Final #1 – Part II

Solutions – Questions 3 and 4

3. (a) Obviously,

$$E(Y) = E(Z_1^2) = V(Z_1) + E^2(Z_1) = 1.$$

Therefore

$$P(Y \geq a) \quad \leq \quad \frac{E(Y)}{a} = \frac{1}{a} \leq \frac{1}{100},$$

which yields $a \geq 100$.

Thus, (iv) is true.

(b) Clearly, $F_Y(t) = 0$ for $t < 0$. For $t \geq 0$ we have:

$$\begin{aligned} F_Y(t) &= P(Y \leq t) = P(Z_1^2 \leq t) \\ &= P(-\sqrt{t} \leq Z_1 \leq \sqrt{t}) \\ &= \Phi(\sqrt{t}) - \Phi(-\sqrt{t}) \\ &= 2\Phi(\sqrt{t}) - 1. \end{aligned}$$

Therefore the density function of Y is

$$f_Y(t) = (F_Y(t))' = 2 f_{Z_1}(\sqrt{t}) \cdot \frac{1}{2\sqrt{t}} = \frac{1}{\sqrt{2\pi}} t^{-1/2} e^{-t/2}, \quad t \geq 0,$$

and 0 otherwise. Hence $Y \sim \Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$.
Thus, (i) is true.

(c) We have $T \sim N(0, 100)$, so that $E(T^2) = V(T) = 100$.
Thus, (i) is true.

(d)

$$\begin{aligned} P(T^2 \leq 100) &= P(-10 \leq T \leq 10) \\ &= P(-1 \leq \frac{T-0}{10} \leq 1) \\ &= \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.6826. \end{aligned}$$

Thus, (iii) is true.

(e)

$$\begin{aligned} F_S(0) &= P(W \leq 0, Z_3^3 \leq 0) \\ &= P(W \leq 0) \cdot P(Z_3^3 \leq 0) \\ &= (1 - P(W > 0)) \cdot P(Z_3 \leq 0) \\ &= (1 - P(Z_1 > 0, Z_2 > 0)) \cdot P(Z_3 \leq 0) \\ &= (1 - P(Z_1 > 0) \cdot P(Z_2 > 0)) \cdot P(Z_3 \leq 0) \\ &= (1 - (1 - \Phi(0))^2) \cdot \Phi(0) \\ &= (1 - 1/4) \cdot 1/2 = 3/8. \end{aligned}$$

Thus, (iv) is true.

4. First, note that (X, Y) is uniformly distributed in the trapezoid $T = \{-1 \leq x \leq 2,\ 0 \leq y \leq 1, y \leq 2 - x\}$. Therefore, various calculations can be done also geometrically. For example,

$$c = \frac{1}{\text{area (T)}} = \frac{1}{(3+2)/2 \cdot 1} = \frac{2}{5}.$$

(a)

$$\begin{aligned}P(X+Y \le 1 | X-Y \ge 0) &= \frac{P(X+Y \le 1, X-Y \ge 0)}{P(X-Y \ge 0)} \\ &= \frac{P(Y \le 1-X, Y \le X)}{P(Y \le X)}.\end{aligned}$$

Obviously,

$$T_1 = \{0 \le x \le 1,\ 0 \le y \le 0.5,\ y \le 1-x, y \le x\}$$

and

$$T_2 = \{0 \le x \le 2,\ 0 \le y \le 1,\ y \le x, y \le 2-x\},$$

are triangles, and

$$\frac{P(Y \le 1-X, Y \le X)}{P(Y \le X)} = \frac{c \cdot \text{area }(\text{T}_1)}{c \cdot \text{area }(\text{T}_2)} = \frac{1 \cdot 0.5 \cdot 0.5}{2 \cdot 1 \cdot 0.5} = \frac{1}{4}.$$

Therefore:

$$P(X+Y \le 1 | X-Y \ge 0) = \tfrac{1}{4}.$$

Thus, (iii) is true.

(b) Obviously, X and Y are not independent. Moreover, the large values of X correspond to the small values of Y, and therefore it is intuitively clear that the covariance between X and Y should be negative. However, the explicit calculations provided here support the intuition. The marginal density functions of X is

$$\begin{aligned}f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy \\ &= \begin{cases} c\int_0^1 dy, & -1 \le x \le 1, \\ c\int_0^{2-x} dy, & 1 < x \le 2, \\ 0, & \text{otherwise}, \end{cases} \\ &= \begin{cases} c, & -1 \le x \le 1, \\ c(2-x), & 1 < x \le 2, \\ 0, & \text{otherwise}, \end{cases}\end{aligned}$$

and that of Y is similarly

$$f_Y(y) = \begin{cases} c\int_{-1}^{2-y} dx, & 0 \le y \le 1, \\ 0, & \text{otherwise}, \end{cases} = \begin{cases} c(3-y), & 0 \le y \le 1, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore

$$E(X) = c\int_{-1}^{1} x dx + c\int_{1}^{2} x(2-x)dx = \frac{2}{3}c = \frac{4}{15},$$

$$E(Y) = c\int_{0}^{1} y(3-y)dy = \frac{7}{6}c = \frac{14}{30},$$

and

$$\begin{aligned} E(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{XY}(x,y)dxdy \\ &= c\int_0^1 y \int_{-1}^{2-y} x\ dxdy \\ &= \frac{c}{2}\int_0^1 y((2-y)^2-1)dy \\ &= \frac{c}{2}\int_0^1 y(3-4y+y^2)dy \\ &= \frac{c}{2}\left(\frac{3}{2}-\frac{4}{3}+\frac{1}{4}\right) = \frac{5}{24}c. \end{aligned}$$

Finally

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{5}{24}c - \frac{2}{3}c \cdot \frac{7}{6}c = c\left(\frac{5}{24} - \frac{28}{90}\right) < 0.$$

Thus, (iv) is true.

(c)

$$E(X^2) = c\int_{-1}^{1} x^2 dx + c\int_{1}^{2} x^2(2-x)dx = \frac{19}{12}c.$$

Therefore:

$$V(X) = E(X^2) - E^2(X) = \frac{19}{12}c - \left(\frac{2}{3}c\right)^2 = \frac{19}{12}c - \frac{4}{9}c^2.$$

Thus, (i) is true.

(d) For an arbitrary $0 \le t \le 1$:

$$P(\min(X,Y) \le t) \quad = \quad 1 - P(\min(X,Y) > t) = 1 - P(X > t, Y > t).$$

Denote by T_3 the trapezoid

$$T_3 = \{t \le x \le 2-t,\ t \le y \le 1, y \le 2-x\}.$$

Clearly

$$P(X > t, Y > t) = c \cdot \text{area } (T_3) = c \cdot \frac{2-t-t+1-t}{2} \cdot (1-t) = \frac{3}{5}(1-t)^2.$$

Therefore:

$$P(\min(X,Y) \le t) = 1 - \frac{3}{5}(1-t)^2.$$

Thus, (iv) is true.