Midterm

Mark all correct answers in each of the following questions.

1. Consider the problem, mentioned in class, of $2n$ people standing in line for a show, where n of them have IS50 bills and the others have IS100 bills, tickets cost IS50 and the cashier has no money at the beginning. Let X denote the number of people, out of the n who have IS100 bills, who have to wait for change.

   (a) X is binomially distributed.

   (b) $P(X = n) = 1/2^n$.

   (c) $E(X) = n/2$.

   (d) Selecting randomly one of the people with IS100 bills, the probability for that person to have to wait for change is slightly over $1/2$, but it converges to $1/2$ as $n \longrightarrow \infty$.

   (e) The probability that none of the people needs to wait for change and the cashier never has more than two IS50 bills during the process is $n/\binom{2n}{n}$.

2. An urn contains n balls, enumerated by the numbers $1, 2, \ldots, n$. The balls are drawn with replacement until each ball has been drawn at least once. For $1 \leq i \leq n$, denote by X_i the number of drawings until each of the balls $1, 2, \ldots, i$ has been drawn at least once. (For example, if $n = 4$ and the balls drawn were 3,3,1,2,3,4, then $X_1 = 3$, $X_2 = X_3 = 4$, $X_4 = 6$.)

   (a) For sufficiently large n we have $0.35 \leq P(X_1 > n) \leq 0.4$.

   (b) For sufficiently large n we have $0.55 \leq P(X_2 > n) \leq 0.65$.

(c) $E(X_1) = n$.

(d) None of the random variables X_i is geometrically distributed.

(e) The events $\{X_2 = 3\}$ and $\{X_3 = 4\}$ are independent.

(f) As the number of balls $n \longrightarrow \infty$, the expected length of the process becomes very large when compared to n. More precisely, $E(X_n)/n \underset{n\to\infty}{\longrightarrow} \infty$.

3. Reuven tosses $n + 1$ dice and Shim'on tosses n.

(a) The probability for Reuven to have more of the outcomes divisible by 3 than Shim'on is $1/2$.

(b) The same for even outcomes.

(c) Let X be the sum of upfaces of Reuven's dice, and Y the corresponding number for Shim'on. Then the number of discontinuity points of the distribution function F_X of X is greater than that of the distribution function F_Y of Y.

(d) Denote by p_n the probability for Reuven and Shim'on to have the same number of even outcomes. Then $\sqrt{n}p_n \underset{n\to\infty}{\longrightarrow} c$ for some real $c > 0$.

(e) Let q_n denote the probability for Reuven to have more even outcomes than Shim'on, given that he has at least as many even outcomes as Shim'on. Then $q_n \underset{n\to\infty}{\longrightarrow} 1$.

4. All cards are drawn without replacement from a full deck of 52 cards.

(a) The probability for at least two of the queens to be drawn consecutively is $\dfrac{1207}{13 \cdot 17 \cdot 25}$.

(b) The event that the queen of hearts and the queen of diamonds are drawn consecutively and the event that the queen of spades and the queen of clubs are drawn consecutively are independent.

(c) The event that the queen of hearts and the queen of diamonds are drawn consecutively and the event that the queen of hearts and the queen of clubs are drawn consecutively are independent.

(d) Let X be the sum of the step numbers at which the four queens are drawn. Then $E(X) = 104$.

Solutions

1. The k-th person among those with IS100 bills needs to wait for change if and only if he arrives at the cashier's before the k-th person among those with IS50 bills. By symmetry, the probability for this is $1/2$. Letting $X_k = 1$ if that person waits and $X_k = 0$ otherwise, we have therefore $E(X_k) = 1/2$. Since $X = \sum_{k=1}^{n} X_i$, we have $E(X) = n/2$.

   Obviously, we have r people waiting in case of a specific ordering if and only if $n - r$ people are waiting in case of the reverse ordering. In particular,
$$P(X = n) = P(X = 0) = \frac{1}{n+1}.$$
   (We mention in passing that the symmetry about $n/2$ provides another way of showing that $E(X) = n/2$.)

   Suppose X is binomially distributed. Since X assumes the values $0, 1, \ldots, n$, and $P(X = 0) = P(X = n)$, we must have $X \sim B(n, 1/2)$. But then we should have $P(X = 0) = 1/2^n$, and not $P(X = 0) = 1/(n+1)$.

   An ordering of the people has the property that none of the people needs to wait for change and the cashier never has more than two IS50 bills if and only if the first person in line has an IS50 bill, the last has an IS100 bill, and exactly one person out of each of the pairs of people $(2, 3)$, $(4, 5)$, $\ldots$, $(2n - 2, 2n - 1)$ has an IS50 bill. Hence the number of possibilities of choosing places for the people, for which the condition is satisfied, is 2^{n-1}. It follows that the probability of this event is $2^{n-1}/\binom{2n}{n}$.

   Thus, only (c) is true.

2. We have $X_1 > n$ if none of the first n drawings was of ball 1. Therefore:

$$P(X_1 > n) = \left(1 - \frac{1}{n}\right)^n \underset{n\to\infty}{\longrightarrow} \frac{1}{e} \approx 0.3678.$$

Similarly, we have $X_2 > n$ if either none of the first n drawings was of ball 1 or none was of ball 2. Consequently:

$$P(X_2 > n) = 2\left(1 - \frac{1}{n}\right)^n - \left(1 - \frac{2}{n}\right)^n \underset{n\to\infty}{\longrightarrow} \frac{2}{e} - \frac{1}{e^2} \approx 0.6004.$$

Defining a drawing of ball 1 as a success, X_1 becomes the number of trials until the first success in a sequence of independent trials, with a probability of success $1/n$ in each. Hence $X_1 \sim G(1/n)$, and in particular $E(X_1) = n$. (Note that, for $i \geq 2$, the variable X_i is not geometrically distributed, as the minimal value it may assume is i.)

Intuitively, the events $\{X_2 = 3\}$ and $\{X_3 = 4\}$ should be dependent since, if we know that $X_3 = 4$, we know in particular that all first three balls were drawn at one of the first four drawings, which then gives a relatively high probability to the event $\{X_2 = 3\}$. More formally, we can verify it as follows without calculating exactly the probabilities in question. Denote by Y_i the number of the ball drawn at the i-th step. Then:

$$\begin{aligned} P(X_2 = 3) &\leq P(Y_1 = 1, Y_3 = 2) + P(Y_2 = 1, Y_3 = 2) \\ &\quad + P(Y_1 = 2, Y_3 = 1) + P(Y_2 = 2, Y_3 = 1) \\ &= \tfrac{4}{n^2}\,. \end{aligned}$$

Similarly:

$$P(X_3 = 4) \leq 18P(Y_1 = 1, Y_2 = 2, Y_4 = 3) = \frac{18}{n^3}\,.$$

On the other hand:

$$P(X_2 = 3, X_3 = 4) \geq P(Y_1 = Y_2 = 1, Y_3 = 2, Y_4 = 3) = \frac{1}{n^4}\,.$$

Thus for sufficiently large n we have

$$P(X_2 = 3, X_3 = 4) \geq \frac{1}{n^4} > \frac{4}{n^2} \cdot \frac{18}{n^3} = P(X_2 = 3)P(X_3 = 4),$$

so that the events in question are dependent.

To calculate the expected length of the process, we may proceed as follows. At the first step we certainly draw a "new" ball. At the second drawing there is a probability $1 - \frac{1}{n}$ of drawing a different ball. Hence the number of steps until a ball different than the one drawn at the first step is drawn is distributed $G\left(1 - \frac{1}{n}\right)$. Similarly, once two distinct balls have been drawn, the time until we draw a ball distinct from both is distributed $G\left(1 - \frac{2}{n}\right)$. Thus, we may represent the length of the process as a sum of the form $T_1 + T_2 + \ldots + T_n$, where $T_i \sim G\left(1 - \frac{i-1}{n}\right)$ for each i. It follows that the expected length of the process is

$$E(T_1)+E(T_2)+\ldots+E(T_n) = n+\frac{n}{n-1}+\ldots+\frac{n}{1} = n\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right).$$

Since the harmonic series diverges, the expression on the right hand side is much larger than n.

Thus, (a), (b), (c) and (f) are true.

3. When considering only the number of even outcomes each of the players gets, the problem reduces to the analogous one with coins, discussed in class, so that the probability for Reuven to have more even outcomes than Shim'on is $1/2$. Now consider the number of outcomes divisible by 3 of each player. Denote by U the number of such outcomes of Reuven and by V the number of those of Shim'on. Let U' be the number of outcomes divisible by 3 out of the first n tosses of Reuven. Then:

$$P(U > V) = P(U' > V) + \frac{1}{3} \cdot P(U' = V). \qquad (1)$$

By symmetry, we clearly have

$$P(U' > V) = P(U' < V),$$

and therefore

$$P(U' > V) = \frac{1 - P(U' = V)}{2}.$$

Substituting in (1), we obtain

$$P(U > V) = \frac{1}{2} - \frac{P(U' = V)}{6} < \frac{1}{2}.$$

The variable X is discrete and assumes with positive probabilities the $5n+6$ values $n+1$, $n+2$, $\ldots$, $6(n+1)$. Hence these $5n+6$ are all the discontinuity points of F_X. Similarly, F_Y has $5n+1$ discontinuity points.

Let T and S denote the number of even outcomes of Reuven and of Shim'on, respectively. Then:

$$p_n = P(T = S) = \sum_{k=0}^{n} \frac{\binom{n+1}{k}}{2^{n+1}} \cdot \frac{\binom{n}{k}}{2^n} = \frac{1}{4^n} \sum_{k=0}^{n} \binom{n+1}{k} \binom{n}{k} .$$

Now

$$\sum_{k=0}^{n} \binom{n+1}{k} \binom{n}{k} = \sum_{k=0}^{n} \binom{n+1}{k} \binom{n}{n-k} = \binom{2n+1}{n} .$$

By Stirling's formula

$$\binom{2n}{n} \approx \frac{\sqrt{2\pi \cdot 2n}(2n/e)^{2n}}{\left(\sqrt{2\pi n}(n/e)^n\right)^2} = \frac{2^{2n}}{\sqrt{\pi n}} ,$$

and therefore:

$$\binom{2n+1}{n} = \frac{1}{2} \cdot \binom{2(n+1)}{n+1} \approx \frac{2^{2n+1}}{\sqrt{\pi n}} .$$

Thus we finally obtain:

$$p_n \approx \frac{2}{\sqrt{\pi n}} .$$

Clearly:

$$q_n = P(T > S | T \geq S) = \frac{P(T > S)}{P(T \geq S)} = \frac{1/2}{1/2 + p_n} \underset{n \to \infty}{\longrightarrow} 1.$$

Thus, (b), (c), (d) are (e) are true.

4. We may alternatively view the experiment as consisting of ordering all cards of the deck.

   To count the number of possibilities of choosing places for the queens so that no pair is placed in succession, let us view the set of all locations as consisting of 49 locations. We have to choose 4 of these locations, in the first 3 of them put 2 cards in each, the first being a queen and the second not, and in the fourth put a queen. Thus there are $\binom{49}{4}$ ways to choose locations for the queens. Hence the probability for no two queens to be adjacent is

$$\frac{\binom{49}{4}}{\binom{52}{4}} = \frac{4324}{13 \cdot 17 \cdot 25}.$$

   The event whereby two of the queens are next to each other is the complementary event, and therefore its probability is

$$1 - \frac{4324}{13 \cdot 17 \cdot 25} = \frac{1201}{13 \cdot 17 \cdot 25}.$$

   The probability for any two specified queens to be adjacent is

$$\frac{2 \cdot 51}{52 \cdot 51} = \frac{2}{52},$$

   and the probability for both these two being adjacent and the other two being adjacent is

$$\frac{2 \cdot 50 \cdot 2 \cdot 49}{52 \cdot 51 \cdot 50 \cdot 49} = \frac{2^2}{52 \cdot 51} \neq \left(\frac{2}{52}\right)^2.$$

   Hence the events in (b) are dependent. The probability that the queen of hearts is adjacent to both the queen of diamond and the queen of clubs is

$$\frac{2 \cdot 50}{52 \cdot 51 \cdot 50} = \frac{2}{52 \cdot 51} \neq \left(\frac{2}{52}\right)^2,$$

   and hence the events in (c) are dependent as well.

   The location of each queen is distributed $U[1, 52]$, and therefore its expected value is $53/2$. The variable in (d) is the sum of the locations of the four queens, and thus its expected value is

$$53/2 + 53/2 + 53/2 + 53/2 = 106.$$

   Thus, none of the claims is true.