Final #2 – Part II

Solutions – Questions 3 and 4

3. (a) For $t \geq 0$ we have:

$$\begin{aligned} F_Y(t) &= P(Y \leq t) = P(X_1 \leq t, X_2 \leq t) \\ &= F_{X_1}(t) \cdot F_{X_2}(t) = F_{X_1}^2(t) \\ &= (1 - e^{-t})^2 = 1 - 2e^{-t} + e^{-2t}. \end{aligned}$$

Therefore:

$$E(Y) = \int_0^\infty (1 - F_Y(t))dt = \int_0^\infty 2e^{-t}dt - \int_0^\infty e^{-2t}dt = \frac{3}{2}.$$

By Markov's Inequality:Therefore:

$$P(Y \geq a) \leq \frac{E(Y)}{a} = \frac{3}{2a}.$$

For the right-hand side to be bounded by $\frac{1}{100}$, we need $a \geq 150$.

Thus, (iii) is true.

(b) Clearly, $F_T(t) = 0$ for $t < 0$. For $t \geq 0$ we have:

$$\begin{aligned} F_T(t) &= 1 - P(Y > t) = 1 - (1 - F_{X_1}(t))^2 \\ &= 1 - e^{-2t}. \end{aligned}$$

Therefore $T \sim \text{Exp}(2)$.
Thus, (ii) is true.

(c) Since the function $f(t) = e^{-t}$ is decreasing, the density function of W is given by

$$\begin{aligned} f_W(w) &= f_{X_1}(-\ln w) \cdot |(-\ln w)'| \\ &= e^{-(-\ln w)} \cdot \frac{1}{w} = w \cdot \left| -\frac{1}{w} \right| = 1, \quad 0 \le w \le 1, \end{aligned}$$

and $f_W(w) = 0$ for $w \notin [0, 1]$.
Thus, (ii) is true

(d)

$$\begin{aligned} V(X_1X_2) &= E(X_1^2X_2^2) - E^2(X_1X_2) \\ &= E(X_1^2)E(X_2^2) - E^2(X_1)E^2(X_2) \\ &= E^2(X_1^2) - E^4(X_1) \\ &= (V(X_1) + E^2(X_1))^2 - E^4(X_1) \\ &= (1 + 1^2)^2 - 1^4 = 3 \end{aligned}$$

Thus, (iii) is true.

(e) Since $E(X_i^2) = 2$ and

$$V(X_i^2) = E(X_i^4) - E^2(X_i^2) = \int_0^\infty t^4 e^{-t} dt - 2^2 = \Gamma(5) - 4 = 4! - 4,$$

we have:

$$P\left(\sum_{i=1}^{100} X_i^2 \le 200\right) = P\left(\frac{\sum_{i=1}^{100} X_i^2 - 2 \cdot 100}{\sqrt{(4! - 4) \cdot 100}} \le 0\right).$$

Using the normal approximation we obtain

$$P(\sum_{i=1}^{100} X_i^2 \le 200) \approx \Phi(0) = \frac{1}{2}.$$

Thus, (iii) is true.

(f)

$$\rho(X_1, e^{-X_1}) = \frac{\text{Cov}(X_1, e^{-X_1})}{\sqrt{V(X_1)V(e^{-X_1})}}.$$

Since $W = e^{-X_1} \sim U(0,1)$, we have $E(W) = \frac{1}{2}$ and $V(W) = \frac{1}{12}$, and

$$\begin{aligned}\text{Cov}(X_1, e^{-X_1}) &= E(X_1 \cdot e^{-X_1}) - E(X_1) \cdot E(W) \\ &= \int_0^\infty te^{-2t}dt - 1 \cdot \tfrac{1}{2} = 1/4 - 1/2 = -1/4.\end{aligned}$$

Therefore

$$\rho(X_1, e^{-X_1}) = \frac{-1/4}{\sqrt{1 \cdot 1/12}} = -\frac{\sqrt{3}}{2}.$$

Thus, (ii) is true.

4. First, note that (X, Y) is uniformly distributed in the region

$$T = \{-1 \leq x \leq 0,\ -1 \leq y \leq 0\} \cup \{0 \leq x \leq 1,\ 0 \leq y \leq 1\}.$$

Therefore, various calculations can be done also using geometricl considerations. For example,

$$c = \frac{1}{\text{area (T)}} = \frac{1}{1+1} = \frac{1}{2}.$$

(a)

$$P(X^2 + Y^2 \leq 1) = c \cdot \frac{\pi \cdot 1^2}{2} = \frac{\pi}{4}.$$

Thus, (iii) is true.

(b) Obviously, X and Y are not independent. Moreover, large values of X correspond to large values of Y, so it is intuitively clear that the covariance between X and Y should be positive. The explicit calculations provided below confirm this intuition. Obviously $E(X) = E(Y) = 0$, and

$$\begin{aligned}E(XY) &= \int_{-1}^{0}\int_{-1}^{0} cxydxdy + \int_{0}^{1}\int_{0}^{1} cxydxdy \\ &= c(1/4 + 1/4) = 1/4.\end{aligned}$$

Finally

$$\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{4} > 0.$$

Thus, (iii) is true.

(c) By symmetry, obviously $Y \sim U(-1,1)$. Therefore:

$$V(X) = \frac{2^2}{12} = \frac{1}{3}.$$

Thus, (iii) is true.

(d) For an arbitrary $0 \leq t \leq 1$ we have:

$$P(Z \leq t) \quad = \quad P(X \leq t, Y \leq t).$$

Denote by R the following region:

$$R = \{x \leq t, y \leq t\}.$$

Clearly:

$$P(X \leq t, Y \leq t) = c \cdot \text{area }(R) = c \cdot (1+t^2) = \frac{1}{2}(1+t^2), \;\; t \in [0,1].$$

Thus, (i) is true.