Final #1

Mark the correct answer in each part of the following questions.

1. A die is rolled until 6 is obtained 10 times. Let X be the number of rolls until 6 was obtained for the third time and Y the total number of rolls. (For example, if we obtained a 6 in all odd-numbered rolls and other results in all even-numbered rolls, then $X = 5, Y = 19$.)

   (a) $P(X = 4|Y = 100) =$

   (i) $\frac{\binom{4}{3}\binom{96}{7}}{\binom{100}{10}}$.

   (ii) $\frac{\binom{3}{2}\binom{95}{6}}{\binom{99}{9}}$.

   (iii) $\binom{4}{3}\binom{96}{7}\left(\frac{1}{6}\right)^{10}\left(\frac{5}{6}\right)^{90}$.

   (iv) $\binom{3}{2}\binom{95}{6}\left(\frac{1}{6}\right)^{8}\left(\frac{5}{6}\right)^{90}$.

   (v) None of the above.

   (b) $\rho(X, Y) =$

   (i) $\frac{3}{10}$.

   (ii) $\frac{3}{7}$.

   (iii) $\sqrt{\frac{3}{10}}$.

   (iv) $\left(\frac{3}{10}\right)^2$.

   (v) None of the above.

   (c) Let X' and Y' be the same as X and Y, respectively, but not counting the last roll for each, namely $X' = X - 1, Y' = Y - 1$. Then

   (i) $E(X'Y') = E(XY)$ and $\mathrm{Cov}(X', Y') = \mathrm{Cov}(X, Y)$.

   (ii) $E(X'Y') = E(XY)$ and $\mathrm{Cov}(X', Y') < \mathrm{Cov}(X, Y)$.

(iii) $E(X'Y') < E(XY)$ and $\mathrm{Cov}(X', Y') = \mathrm{Cov}(X, Y)$.

(iv) $E(X'Y') < E(XY)$ and $\mathrm{Cov}(X', Y') < \mathrm{Cov}(X, Y)$.

(v) None of the above.

(d) Let Z be the number of rolls until the first 6 is obtained. (In the example above, $Z = 1$.) For an arbitrary positive integer n, denote $p_n = P(Z \geq n)$, and let b_n be the upper bound on $P(Z \geq n)$ given by Markov's inequality. Then $b_n/p_n =$

(i) $\frac{6^n}{5^n \cdot n}$.

(ii) $\frac{6^n}{5^{n-1} \cdot n}$.

(iii) $\frac{6^{n+1}}{5^n \cdot n}$.

(iv) $\frac{6^{n+1}}{5^{n-1} \cdot n}$.

(v) None of the above.

2. An urn contains n white and $2n$ black balls, where $n \geq 2$. A 2-stage experiment is conducted, as follows.

(a) In the first stage we draw two uniformly random balls out of the urn. Let W and B denote the number of white and black balls, respectively, out of the $3n - 2$ balls in the urn.

(i)

$$V(W) = \frac{4}{9} \cdot \frac{3n-1}{3n}$$

and

$$V(B) = V(W).$$

(ii)

$$V(W) = \frac{4}{9} \cdot \frac{3n-1}{3n}$$

and

$$V(B) = 2^2 \cdot V(W).$$

(iii)

$$V(W) = \frac{4}{9} \cdot \frac{3n-2}{3n-1}$$

and

$$V(B) = V(W).$$

(iv)
$$V(W) = \frac{4}{9} \cdot \frac{3n-2}{3n-1}$$
and
$$V(B) = 2^2 \cdot V(W).$$
(v) None of the above.

(b) In the second stage, we add two white balls to the urn and then draw a random ball out of the $3n$ balls in it. The probability that this ball is white is

(i) $\frac{1}{3}$.
(ii) $\frac{1}{3} + \frac{12}{9(3n-1)}$.
(iii) $\frac{1}{3} + \frac{4}{9n}$.
(iv) $\frac{1}{3} + \frac{12n-4}{9n(3n-1)}$.
(v) None of the above.

3. Two players, A and B, compete in marksmanship. They shoot alternately, where A is first, then B, then again A, and so forth. They continue until one of them hits the target for the first time. Both have a probability of 1/4 of hitting the target at any shot, and the results of the shots are independent.

Let X denote the total number of shots. After giving a prize to the winner, we give the loser a chance to get a consolation prize, as follows. The loser gets X shots. Denote by Y the number of hits out of these X shots. The loser gets the consolation prize if $Y \geq X - 1$. (For example, if A hits the target in his third shot, then $X = 5$. Note that, if A hits the target in his first shot, then B will certainly get the consolation prize.)

(a) A's probability of winning is

(i) $\frac{1}{2}$.
(ii) $\frac{4}{7}$.
(iii) $\frac{3}{5}$.
(iv) $\frac{2}{3}$.

(v) None of the above.

(b) The probability function of (X, Y) is given by:

(i) $P_{X,Y}(m,n) = \binom{m}{n} \cdot \frac{3^{2m-n-1}}{4^{2m}}, \qquad m \geq 1, 0 \leq n \leq m.$

(ii) $P_{X,Y}(m,n) = \binom{m}{n} \cdot \frac{3^{m+n-1}}{4^{2m}}, \qquad m \geq 1, 0 \leq n \leq m.$

(iii) $P_{X,Y}(m,n) = \binom{m}{n} \cdot \frac{3^{2m-1}}{4^{2m+n}}, \qquad m \geq 1, 0 \leq n \leq m.$

(iv) $P_{X,Y}(m,n) = \binom{m}{n} \cdot \frac{3^{m-1}}{4^{2m-n}}, \qquad m \geq 1, 0 \leq n \leq m.$

(v) None of the above.

(c) The probability of the loser to win the consolation prize is:

(i) $\frac{60}{13^2}$.

(ii) $\frac{61}{13^2}$.

(iii) $\frac{62}{13^2}$.

(iv) $\frac{63}{13^2}$.

(v) None of the above.

(d) To keep in shape, A and B train every day with five shots each. Suppose they stick with this training schedule for 100,000 consecutive days. Their probability of hitting the target does not change during this period. The probability that there will be exactly one day out of these 100,000 in which they hit the target exactly in 9 out of their 10 shots is approximately

(i) e^{-3}.

(ii) $3e^{-3}$.

(iii) e^{-1}.

(iv) $2e^{-1}$.

(v) None of the above.

Remark: In the spirit of computer science, refer to 2^{10} as 10^3 and to 2^{20} as 10^6.

4. $X_1, X_2 \sim U[1, n]$ are independent random variables, where n is some positive integer. Let $Y = \max(X_1, X_2)$.

(a) $E(Y) =$

(i) $\frac{n+1}{2}$.

(ii) $\frac{2n+1}{3}$.

(iii) $\frac{4n^2+3n-1}{6n}$.

(iv) $\frac{3n+1}{4}$.

(v) None of the above.

(b) $V(2X_1 + 3X_2) =$

(i) $\frac{5(n^2-1)}{12}$.

(ii) $\frac{6(n^2-1)}{12}$.

(iii) $\frac{9(n^2-1)}{12}$.

(iv) $\frac{13(n^2-1)}{12}$.

(v) None of the above.

Solutions

1. (a) The condition $Y = 100$ means that the die showed 6 for the tenth time in the 100-th roll. Thus, in the first 99 rolls, it showed 6 nine times. There are $\binom{99}{9}$ possibilities for the locations of the nine 6-s, all having the same probability by symmetry.

   To have $X = 4$ then, we need two 6-s in the first three rolls, a 6 in the fourth roll, and six 6-s from the fifth roll until the 99-th roll. It follows that

$$P(X = 4|Y = 100) = \frac{\binom{3}{2}\binom{95}{6}}{\binom{99}{9}}.$$

   Thus, (ii) is true.

   (b) Let $Z_i, 1 \leq i \leq 10$, denote the number of rolls between the $(i-1)$-st 6 and the i-th 6. The Z_i-s are independent and $G(1/6)$-distributed. We have:

$$X = \sum_{i=1}^{3} Z_i, \qquad Y = \sum_{i=1}^{10} Z_i.$$

   Therefore, by symmetry:

$$E(X) = 3E(Z_1), \qquad V(X) = 3V(Z_1),$$

   and

$$E(Y) = 10E(Z_1), \qquad V(Y) = 10V(Z_1).$$

   It follows that:

$$E(X)E(Y) = 30E^2(Z_1).$$

   Now:

$$E(XY) = \sum_{i=1}^{3}\sum_{j=1}^{10} E(Z_iZ_j) = 3E(Z_1^2) + 27E(Z_1Z_2) = 3V(Z_1) + 30E^2(Z_1).$$

   Hence

$$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) = 3V(Z_1),$$

and finally:

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{V(X)V(Y)}} = \frac{3V(Z_1)}{\sqrt{30}V(Z_1)} = \sqrt{\frac{3}{10}}.$$

Thus, (iii) is true.

(c) The covariance is invariant under translations, so that

$$\mathrm{Cov}(X',Y') = \mathrm{Cov}(X,Y).$$

Now $0 \le X' < X$ and $0 \le Y' < Y$ on all the sample space. Therefore $0 \le X'Y' < XY$, and in particular $E(X'Y') < E(XY)$. Thus, (iii) is true.

(d) We have

$$p_n = P(Z \ge n) = (5/6)^{n-1}.$$

Markov's inequality gives:

$$P(Z \ge n) \le \frac{E(Z)}{n} = \frac{6}{n}.$$

Hence:

$$\frac{b_n}{p_n} = \frac{6^n}{5^{n-1}n}.$$

Thus, (ii) is true.

2. (a) Let W' be the number of white balls drawn from the urn. We have $W' \sim H(2,n,2n)$, and therefore:

$$V(W') = \frac{2 \cdot n \cdot 2n}{(n+2n)^2} \cdot \left(1 - \frac{2-1}{n+2n-1}\right) = \frac{4}{9} \cdot \frac{3n-2}{3n-1}.$$

Now $W = n - W'$ and $B = 3n - 2 - W$, and consequently

$$V(B) = V(W) = V(W').$$

Thus, (iii) is true.

(b) With W' as above and p denoting the required probability, the law of total probability yields:

$$\begin{aligned} p &= P(W'=0)\cdot\tfrac{n+2}{3n}+P(W'=1)\cdot\tfrac{n+1}{3n}+P(W'=2)\cdot\tfrac{n}{3n} \\ &= \tfrac{2n(2n-1)}{3n(3n-1)}\cdot\tfrac{n+2}{3n}+\tfrac{2n\cdot 2n}{3n(3n-1)}\cdot\tfrac{n+1}{3n}+\tfrac{n(n-1)}{3n(3n-1)}\cdot\tfrac{n}{3n} \\ &= \tfrac{1}{3}+\tfrac{12n-4}{9n(3n-1)}. \end{aligned}$$

Thus, both (iii) and (iv) are true.

3. (a) A wins if, for some $k \geq 1$, both A and B miss $k-1$ times, and then A gets a hit at the k-th try. Hence the required probability p is:

$$\begin{aligned} p = {} & 1/4+(1-1/4)^2\cdot 1/4+(1-1/4)^4\cdot 1/4+\cdots \\ = {} & \frac{1/4}{1-(1-1/4)^2}=\frac{4}{7}. \end{aligned}$$

Thus, (ii) is true.

(b) Clearly, $X \sim G(1/4)$. If $X=m$, then $Y \sim B(m, 1/4)$. It follows that:

$$\begin{aligned} P_{X,Y}(m,n) &= (1-1/4)^{m-1}\cdot 1/4\cdot\tbinom{m}{n}(1/4)^n(1-1/4)^{m-n} \\ &= \tbinom{m}{n}3^{2m-n-1}/4^{2m}, \qquad m\geq 1,\ m\geq n\geq 0. \end{aligned}$$

Thus, (i) is true.

(c) If $X=m$, then the probability of the loser to win the consolation prize is:

$$\begin{aligned} P(Y\geq m-1) &= \tbinom{m}{m-1}\cdot 3/4\cdot(1/4)^{m-1}+\tbinom{m}{m}\cdot(1/4)^m \\ &= 3m(1/4)^m+(1/4)^m. \end{aligned}$$

It follows that:

$$\begin{aligned}
P(Y \geq X-1) &= \textstyle\sum_{m=1}^{\infty} (3/4)^{m-1} \cdot 1/4 \cdot (3m+1) \cdot (1/4)^m \\
&= \textstyle\sum_{m=1}^{\infty} m \cdot (3/16)^m + 1/3 \cdot \sum_{m=1}^{\infty} (3/16)^m \\
&= \tfrac{3/16}{(1-3/16)^2} + \tfrac{1/16}{1-3/16} \\
&= \tfrac{3 \cdot 16 + 13}{13^2} \\
&= \tfrac{61}{13^2}.
\end{aligned}$$

Thus, (ii) is true.

(d) The probability of exactly nine hits out of 10 shots on any specific day is

$$p' = \binom{10}{9} \cdot (1/4)^9 \cdot 3/4 = 30/4^{10} \approx 3/10^5.$$

Hence the number of days out of 100,000 in which they have exactly nine hits is distributed $B(10^5, 30/4^{10})$, which is approximately $P(3)$. Hence the required probability is approximately the probability that a $P(3)$-distributed variable assumes the value 1, that is

$$e^{-3} \cdot \frac{3^1}{1!} = 3e^{-3}.$$

Thus, (ii) is true.

4. (a) For $1 \leq k \leq n$, we have $Y = k$ if one of the X_i-s is k and the other is at most k. Thus, $Y = k$ for

$$(X_1, X_2) = (1, k), (2, k), \ldots, (k, k), (k, k-1), \ldots, (k, 1),$$

altogether $2k - 1$ pairs. Hence:

$$P_Y(k) = \frac{2k-1}{n^2}, \qquad 1 \leq k \leq n.$$

Thus:

$$\begin{aligned}
E(Y) &= \textstyle\sum_{k=1}^{n} \tfrac{2k-1}{n^2} \cdot k \\
&= \tfrac{1}{n} \left(2 \sum_{k=1}^{n} k^2 \cdot \tfrac{1}{n} - \sum_{k=1}^{n} k \cdot \tfrac{1}{n} \right).
\end{aligned}$$

The first sum on the right-hand side is $E(U^2)$ and the second is $E(U)$ for a random variable $U \sim U[1,n]$. This yields:

$$\begin{aligned} E(Y) &= \tfrac{1}{n}\left(2V(U) + 2E^2(U) - E(U)\right) \\ &= \tfrac{1}{n}\left(2 \cdot \tfrac{n^2-1}{12} + 2\left(\tfrac{n+1}{2}\right)^2 - \tfrac{n+1}{2}\right) \\ &= \tfrac{4n^2+3n-1}{6n}. \end{aligned}$$

Thus, (iii) is true.

(b) Since X_1, X_2 are independent,

$$\begin{aligned} V(2X_1 + 3X_2) &= V(2X_1) + V(3X_2) \\ &= 2^2 V(X_1) + 3^2 V(X_2) = 13 \cdot \frac{n^2-1}{2}. \end{aligned}$$

Thus, (iv) is true.