Midterm

Mark the correct answer in each part of the following questions.

1. Two players – A and B – draw each two cards **with replacement** from a full deck of 52 cards. Let p denote the probability that they will draw the same cards. (The order is not important. If they draw the same cards in reverse order, the condition is satisfied.)

   (a) $p =$

   (i) $102/52^3$.

   (ii) $2/52^2$.

   (iii) $103/52^3$.

   (iv) $1/52^2$.

   (v) None of the above.

   (b) The probability that none of the cards A draws is the same as any of the cards that B draws is:

   (i) $\frac{50^2+51}{52^3}$.

   (ii) $\frac{50^2}{52^2}$.

   (iii) $\frac{51^2}{52^2}$.

   (iv) $\frac{51(50^2+51)}{52^3}$.

   (v) None of the above.

   (c) The probability that the four cards will be altogether of two distinct values (for example, A will draw twice the 7 of hearts, and B will draw the 3 of diamonds and the 7 of spades) is:

   (i) $\frac{4\cdot 12}{13^3}$.

   (ii) $\frac{7\cdot 12}{13^3}$.

   (iii) $\frac{5\cdot 12}{13^3}$.

(iv) $\frac{6\cdot 12}{13^3}$.

(v) None of the above.

2. We perform a 2-stage experiment, as follows. In the first stage, a fair die is rolled again and again until it shows 6 for the first time. Next, if the die has been rolled n times, we draw randomly a permutation of the numbers $1, 2, \ldots, n$ (each permutation with the same probability). Let X be the number of rolls of the die.

   (a) X is distributed

      (i) $U[1, 6]$.

      (ii) $G(1/6)$.

      (iii) $B(6, 1/6)$.

      (iv) $\bar{B}(6, 1/6)$.

      (v) None of the above.

   (b) Then $F_X(4) =$

      (i) $(5/6)^3 \cdot 1/6$.

      (ii) $(5/6)^4$.

      (iii) $1 - (5/6)^4$.

      (iv) $1 - (5/6)^3$.

      (v) None of the above.

   (c) Let p denote the probability that the resulting permutation is the identity permutation. (For example, the results of the die are 5, 1, 1, 4, 6, and the permutation is $(1, 2, 3, 4, 5)$.) Then $p =$

      (i) $\frac{e^{5/6}-1}{5}$.

      (ii) $\frac{e^{5/6}-1}{6}$.

      (iii) $e^{1/6} - 1$.

      (iv) $\frac{5(e^{5/6}-1)}{6}$.

      (v) None of the above.

   (d) Suppose it is known that the permutation obtained is the identity permutation for some unknown number n of rolls of the die. The probability that the number of rolls was 3 is:

(i) $\frac{1/6}{p}$.
(ii) $\frac{(5/6)^3}{p}$.
(iii) $\frac{(5/6)^3 \cdot 1/6}{p}$.
(iv) $\frac{(5/6)^2 \cdot 1/6}{p}$.
(v) None of the above.

(e) Let p' denote the probability that the resulting permutation starts with 1. (For example, the results of the die are 5, 1, 1, 4, 6, and the permutation is $(1, 5, 3, 2, 4)$.) Then $p' =$

(i) $\log \frac{6}{5}$.
(ii) $\frac{5}{6} \cdot \log \frac{6}{5}$.
(iii) $\frac{1}{6} \cdot \log 6$.
(iv) $\frac{1}{5} \cdot \log 6$.
(v) None of the above.

3. Two-thirds of Mars's population are women and only one-third are men. Two-thirds of the women and one-sixth of the men have blue eyes.

(a) One person is selected randomly (equal probabilities to all people). Consider the following events:
A – a woman has been selected.
B – the eyes of the selected person are not blue.

(i) The events A and B are disjoint.
(ii) The events A and B are independent.
(iii) $P(B|A) > P(B)$.
(iv) $P(A|B) < P(A)$.
(v) None of the above.

(b) Six people are chosen randomly (equal probabilities to all people). Let X be the number of those with blue eyes. X is distributed

(i) $B(6, 1/2)$.
(ii) $B(6, 5/12)$.
(iii) $B(6, 1/3)$.

(iv) $\bar{B}(6, 1/2)$.

(v) None of the above.

4. We draw a random card from a full deck of 52 cards. We return the selected card, add another full deck, and draw again a random card (out of all 104 cards). We continue the process over and over, each time returning the selected card and adding another deck to the pile. Each time, all cards have the same probability of being drawn. We mark all cards in the original deck by 1, those in the second deck by 2, and so forth. (Thus, for example, the queen of hearts of the second deck and the queen of hearts of the fifth deck are distinct.)

   (a) For each $n \geq 1$, the probability that the first n cards to be selected are all distinct is

   (i) $\left(\frac{51}{52}\right)^n$.

   (ii) $\prod_{k=1}^{n} \frac{52k-1}{52k}$.

   (iii) $\prod_{k=1}^{n} \frac{51k-1}{52k}$.

   (iv) $\prod_{k=1}^{n} \frac{51k+1}{52k}$.

   (v) None of the above.

   (b) Let p denote the probability that the queen of hearts from the first deck will never be drawn. Then

   (i) $p = 0$.

   (ii) $p \in (0, 1/e]$.

   (iii) $p \in (1/e, 1/2]$.

   (iv) $p \in (1 - 2/52, 1 - 1/52]$.

   (v) None of the above.

Solutions

1. (a) Each player has 52 possibilities in each drawing, so that the sample space consists of 52^4 points.

   The event in the question is the union of two disjoint events:

   - A draws two distinct cards, for which there are $52 \cdot 51$ possibilities, and B draws the same very cards in any order, for which there are 2 possibilities. Altogether, $52 \cdot 51 \cdot 2$ possibilities.
   - A draws the same card in both drawings, for which there are 52 possibilities, and B draws the same card in both drawings, for which there is just 1 possibility. Altogether, 52 possibilities.

   Hence:

$$p = \frac{52 \cdot 51 \cdot 2 + 52}{52^4} = \frac{103}{52^3}.$$

   Thus, (iii) is true.

   (b) The event is the union of two disjoint events:

   - A draws two distinct cards, for which there are $52 \cdot 51$ possibilities, and B draws each time any of the other 50 cards, for which there are 50^2 possibilities. Altogether, $52 \cdot 51 \cdot 50^2$ possibilities.
   - A draws the same card in both drawings, for which there are 52 possibilities, and B draws each time any of the other 51 cards, for which there are 51^2 possibilities. Altogether, $52 \cdot 51^2$ possibilities.

   Hence the required probability is:

$$\frac{52 \cdot 51 \cdot 50^2 + 52 \cdot 51^2}{52^4} = \frac{51(50^2 + 51)}{52^3}.$$

   Thus, (iv) is true.

   (c) As we care only about the values of the cards, we may regard the sample space as consisting of 13^4 points.

   The event in question is the union of four pairwise disjoint events:

- A draws cards of two distinct values, for which there are $13 \cdot 12$ possibilities, and B draws cards of the same very values in any order, for which there are 2 possibilities. Altogether, $13 \cdot 12 \cdot 2$ possibilities.
- A draws two cards of the same value, for which there are 13 possibilities, and B draws two cards of some other value (same value both times), for which there are 12 possibilities. Altogether, $13 \cdot 12$ possibilities.
- A draws two cards of the same value, for which there are 13 possibilities, and B draws one card of the same value and one card of some other value, for which there are $12 \cdot 2$ possibilities. Altogether, $13 \cdot 12 \cdot 2$ possibilities.
- Same as the previous event, with A and B interchanged.

Altogether, the probability is:

$$p = \frac{13 \cdot 12 \cdot 7}{13^4} = \frac{7 \cdot 12}{13^3}.$$

Alternatively, we can first select the values that will be drawn, which can be done in $\binom{13}{2}$ possibilities. Then we need to choose, for each drawing, which of the two values will be drawn. This gives $2^4 = 16$ possibilities. However, in two of them, all cards have the same value. Thus, the total number of possibilities is $\binom{13}{2} \cdot (16 - 2)$, which leads to the same probability as above.

Thus, (ii) is true.

2. (a) X counts the rolls of a die until obtaining an outcome of 6. Referring to this outcome as a success and to any other as a failure, we count the trials in a sequence of independent events until the first success. As the success probability is $1/6$, we have $X \sim G(1/6)$.

   Thus, (ii) is true.

   (b) By the definition of the distribution function,

$$F_X(4) = P(X \leq 4).$$

   The complementary event, namely $\{X > 4\}$, occurs if none of the first four outcomes is 6, which happens with probability $(5/6)^4$.

It follows that

$$F_X(4) = 1 - (5/6)^4.$$

Thus, (iii) is true.

(c) Let I be the event whereby the resulting permutation is the identity permutation. By the law of total probability:

$$\begin{aligned}
P(I) &= \textstyle\sum_{n=1}^{\infty} P(X=n)P(I|X=n) \\
&= \textstyle\sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^{n-1} \cdot \frac{1}{6} \cdot \frac{1}{n!} \\
&= \frac{1}{5} \textstyle\sum_{n=1}^{\infty} \frac{(5/6)^n}{n!} \\
&= \frac{e^{5/6}-1}{5}.
\end{aligned}$$

Thus, (i) is true.

(d)

$$\begin{aligned}
P(X=3|I) &= \frac{P(\{X=3\} \cap I)}{P(I)} \\
&= \frac{P(X=3)P(I|X=3)}{P(I)} \\
&= \frac{(5/6)^2 \cdot 1/6 \cdot 1/3!}{p} \\
&= \frac{5^2/6^4}{p}.
\end{aligned}$$

Thus, (v) is true.

(e) We proceed similarly to part (c), except that now the conditional probability of obtaining a permutation with the requested property, given that $X = n$, is $1/n$ instead of $1/n!$. Hence, denoting by I' the event in question, we have:

$$\begin{aligned}
P(I) &= \textstyle\sum_{n=1}^{\infty} P(X=n)P(I'|X=n) \\
&= \textstyle\sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^{n-1} \cdot \frac{1}{6} \cdot \frac{1}{n} \\
&= -\tfrac{1}{5} \textstyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-5/6)^n}{n} \\
&= -\tfrac{1}{5} \log(1-5/6) \\
&= \tfrac{\log 6}{5}.
\end{aligned}$$

Thus, (iv) is true.

3. (a) By the law of total probability:

$$P(B) = \frac{2}{3} \cdot \left(1 - \frac{2}{3}\right) + \frac{1}{3} \cdot \left(1 - \frac{1}{6}\right) = \frac{1}{2}.$$

Therefore:

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)} = \frac{2/3 \cdot (1-2/3)}{1/2} = \frac{4}{9} < P(A).$$

Thus, only (iv) is true.

(b) The probability that a random person is blue-eyed is, as seen in part (a), $1/2$. Therefore, X is distributed as the number of successes in 6 independent trials with a probability of success of $1/2$.
Thus, (i) is true.

4. (a) Denote by $D_k, k \geq 1$, the event whereby the card drawn at stage k is different from all previously drawn cards. The event in question is $\cap_{k=1}^{n} D_k$. We have:

$$P\left(\cap_{k=1}^{n} D_k\right) = \prod_{k=1}^{n} P\left(D_k | \cap_{i=1}^{k-1} D_i\right).$$

Now, when we draw the k-th card, given the event $\cap_{i=1}^{k-1} D_i$, we draw a card from a pile of $52k$ cards, of which we have previously drawn exactly $k-1$. It follows that

$$P\left(D_k | \cap_{i=1}^{k-1} D_i\right) = \frac{52k - (k-1)}{52k} = \frac{51k+1}{52k}.$$

Thus, (iv) is true.

(b) Denote by $Q_k, k \geq 1$, the event whereby the card drawn at stage k is not the queen of hearts. The event in question is $\cap_{k=1}^{\infty} Q_k$. We have:

$$P\left(\cap_{k=1}^{\infty} Q_k\right) = \prod_{k=1}^{\infty} \left(1 - \frac{1}{52k}\right).$$

Now

$$\sum_{k=1}^{\infty} \frac{1}{52k} = \frac{1}{52} \sum_{k=1}^{\infty} \frac{1}{k} = \infty,$$

whence the infinite product above vanishes.

Thus, (i) is true.