Preferences on Intervals

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Outline

Preferences and Numbers

Comparison

Given objects having a numerical representation how do these compare (before, after, near, better, worst, similar)?

Representation

Given a binary relation among objects what is a suitable numerical representation for it?

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Intervals comparison

A general framework under which objects represented by n points of the reals can be compared.

Representation Theorems

Necessary and sufficient conditions for which a numerical representation (using intervals) fits a certain binary relation.

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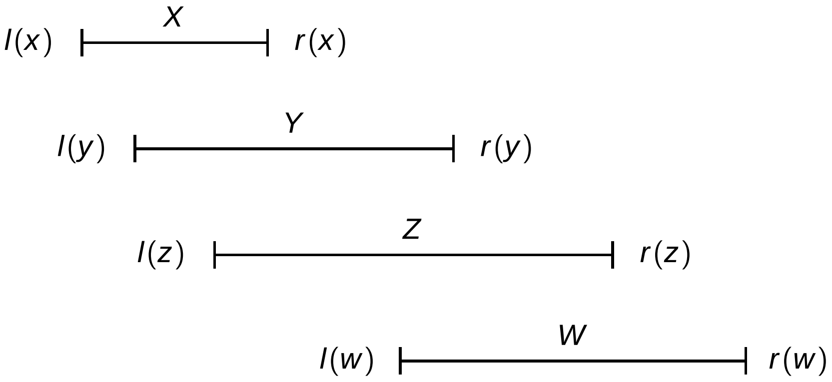
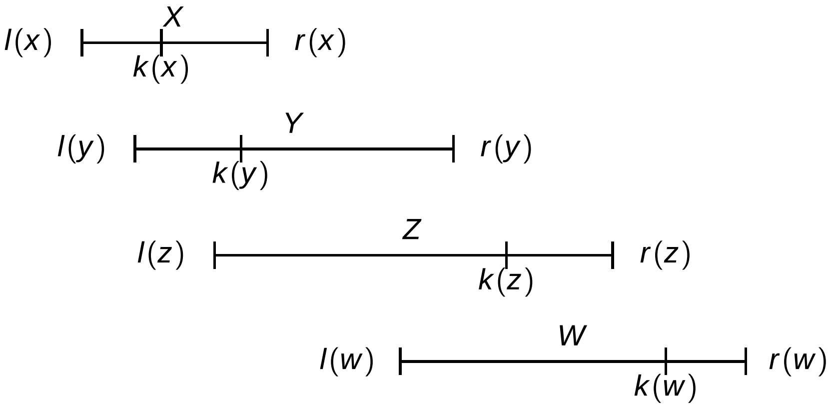
An example: 2 points



An example: 3 points



Where do they come from?

- Imprecision in measurement and/or information (length 10cm$\pm$2mm).
- Uncertain information (price between 10 € and 12 €).
- Uncertain assessments (quality between average and good).
- Positive and negative reasons in evaluation.

Two “different” approaches

Preference and Indifference

Use only two binary relations, an asymmetric one (preference) and a symmetric one (indifference). The symmetric relation can always been seen as the union of the identity relation, I_o and two inverse asymmetric relations. We call that a $\langle P, I \rangle$ preference structure.

More preference relations

Use n ($n > 2$) asymmetric relations and the identity relation I_o. This amounts getting an indifference relation I and $n - 1$ preference relations. A well known case are the $\langle P, Q, I \rangle$ preference structures.

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Well known structures

Interval Orders

$P(x,y) \quad \Leftrightarrow \quad l(x) > r(y)$

$P.I.P \subset P$

Semi Orders

$P(x,y) \quad \Leftrightarrow \quad l(x) > l(y) + k$

$P.I.P \subset P$

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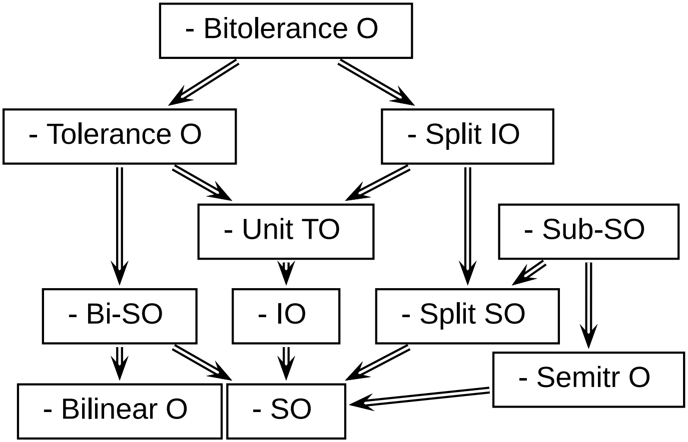
Fishburn's classification



Problems

1. Is this classification exhaustive?
2. There is no unique characterisation of these structures. Some are characterised using forbidden posets, some characterising the binary relations, some through their numerical representation.
3. There is no general framework within which such structures

Example: Subsemiorders

Proposition

Let $R = P \cup I$ be a complete binary relation then the following assertions are equivalent :

i. *R is a subsemiorder;*

ii. *R is reflexive, complete , (1+2)-Ferrers and (0+3)-Ferrers;*

iii. $\begin{cases} R.R^d.R^2 \subset R \text{ (or } R^2.R^d.R \subset R), \\ R^3.I \subset R \text{ (or } I.R^3 \subset R), \end{cases}$

iv. $\begin{cases} P.I.P^2 \subset P \text{ (or } P^2.I.P \subset P), \\ P^3.I \subset P \text{ (or } I.P^3 \subset P), \\ P \text{ is transitive}, \end{cases}$

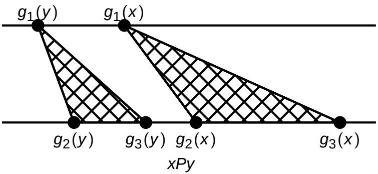
v. $\begin{cases} P.P^d.P^2 \subset P \text{ (or } P^2.P^d.P \subset P), \\ P^3.P^d \subset P \text{ (or } P^d.P^3 \subset P), \end{cases}$

Example : Triangle orders

Definition

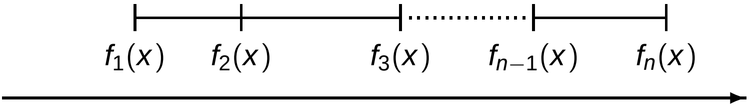
$P \cup I$ is a triangle order if it is defined as the intersection of one weak order and one interval order.

$$\begin{cases} xPy \iff \begin{cases} g_1(x) > g_1(y), \\ g_2(x) > g_3(y), \end{cases} \\ \forall x,\ \forall i \in \{1, 2\},\ g_{i+1}(x) \geq g_i(x). \end{cases}$$



n-point intervals

n-point interval : n ordered points, $f_1(x), f_2(x), \ldots f_n(x)$, such that for all $x \in A$ and all i in $\{1, \ldots, n-1\}$, $f_i(x) < f_{i+1}(x)$.



Relative Positions

Definition (Relative position)

A relative position $\varphi(x, y)$ *is the n-tuple* $\langle \varphi_1(x, y), \ldots, \varphi_n(x, y) \rangle)$ *where* $\varphi_i(x, y)$ *represents the number of j such that* $f_i(x) \leq f_j(y)$

$\varphi(x, y) = (1, 0, 0)$ $\quad$ $\varphi^T(x, y) = \varphi(y, x) = (3, 3, 2)$

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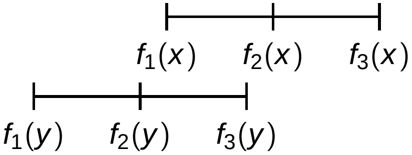

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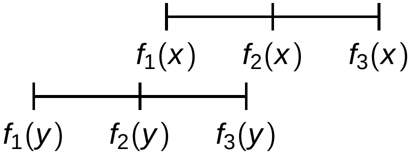


$$\varphi(x, y) = (1, 0, 0) \qquad \varphi^T(x, y) = \varphi(y, x) = (3, 3, 2)$$

Relative positions and inverse

Proposition

Let $\varphi(x, y)$ be the relative position of the n-point interval x with respect to the n-point interval y then $\forall i$,

$$\begin{cases} \varphi_i^T(x,y) = n + 1 - |j, \varphi_j(x,y) \geq (n+1-i)| & \text{if } \exists k, f_i(y) = f_k(x) \\ \varphi_i^T(x,y) = n - |j, \varphi_j(x,y) \geq (n+1-i)| & \text{otherwise} \end{cases}$$

Intervals of Time

Allen introduced 13 relations

7 of them are "basic", the others are the inverse:

Name	***Not.***	***Position***	$\varphi(x,y)$	$\varphi(y,x)$
Equal	$x = y$	$f_1(x) = f_1(y) \wedge f_2(x) = f_2(y)$	$(2,1)$	$(2,1)$
Before	$x\underline{b}y$	$f_1(x) > f_2(y)$	$(0,0)$	$(2,2)$
Overlap	$x\underline{o}y$	$f_1(x) > f_1(y) \wedge f_2(y) > f_1(x) \wedge f_2(x) > f_2(y)$	$(1,0)$	$(2,1)$
Meets	$x\underline{m}y$	$f_1(x) = f_2(y)$	$(1,0)$	$(2,2)$
During	$x\underline{d}y$	$f_1(y) > f_1(x) \wedge f_2(x) > f_2(y)$	$(2,0)$	$(1,1)$
Starts	$x\underline{s}y$	$f_1(x) = f_1(y) \wedge f_2(x) > f_2(y)$	$(2,0)$	$(2,1)$
Finishes	$x\underline{f}y$	$f_1(y) > f_1(x) \wedge f_2(x) = f_2(y)$	$(2,1)$	$(1,1)$

How many are they?

Proposition

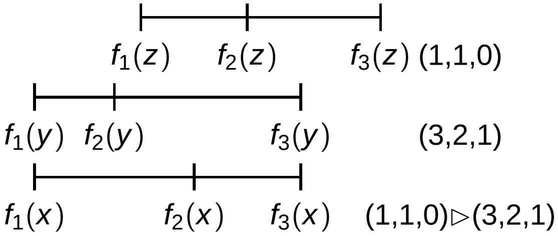
Let x and y be two n-point intervals then the number of possible relative positions $\varphi(x, y)$ is $m = \frac{(2n)!}{(n!)^2}$.

	$n = 2$	$n = 3$	$n = 4$	n
Relative positions	6	20	70	$\frac{(2n)!}{(n!)^2}$

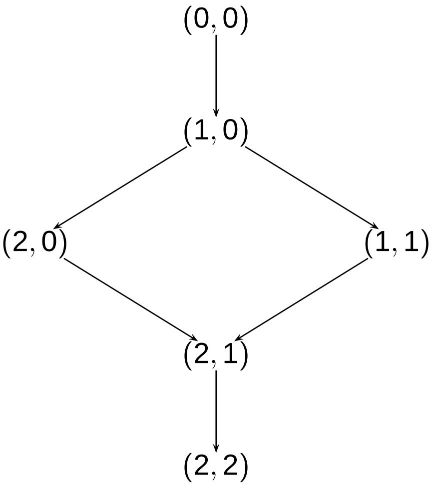
“Stronger than” Relation: $\rhd$

Definition (“Stronger than” relation)

Let φ and φ' be two relative positions, then we say that φ is “stronger than” φ' and note $\varphi \rhd \varphi'$ if $\forall i \in \{1, \dots, n\}$, $\varphi_i \leq \varphi'_i$.



The graph of $\rhd$



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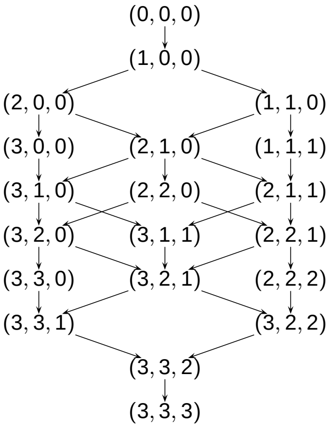

Axioms

- ***Axiom 1*** The relation $P \cup I$ is complete and I is the complement of P.
- ***Axiom 2*** $P(x, y)$ and $I(x, y)$ depend only on the relative position of x and y.
- ***Axiom 3*** For all x, y in A, if $f_i(x) \leq f_i(y)$ for all i then $P(x, y)$ does not hold.
- ***Axiom 4*** Given a relative position $\varphi(x, y)$, if $P(x, y)$ holds then $\forall z, t$ such that $\varphi(z, t) \triangleright \varphi(x, y)$, $P(z, t)$ holds.
- ***Axiom 5*** Let Θ be the set of relative positions $\varphi(x, y)$ such that $P(x, y)$ holds. Then the sublattice formed by the elements of Θ has one and only one lower bound.

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How many such sets of relative positions?

Proposition

Let m be the number of sets of relative positions satisfying axioms 1-5 then

$$m = \frac{(2n)!}{(n!)^2} - \frac{1}{n+1}\binom{2n}{n}$$

	$n=2$	$n=3$	$n=4$	n
Set of relative positions	4	15	56	$\frac{(2n)!}{(n!)^2} - \frac{1}{n+1}\binom{2n}{n}$

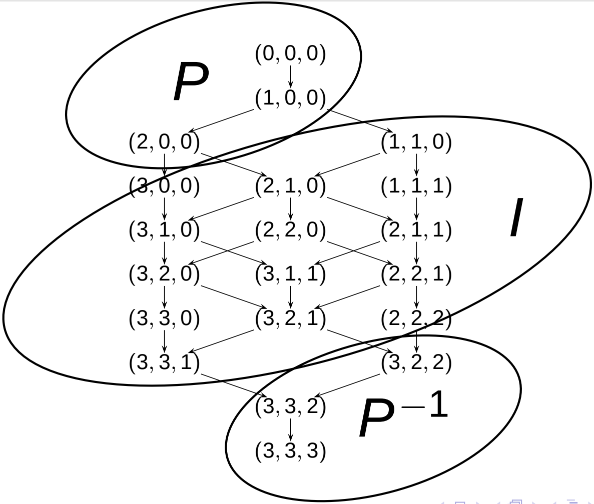
Preference Structure

Definition

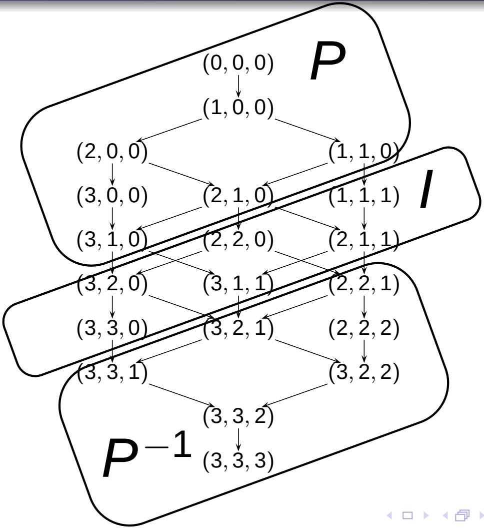
Let φ be an n-tuple and x and y 2 n-point intervals. Relations $P_{\leq\varphi}$ and $I_{\leq\varphi}$ where $(n, n-1, n-2, \ldots, 1) \not\triangleright \varphi$ are defined as

$$P_{\leq\varphi}(x, y) \iff \varphi(x, y) \triangleright \varphi,$$
$$I_{\leq\varphi}(x, y) \iff \neg P_{\leq\varphi}(x, y) \wedge \neg P_{\leq\varphi}(y, x).$$

$P_{\leq(2,0,0)}$


P
I
P^{-1}
(0, 0, 0)
(1, 0, 0)
(2, 0, 0)
(1, 1, 0)
(3, 0, 0)
(2, 1, 0)
(1, 1, 1)
(3, 1, 0)
(2, 2, 0)
(2, 1, 1)
(3, 2, 0)
(3, 1, 1)
(2, 2, 1)
(3, 3, 0)
(3, 2, 1)
(2, 2, 2)
(3, 3, 1)
(3, 2, 2)
(3, 3, 2)
(3, 3, 3)

$P_{\leq(3,1,0)}$


P
I
P−1
(0, 0, 0)
(1, 0, 0)
(2, 0, 0)
(1, 1, 0)
(3, 0, 0)
(2, 1, 0)
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(3, 1, 0)
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(3, 3, 1)
(3, 2, 2)
(3, 3, 2)
(3, 3, 3)

Preference Structure

Proposition

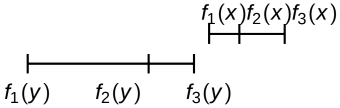
The strict preference and the indifference of Definition 4 form a ***preference structure****, i.e. the strict preference is asymmetric, the indifference is symmetric and reflexive, their union is complete and their intersection is empty. And this preference structure verifies Axioms 1-5.*

Component set

Definition

The component set $Cp_{\leq\varphi}$ is the set of couples $(n - \varphi_i, i)$ such that $\varphi_i \neq n$ and there is no $i' < i$ with $\varphi_{i'} = \varphi_i$

$$P_{\leq(2,0,0)} : (0,0,0)$$



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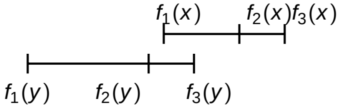

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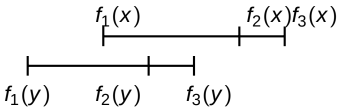

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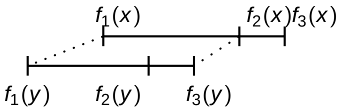

$$Cp_{\leq(2,0,0)} = \{(1,1),(3,2)\}$$

Some characterisations

	$\lvert Cp_{\leq\varphi}\rvert$	$(i,j) \in Cp_{\leq\varphi}$
$P_{\leq\varphi}$ is *transitive*		$\forall(i,j),\ i \geq j,$
$I_{\leq\varphi}$ is *transitive*	1	$Cp_{\leq\varphi} = \{(i,i)\}$
$P_{\leq\varphi} \cup I_{\leq\varphi}$ is a *weak oder*	1	$Cp_{\leq\varphi} = \{(i,i)\}$
$P_{\leq\varphi} \cup I_{\leq\varphi}$ is a *d-weak order*	d	$\forall(i,j),\ i = j$
$P_{\leq\varphi} \cup I_{\leq\varphi}$ is an *interval order*	1	
$P_{\leq\varphi} \cup I_{\leq\varphi}$ is a *"bi-tolerance order"*	2	$\forall(i,j),\ i \geq j,$
$P_{\leq\varphi} \cup I_{\leq\varphi}$ is a *triangle order*	2	$Cp_{\leq\varphi} = \{(l,l),(i,j)\}$ where $i \geq j$

How many representations?

	$n = 2$	$n = 3$	$n = 4$	n
weak order	2	3	4	n
d-weak order	$\binom{2}{d}$	$\binom{3}{d}$	$\binom{4}{d}$	$\binom{n}{d}$
bi-weak order	1	3	6	$\frac{n(n-1)}{2}$
3-weak order	0	1	4	$\binom{n}{3}$
interval order	1	3	6	$\frac{n(n-1)}{2}$
bitolerance order	0	0	6	
triangle order	0	2	8	

Results for 3-point intervals

Preference Structure	$\langle P_{\preceq_\varphi}, I_{\preceq_\varphi} \rangle$ *interval representation*
Weak Orders	$Cp_{\preceq(3,3,0)} = \{(3,3)\}$ $Cp_{\preceq(3,1,1)} = \{(2,2)\}$ $Cp_{\preceq(2,2,2)} = \{(1,1)\}$
Bi-weak Orders	$Cp_{\preceq(3,1,0)} = \{(2,2),(3,3)\}$ $Cp_{\preceq(2,1,1)} = \{(1,1),(2,2)\}$ $Cp_{\preceq(2,2,0)} = \{(1,1),(3,3)\}$
Three-Weak Orders	$Cp_{\preceq(2,1,0)} = \{(1,1),(2,2),(3,3)\}$

Results for 3-point intervals

Preference Structure	$\langle P_{\leq_\varphi}, I_{\leq_\varphi}\rangle$ *interval representation*
Interval Orders	$Cp_{\leq(0,0,0)} = \{(3,1)\}$ $Cp_{\leq(3,0,0)} = \{(3,2)\}$ $Cp_{\leq(1,1,1)} = \{(2,1)\}$
Split Interval Orders	$Cp_{\leq(1,0,0)} = \{(3,2),(2,1)\}$
Triangle Orders	$Cp_{\leq(1,1,0)} = \{(2,1),(3,3)\}$ $Cp_{\leq(2,0,0)} = \{(1,1),(3,2)\}$
Intransitive Orders	$Cp_{\leq(3,2,0)} = \{(3,3),(1,2)\}$ $Cp_{\leq(2,2,1)} = \{(1,1),(2,3)\}$

$\langle P, Q, I \rangle$ Interval Orders

Definition

A $\langle P, Q, I \rangle$ IO is a $\langle P, Q, I \rangle$ Preference Structure such that:

$$P(x,y) \Leftrightarrow l(x) > r(y)$$
$$Q(x,y) \Leftrightarrow r(x) > r(y) > l(x) > l(y)$$
$$I(x,y) \Leftrightarrow r(x) > r(y) > l(y) > l(x) \text{ or the inverse.}$$

Theorem

A $\langle P, Q, I \rangle$ Preference Structure is a $\langle P, Q, I \rangle$ Interval Order iff:

$$I = I_l \cup I_r \cup I_o;\ I_r = I_l^{-1}$$
$$(P \cup Q \cup I_l).P \subseteq P$$
$$P.(P \cup Q \cup I_r) \subseteq P$$
$$(P \cup Q \cup I_l).Q \subseteq P \cup Q \cup I_l$$
$$Q.(P \cup Q \cup I_r) \subseteq P \cup Q \cup I_r$$

Double-threshold Orders

Definition

A Double-threshold Order is a $\langle P, Q, I \rangle$ Preference Structure such that:

$$P(x,y) \Leftrightarrow l(x) > r(y)$$
$$Q(x,y) \Leftrightarrow r(y) > l(x) > k(y)$$
$$I(x,y) \Leftrightarrow k(y) > l(x) \wedge k(x) > l(y)$$

Theorem

A $\langle P, Q, I \rangle$ Preference Structure is a Double-threshold Order iff:

$$Q.I.Q \subseteq P \cup Q$$
$$Q.I.P \subseteq P$$
$$P.I.P \subseteq P$$
$$P.Q^{-1}.P \subseteq P$$

Generalised Transitivity

Consider three asymmetric relations P_3, P_2, P_1 such that:
- $P = P_3$
- $Q = P_2$
- $I = P_1 \cup I_o \cup P_1^{-1}$

What happens to the preference structures we introduced before?

Generalised Transitivity

Theorem

A $\langle P, Q, I \rangle$ Preference Structure is a $\langle P, Q, I \rangle$ Interval Order iff:

$P_3.P_3 \subseteq P_3$
$P_2.P_3 \subseteq P_3$
$P_3.P_2 \subseteq P_3$
$P_3.P_1 \subseteq P_3$
$P_1^{-1}.P_3 \subseteq P_3$
$P_2.P_2 \subseteq P_2 \cup P_3$
$P_1.P_2 \subseteq P_2 \cup P_1$
$P_2.P_1^{-1} \subseteq P_2 \cup P_1^{-1}$

Generalised Transitivity

Theorem

A $\langle P, Q, I \rangle$ Preference Structure is a Double-threshold Order iff:

$P_3.P_3 \subseteq P_3$
$P_2.P_3 \subseteq P_3$
$P_3.P_1 \subseteq P_3$
$P_1^{-1}.P_3 \subseteq P_3$
$P_2^{-1}.P_3 \subseteq P_3$
$P_2.P_2 \subseteq P_2 \cup P_3$
$P_2.P_1 \subseteq P_2 \cup P_3$
$P_1^{-1}.P_2 \subseteq P_2 \cup P_3$

So what?

- A general framework for comparing n-point intervals and for characterising such relations.
- Some representation theorems for $\langle P, Q, I \rangle$ preference structures.
- A framework for continuous valuation of intervals comparison.
- Coherence Conditions
- Algorithmic and Complexity Issues.
- Further generalisations.

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