done for the *additive value (utility) function* model ?

In a sense, the answer is “YES” but there are also good reasons . . .

Axiomatic analysis is the key to the elaboration of rigorous elicitation methods

The main virtues of an axiomatic analysis are that

- the model is completely described by one or several system(s) of properties
- the axioms may provide means for testing the applicability of the model
- it focuses the attention on primitives (e.g. tradeoffs, marginal preferences, ...) on which the elicitation procedure can rely

Illustration : using the *Standard sequence* method to build an additive value fonction

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i)$$

Well-designed sequence of *indifference judgments*

Example (Bouyssou et al. [00]): a student buys a second-hand sportive car

A standard sequence is used for building an (approximate) value function on the criterion "acceleration" on the basis of a *standard* price difference: 15,000 € to 16,000 €

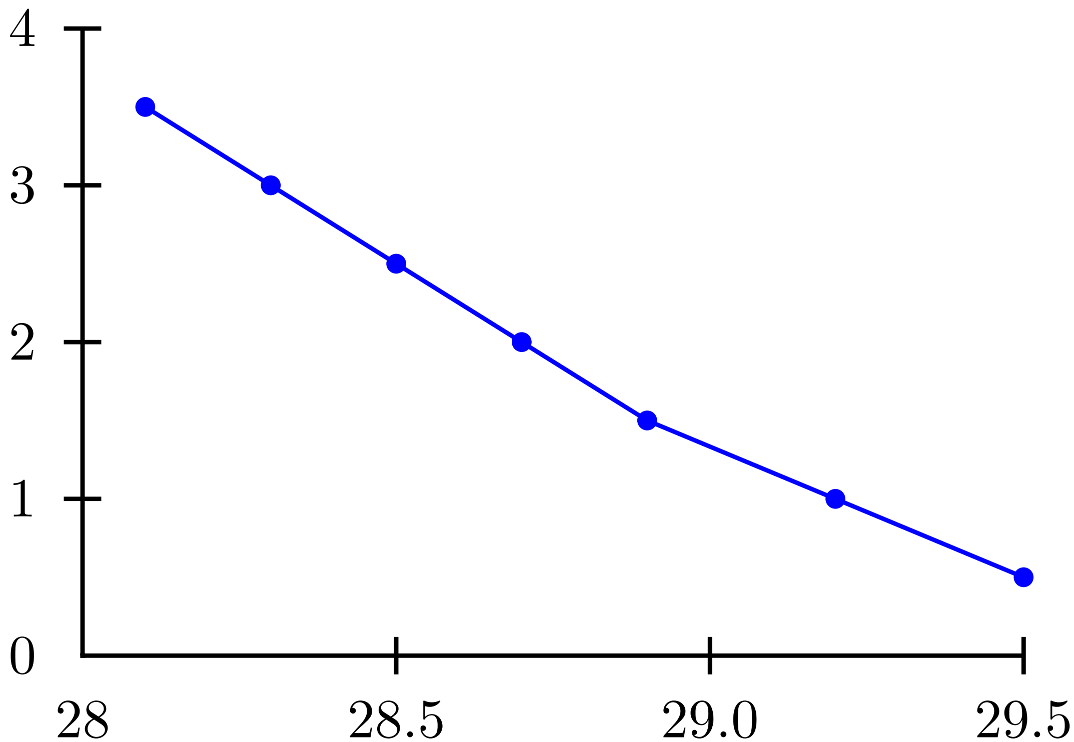
As a sportsman, the student is interested in descending under 29 sec.

Two attributes considered here: *price* and *acceleration* (time needed to cover 1 km, in seconds)

The DM is asked to equilibrate a balance in the following cases in turn:

- (15,000 ; 29.5) $\sim$ (16,000; ?) Answer = 29.2
- (15,000 ; 29.2) $\sim$ (16,000; ?) Answer = 28.9
- (15,000 ; 28.9) $\sim$ (16,000; ?) Answer = 28.7
- ...
- (15,000 ; 28.3) $\sim$ (16,000; ?) Answer = 28.1

This sequence allows to construct an approximation of the function u_2 recoding the attribute “acceleration”


4
3
2
1
0
28
28.5
29.0
29.5

Question : Is it possible to do something similar with outranking methods and especially with ELECTRE-TRI ?

We use a simplified and axiomatized version of ELECTRE-TRI: the Non-Compensatory Sorting Model

Summary

- The Non-Compensatory Sorting Model
- Elicitation issues
- Eliciting the set of satisfactory levels
- Eliciting the set of sufficient coalitions
- Further research and perspectives

The Non-Compensatory Sorting Model

What is NCSM ? Conjoint measurement model for sorting alternatives in pre-defined ordered categories

- inspired by Electre-Tri (pessimistic version): the assignment of an alternative to a category is done by comparing the alternative to a limiting profile by using an outranking rule
- in conjoint measurement models, alternatives are usually the elements of a cartesian product.
- here, the primitive object is an assignment of the alternatives in ordered categories, i.e. an ordered partition
- the model consists in specifying a particular way of making the assignment (= assignment rule = model)

- one seeks to describe completely this type of assignment by a set of characteristic properties (axioms)

- Developed by D. Bouyssou and Th. Marchant (EJOR 07, 2 papers: two categories & more than two categories)
- Following previous work by Goldstein (JMP 91) and Greco et al (01), Slowinski et al (Control & Cybernetics 02)

The setting

- The alternatives are all the elements of a product set $X = X_1 \times X_2 \times \ldots \times X_n$;
- interpretation: an alternative x is identified to the vector $(x_1, \ldots, x_n)$ of its evaluations on the set $N = \{1, \ldots, n\}$ of attributes
- One can thus build an alternative by mixing up other alternatives x and y; for instance:
  - for a subset $J \subseteq N$ of criteria, we may consider $z = (x_J, y_{-J})$
  - and, abusing notation, we shall often consider $z = (x_i, y_{-i})$

Assignment

- An assignment in two (ordered) categories is an ordered bipartition of X, $\langle \mathcal{A}, \mathcal{U} \rangle$ with:
  - $\mathcal{A}$ denoting the set of “acceptable” alternatives and
  - $\mathcal{U}$ denoting the set of “unacceptable” alternatives
- An assignment in r (ordered) categories is a partition of X in r ordered classes $\langle \mathcal{C}^1, \ldots \mathcal{C}^r \rangle$ with:
  - $\mathcal{C}^1$ denoting the “worst” category of alternatives and
  - $\mathcal{C}^r$ denoting the “best” category of alternatives

In the sequel, we focus on sorting in 2 categories. Sorting in more than 2 categories can be seen as repeated sorting in 2 categories

The case of sorting in two categories

Definition (Bouyssou, Marchant EJOR 07)
An ordered partitioning $\langle \mathcal{A}, \mathcal{U} \rangle$ of X has a representation in the *Non-Compensatory Sorting Model* if

- on each dimension i, there is a set $\mathcal{A}_i \subseteq X_i$ and
- there is a family $\mathcal{F}$ of subsets of N that is final, (i.e. $I \in \mathcal{F}$ and $J \supseteq I \;\Rightarrow\; J \in \mathcal{F}$)

such that:

$$x \in \mathcal{A} \quad \Leftrightarrow \quad \{i \in N : x_i \in \mathcal{A}_i\} \in \mathcal{F}$$

Interpretation

- $\mathcal{A}_i$ is the set of "satisfactory" levels on dimension i
- $\mathcal{F}$ is the family of "sufficient coalitions" of criteria

Justification for the term "Non-Compensatory":
Very rough distinction among "levels" on the scale of each criterion i: only two classes of equivalence

Remark:
We do not consider vetoes here (while Bouyssou-Marchant do also characterize the "Non-Compensatory Sorting Model with Veto")

Characterization result

Theorem (Bouyssou, Marchant EJOR 07)
An ordered partitioning $\langle \mathcal{A}, \mathcal{U} \rangle$ of X has a representation in the *Non-Compensatory Sorting Model* iff

- it is linear and
- it is $2-$graded

Interpretation:

- Linearity is equivalent to assuming that relation $\succsim_i$, defined below, is a complete preorder on each X_i:

$$x_i \succsim_i y_i \;\Leftrightarrow\; [\forall b_{-i},\; (y_i, b_{-i}) \in \mathcal{A} \;\Rightarrow\; (x_i, b_{-i}) \in \mathcal{A}]$$

- under linearity, $2-$gradedness is equivalent to assuming that $\succsim_i$ has at most two equivalence classes

Remark The axioms of linearity and $2-$gradedness are, in principle, testable

Relationship with ELECTRE-TRI

Consider a simple version of ELECTRE-TRI ("pessimistic version") in which

- there are only two classes
- the preference and the indifference thresholds are equal
- there are no discordance effects

For each i, there is a semiorder S_i on X_i ("at least as good" relation on X_i). A weight w_i is associated to each criterion i; these weights can be supposed to be normalized. Let λ be a number between 1/2 and 1.

Given a profile $p = (p_1, \ldots, p_n) \in X$, ELECTRE-TRI determines that

$$x \in \mathcal{A} \Leftrightarrow \sum_{i: x_i S_i p_i} w_i \geq \lambda$$

The obtained partition $\langle \mathcal{A}, \mathcal{U} \rangle$ is representable in the NCSM model with

- $\mathcal{A}_i = \{x_i : x_i S_i p_i\}$
- $\mathcal{F} = \{I \subseteq N : \sum_{i \in I} w_i \geq \lambda\}$

We also have $x_i \succ_i y_i$ iff $x_i S_i p_i$ and Not $[y_i S_i p_i]$

Elicitation of $\mathcal{A}_i$ and $\mathcal{F}$

Assumptions

- for all i, X_i is an interval (a_i, b_i) of the real line, with $a_i < b_i$
- the unknown partition $\langle \mathcal{A}, \mathcal{U} \rangle$ of $X = \prod X_i$ is representable in the NCSM (testable)
- the natural order $\geq_i$ on X_i is compatible with the weak order $\succsim_i$, i.e.

$$x_i \geq_i y_i \;\Rightarrow\; x_i \succsim_i y_i$$

Influent criteria

Definition A criterion i is influent if there are $x_i, y_i \in X_i$ and $z_{-i} \in X_{-i}$ such that

$$(x_i, z_{-i}) \in \mathcal{A} \text{ and } (y_i, z_{-i}) \notin \mathcal{A}$$

In such a case, $x_i \in \mathcal{A}_i$ and $y_i \notin \mathcal{A}_i$.

Assume that we know $\mathcal{F}$, the set of sufficient coalitions. We can find out which criteria are influent by looking at $\mathcal{F}$. Indeed, i is influent if there is $I \in \mathcal{F}$ such that

$$i \in I \text{ and } I \setminus \{i\} \notin \mathcal{F}$$

Note also that

$$(x_i, z_{-i}) \in \mathcal{A} \text{ and } (y_i, z_{-i}) \notin \mathcal{A}$$

entails

$$(b_i, z_{-i}) \in \mathcal{A} \text{ and } (a_i, z_{-i}) \notin \mathcal{A}$$

since $\geq_i$ on $X_i = (a_i, b_i)$ is compatible with $\succsim_i$ and

$$b_i \geq_i x_i >_i y_i \geq_i a_i$$

Eliciting $\mathcal{A}_i$ knowing $\mathcal{F}$

Assume that $\mathcal{F}$ has been determined

Remember that $X_j = (a_j, b_j)$, with $a_j < b_j$. For all $J \subseteq N$, let b_J denote a vector of evaluations along the criteria in J, in which all values are maximal (w.r.t. $\geq_i$), i.e. equal to b_j for all $j \in J$. And a_J will denote a vector of evaluations on J, in which all evaluations are minimal, i.e. equal to a_j for all $j \in J$.

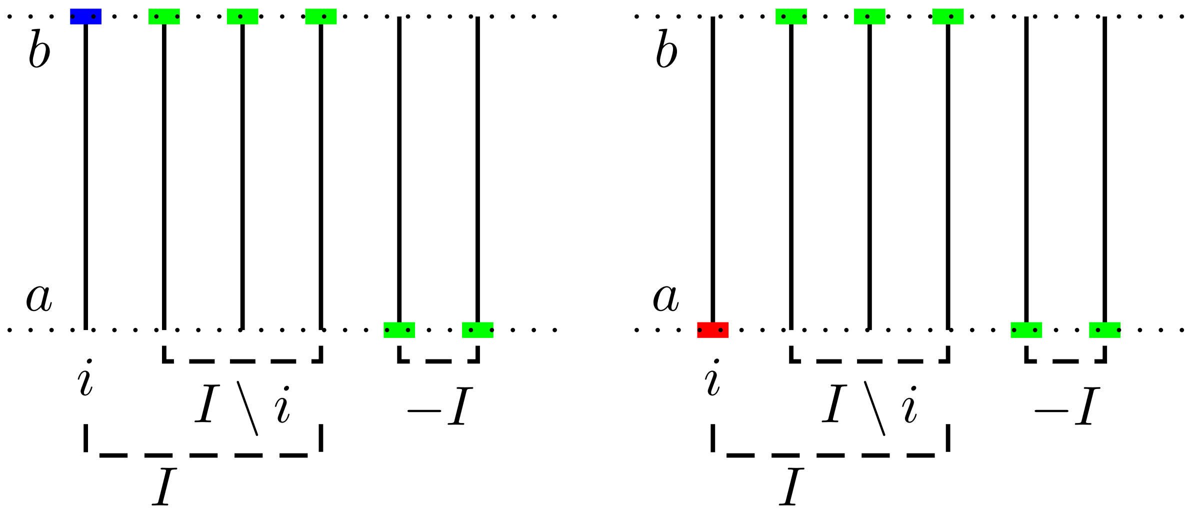
Let i be an influent criterion. Then there is $I \in \mathcal{F}$ such that
$i \in I$ and $I \setminus \{i\} \notin \mathcal{F}$

We have:

$$(b_I, a_{-I}) \in \mathcal{A} \text{ and } (a_i, b_{I \setminus i}, a_{-I}) \notin \mathcal{A}$$

Graphically,

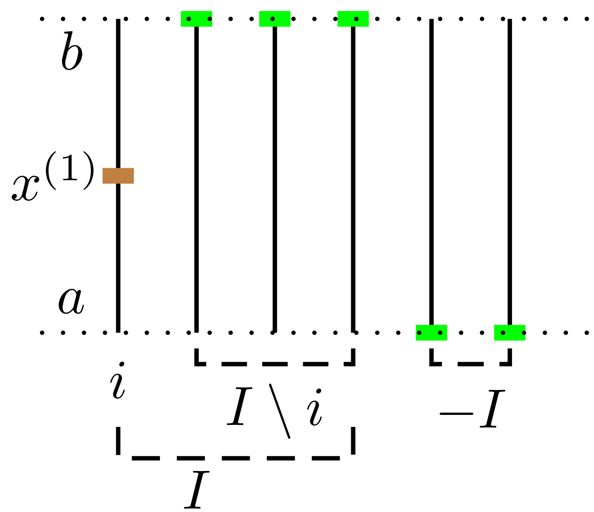
$$(b_I, a_{-I}) \in \mathcal{A} \quad \text{and} \quad (a_i, b_{I\setminus i}, a_{-I}) \notin \mathcal{A}$$



We want to determine (approximately) $\mathcal{A}_i = [z_i, b_{-i}]$ or $]z_i, b_{-i}]$, for some $z_i \in [a_i, b_i]$

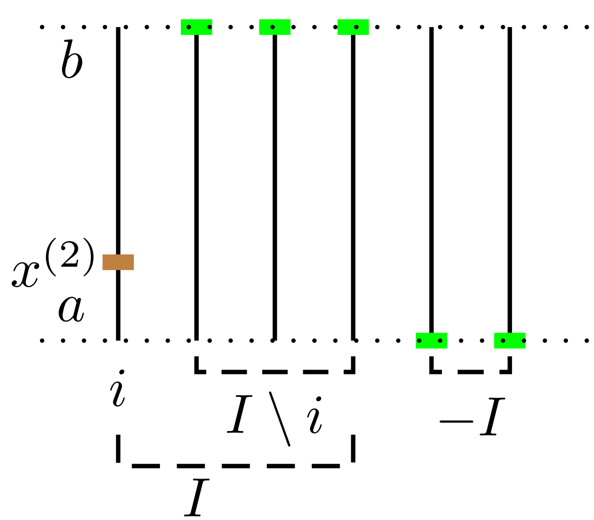
We engage into a dichotomic procedure: let $x_i^{(1)}$ be the middle of $[a_i, b_i]$

Ask whether

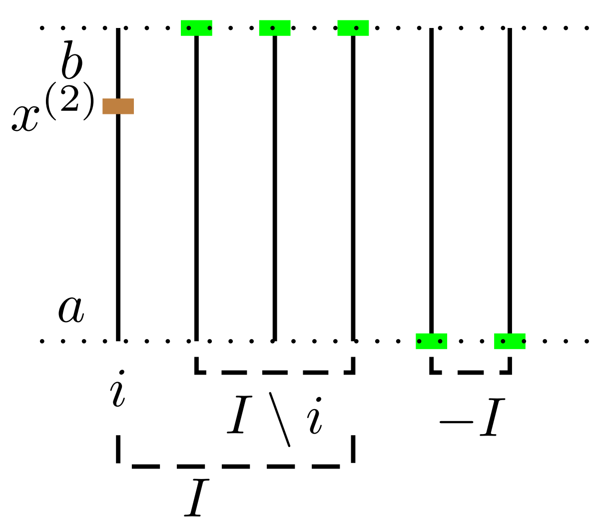


$\in \mathcal{A}$?

If YES $\longrightarrow$

Ask whether  $\in \mathcal{A}$?

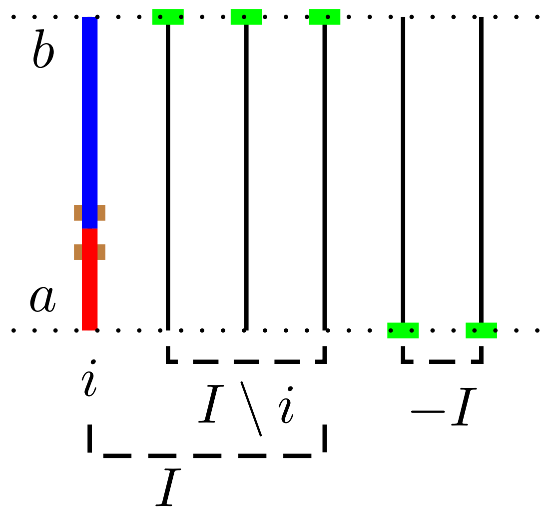
If NO $\longrightarrow$

Ask whether  $\in \mathcal{A}$?

Finally, when we have, after k steps:

- $(x_i^{(k)}, b_{I \setminus i}, a_{-I}) \in \mathcal{A}_i$ and
- $(x_i^{(k+1)}, b_{I \setminus i}, a_{-I}) \notin \mathcal{A}_i$

we know that $z_i \in]x_i^{(k+1)}, x_i^{(k)}]$



Precision: $(\frac{1}{2})^{k+1} \times (b_i - a_i)$

Remarks

- In case there are only finitely many alternatives to be sorted, it is possible to proceed with dichotomization up to the point of determining whether any *observed* level of X_i is in $\mathcal{A}_i$ or not
- if i is not an influent criterion, it makes no difference saying that $\mathcal{A}_i = X_i$ or $\mathcal{A}_i = \emptyset$

Eliciting $\mathcal{F}$

In Electre-Tri, it is assumed that being a sufficient coalition can be determined by using (normalized) weights w_i and a threshold λ in the following rule:

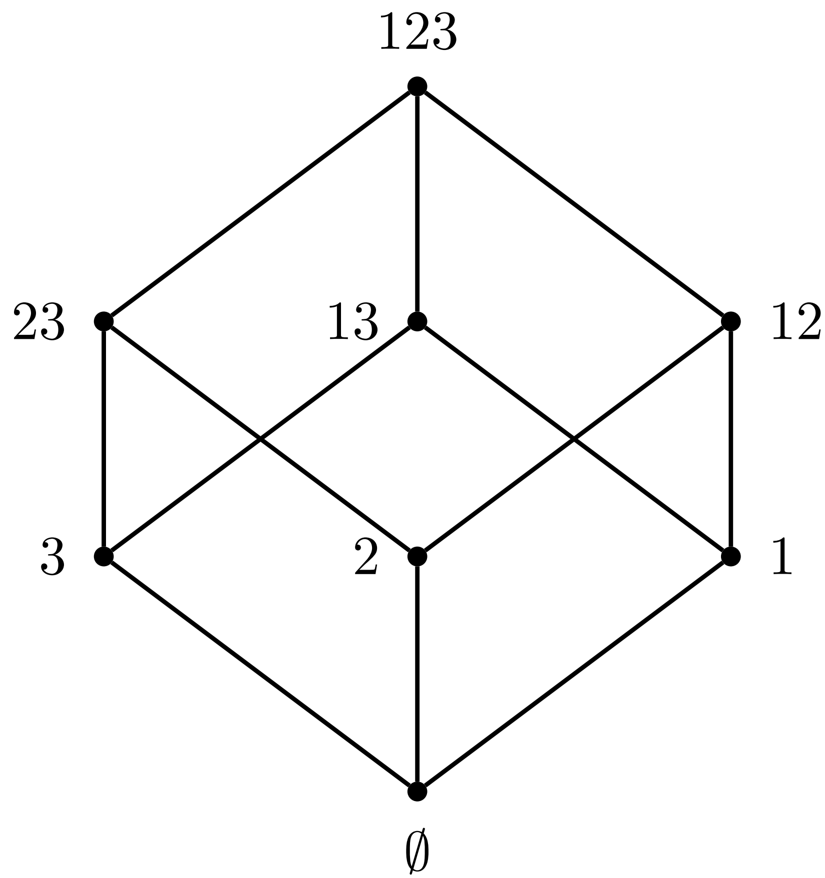
$$I \in \mathcal{F} \;\Leftrightarrow\; \sum_{i \in I} w_i \geq \lambda \tag{1}$$

Axioms guaranteeing the existence of such weights and threshold can possibly be found but they will be hardly interpretable and testable

We choose not to postulate the existence of w_i and λ and to tackle directly the elicitation of $\mathcal{F}$

Highly combinatorial problem

$2^3 = 8$ coalitions



Identifying the set $\mathcal{F}$ of sufficient coalitions $\equiv$ finding a *final* subset of 2^n

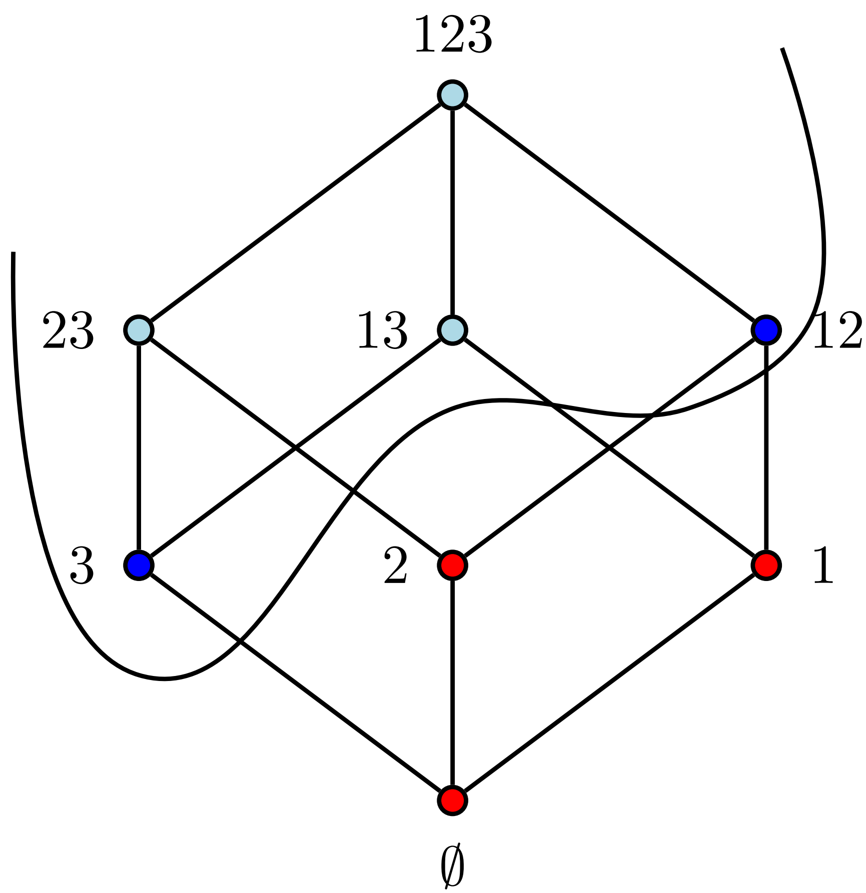
Tantamount to identifying $\mathcal{F}_{min}$ = the set of minimally sufficient coalitions, i.e.

$\mathcal{F}_{min} \subseteq \mathcal{F}$ such that:

- $\forall J \in \mathcal{F},\ \exists I \in \mathcal{F}_{min}$ s.t. $I \subseteq J$ and
- $\forall I, J \in \mathcal{F}_{min},\ I \not\subset J$

The possible $\mathcal{F}_{min}$ are the *antichains* of $2^n, \subseteq$

Example



$\mathcal{F} = \{3, 12, 13, 23, 123\}$

$\mathcal{F}_{min} = \{3, 12\}$

The number of antichains in 2^n forms the integer sequence of *Dedekind* numbers (sequence A000372)

For $n = 2$, there are 6 antichains:

$$\emptyset \mid 1 \mid 2 \mid 12 \mid 1,2 \mid \{\ \} \mid$$

These numbers grow *very fast*:

n	$A000372$
3	20
4	168
5	7581
6	7828354
7	2414682040998

Looking for a needle in a hay stack ...

Finding a particular antichain is a combinatorial search problem like e.g.

- detecting a false coin in a set using a balance
- Master Mind game
- Genomics: reconstruct the right sequencing of pieces of genes $\rightarrow$ problem of finding a particular matching in a graph, or a particular hamiltonian cycle, or more generally, a particular structure

Efficiency of the search:

- False coin: minimal number of weighings
- Master Mind: minimal number of questions
- Genomics: minimal number of tests or experiments, ...

Connections with theory of information

References: V. Grebinski and G. Kucherov [95 to 98], N. Alon et al. [02], N. Alon et V. Asodi [04], etc

Efficient strategy for eliciting $\mathcal{F}$

Type of question considered : $I \in \mathcal{F}$?

Under the guise:

$$(b_I, a_{-I}) \in \mathcal{A} \text{ ?}$$

What is an efficient strategy?

- minimize the number of questions ?
- minimize the cognitive burden on the DM ? $\rightarrow$ raises the question of the *difficulty* of the questions

What about errors in answering the questions? $\rightarrow$ we neglect this issue

Let us concentrate on the number of questions

It is uncertain $\rightarrow$ minimize expected number of questions $\rightarrow$ which distribution ?

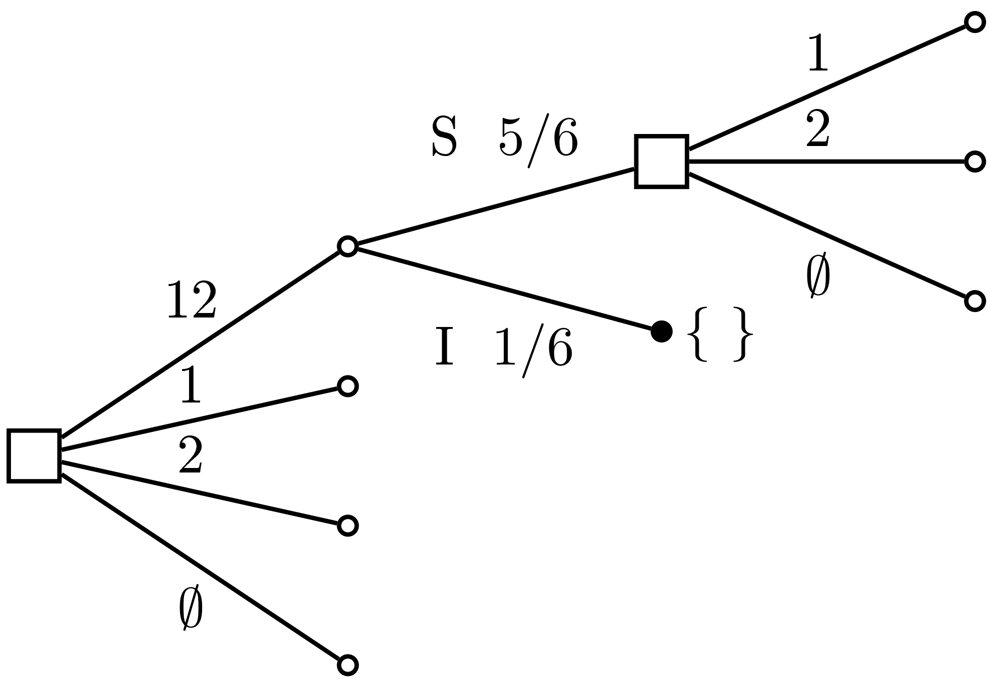
We assume a uniform distribution on the antichains

Exploratory analysis:

- $n = 2$ and $n = 3$
- additional information: $3 \triangleright 2 \triangleright 1$

The case of two criteria

Compute a decision tree for questioning



Conclusion for $n = 2$

On average: $2\frac{2}{3}$ questions

Strategy:

- ask "1 is S ? " (or "2 is S ? ")
- then,
  - if 1 is S, ask "2 is S ?" or "$\emptyset$ is S ? "
  - if 1 is I, ask "2 is S ?" or "12 is S ? "
- etc.

Case $n = 3$ Complicated but feasible by hand; 20 possible $\mathcal{F}_{min}$

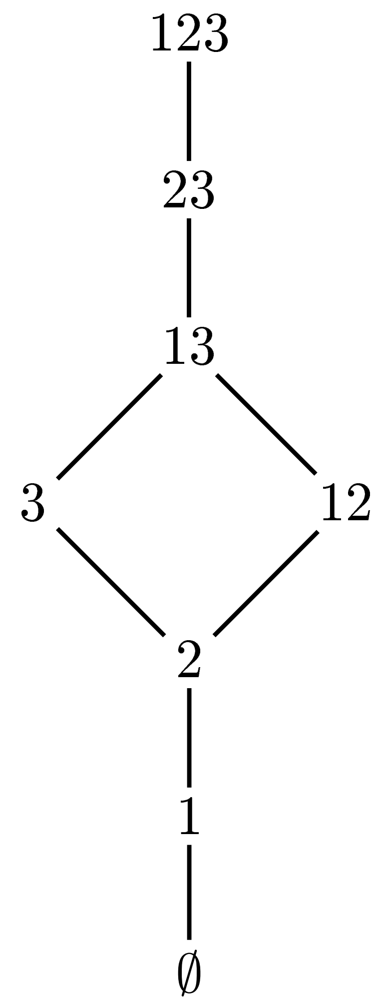
Additional information available

Example: $n = 3$

and $3 \triangleright 2 \triangleright 1$

$\mathcal{F}_{min}$ can be
a singleton
or 3 and 12

10 instead of 20 antichains



Conclusion

Next issues:

- compute the expected number of questions and the questioning strategy for $n = 3, 4$; study the symmetries, the recursiveness, ...
- use (or implement) computer programs to solve the tree for small but larger n
- implement programs to integrate additional information and compute the corresponding decision tree and assess the reduction in the number of questions
- for moderate values of n, combine with other elicitation techniques (find weights compatible with available info (if any) then try to reduce indetermination through questioning)

- how much less expressive are weights w.r.t. sets of sufficient coalitions when n is small ?
- find an upper bound for the number of questions (and an algorithm that stays beyond that bound); we have an algorithm based on depth first search that we suspect (without proof) to cut by two (asymptotically) the number of questions (2^{n-1}). What if additional information is available?

Further issues:

- explore the issue of the cognitive burden on the DM, i.e. take into account the difficulty of the questions in the definition of a questioning strategy
- introduce vetoes
- consider more than two categories