Final #1 – Part I

Mark the correct answer in each part of the following questions.

1. A two-stage experiment is performed. At the first stage, a coin is tossed repeatedly until it shows a head for the first time. Suppose the number of tosses at the first stage has been k for some positive integer k. Then, at the second stage, we use k coins. Each of these coins is tossed repeatedly until it shows a head for the first time. Let X be the number of tosses at the first stage and Y the number of tosses at the second. (For example, suppose at the first stage the coin showed a tail in the first 5 tosses and a head in the sixth. At the second stage, the first coin showed a head in the first toss, the second, third, fourth and fifth coins showed a head for the first time in the second toss each, and the sixth coin showed a head for the first time in the one hundredth toss. Then $X = 6$ and $Y = 109$.)

   (a) The conditional distribution of Y given X is:

   (i) uniform.

   (ii) binomial.

   (iii) negative binomial.

   (iv) Poissonian.

   (v) none of the above.

   (b) $E(Y|X = x) =$

   (i) x.

   (ii) $2x$.

   (iii) 2^{x-1}.

   (iv) 2^x.

(v) none of the above.

(c) $P(X=1|Y=2) =$

(i) 1/6.

(ii) 1/3.

(iii) 1/2.

(iv) 2/3.

(v) none of the above.

(d) $P(Y=X) =$

(i) 1/6.

(ii) 1/3.

(iii) 1/2.

(iv) 2/3.

(v) none of the above.

(e) For each positive integer j we have $P(Y=j) =$

(i) $3^{j-1}/4^{j+1}$.

(ii) $3^j/4^{j+1}$.

(iii) $3^{j-1}/4^j$.

(iv) $3^j/4^j$.

(v) none of the above.

(f) Suppose now that the two-stage experiment is performed n times. Let Y_i be the number of tosses at the second stage of the i-th experiment, $1 \le i \le n$. For sufficiently large n have $P(\frac{1}{n}\sum_{i=1}^{n} Y_i \le 4) \approx$

(i) 0.25.

(ii) 0.5.

(iii) 0.75.

(iv) 0.95.

(v) none of the above.

2. Reuven and Shimon play the following n-stage game. At each stage a continuous roulette is turned, which gives a number (uniformly distributed) between 0 and 1. Denote by x_k the number thus obtained

at the k-th stage, $k = 1, 2, \dots, n$. At each stage k, if $x_k > x_i$ for $i = 1, 2, \dots, k-1$, then Reuven gets from Shimon 1 shekel, while if $x_k < x_{k-1} < \dots < x_2 < x_1$, then Shimon gets from Reuven 2^k shekels. Let R_n denote the total amount received by Reuven and S_n be the total amount received by Shimon. (For example, suppose $n = 10, x_k = 1/(k+1)$ for $1 \le k \le 5$, $x_6 = 0.6$, $x_7 = 0.7, x_8 = 0.1, x_9 = 0.2$, and $x_{10} = 0.9$. Then Reuven gets 1 shekel at each of the stages 1, 6, 7, and 10, so that $R_{10} = 4$ shekels. Shimon gets in this case 2^k shekel at each stage k for of the first 5 stages, and nothing later, so that $S_{10} = 2 + 4 + 8 + 16 + 32 = 62$.

(a) For all sufficiently large n we have $P(S_n = 2 + 2^2 + \dots + 2^n = 2^{n+1} - 2 | R_n = 1) =$

  (i) $1/n!$.
  (ii) $2/n!$.
  (iii) $2^n/n!$.
  (iv) $1/n$.
  (v) none of the above.

(b) $E(R_n) \underset{n\to\infty}{\longrightarrow}$

  (i) $e - 1$.
  (ii) e.
  (iii) e^2.
  (iv) ∞.
  (v) none of the above.

(c) $E(S_n) \underset{n\to\infty}{\longrightarrow}$

  (i) e.
  (ii) $e + 1$.
  (iii) $e^2 - 1$.
  (iv) ∞.
  (v) none of the above.

Solutions

1. (a) Let "success" be the event whereby a coin shows a head in a toss. If $X = x$ for some positive integer x, then the number of tosses at the second stage is the total number of tosses till x-th success (including). Clearly, the tosses are independent. Therefore, the conditional distribution of Y given $X = x$ fits the model of the negative binomial distribution. Namely, $Y|_{X=x} \sim \overline{B}(x, 0.5)$.
Thus, (iii) is true.

(b) Since $Y|_{X=x} \sim \overline{B}(x, 0.5)$,

$$E(Y|X = x) = \frac{x}{0.5} = 2x.$$

Thus, (ii) is true.

(c) The required probability is

$$\begin{aligned} P(X = 1|Y = 2) &= \frac{P(X = 1, Y = 2)}{P(Y = 2)} \\ &= \frac{P(X = 1) \cdot P(Y = 2|X = 1)}{\sum_{i=1}^{\infty} P(X = i) \cdot P(Y = 2|X = i)} \\ &= \frac{P(X = 1) \cdot P(Y = 2|X = 1)}{\sum_{i=1}^{2} P(X = i) \cdot P(Y = 2|X = i)}. \end{aligned}$$

Obviously, $X \sim G(0.5)$. As mentioned above, $Y|_{X=x} \sim \overline{B}(x, 0.5)$, and therefore:

$$P(X = 1|Y = 2) = \frac{0.5 \cdot 0.5^2}{0.5 \cdot 0.5^2 + 0.5^2 \cdot 0.5^2} = \frac{2}{3}.$$

Thus, (iv) is true.

(d) The required probability is

$$\begin{aligned} P(X=Y) &= \sum_{r=1}^{\infty} P(X=r)\cdot P(Y=r|X=r) \\ &= \sum_{r=1}^{\infty} 0.5^r \cdot 0.5^r \\ &= \sum_{r=1}^{\infty} 0.25^r = \frac{1}{3}. \end{aligned}$$

Thus, (ii) is true.

(e) For each integer $j \geq 1$ we have

$$\begin{aligned} P(Y=j) &= \sum_{r=1}^{\infty} P(X=r)\cdot P(Y=j|X=r) \\ &= \sum_{r=1}^{\infty} 0.5^r \cdot \binom{j-1}{r-1} 0.5^r \cdot 0.5^{j-r} \\ &= 0.5^{j+1} \sum_{r=1}^{j} \binom{j-1}{r-1} 0.5^{r-1} \\ &= 0.5^{j+1}\left(0.5+1\right)^{j-1} = \frac{3^{j-1}}{4^j}. \end{aligned}$$

Thus, (iii) is true.

(f) The variables $Y_i,\ \ 1 \leq i \leq n$ are independent. By the previous part, $Y_i \sim G\left(\frac{1}{4}\right),\ 1 \leq i \leq n$, and hence $\mu = E(Y_i) = 4$ and $\sigma^2 = V(Y_i) = \frac{1-1/4}{(1/4)^2} = 12$. Therefore:

$$\begin{aligned} P\left(\tfrac{1}{n}\sum_{i=1}^n Y_i \leq 4\right) &= P\left(\frac{\frac{1}{n}\sum_{i=1}^n Y_i - \mu}{\sigma/\sqrt{n}} \leq \frac{4-\mu}{\sigma/\sqrt{n}}\right) \\ &= P\left(\frac{\frac{1}{n}\sum_{i=1}^n Y_i - \mu}{\sigma/\sqrt{n}} \leq 0\right). \end{aligned}$$

For sufficiently large n, by the Central Limit Theorem we obtain:

$$P\left(\frac{1}{n}\sum_{i=1}^{n} Y_i \leq 4\right) \approx \Phi(0) = 0.5.$$

Thus, (ii) is true.

2. (a) The required probability is

$$\begin{aligned} P\left(S_n = 2^{n+1} - 2 | R_n = 1\right) &= \frac{P\left(S_n = 2^{n+1} - 2, R_n = 1\right)}{P\left(R_n = 1\right)} \\ &= \frac{P\left(S_n = 2^{n+1} - 2\right)}{P\left(R_n = 1\right)} \\ &= \frac{P\left(x_n < x_{n-1} < \ldots < x_1\right)}{P\left(x_1 > x_2, x_3, \ldots, x_n\right)} \\ &= \frac{\frac{1}{n!}}{\frac{(n-1)!}{n!}} = \frac{1}{(n-1)!}. \end{aligned}$$

Thus, (v) is true.

(b) For each stage k of the game $1 \leq k \leq n$, define a random variable Y_k by:

$$Y_1 \equiv 1, \qquad Y_k = \begin{cases} 1, & x_k > x_i \;\; 1 \leq i \leq k-1, \\ 0, & \text{otherwise.} \end{cases} \qquad 2 \leq k \leq n.$$

Clearly, $Y_k \sim B(1, \frac{1}{k})$. In these terms,

$$R_n = Y_1 + Y_2 + \ldots + Y_n.$$

Therefore

$$E(R_n) = \sum_{k=1}^{n} \frac{1}{k} \underset{n\to\infty}{\longrightarrow} \infty.$$

Thus, (iv) is true.

(c) Similarly, for each stage k, $1 \leq k \leq n$, define a random variable T_k by:

$$T_1 \equiv 2, \qquad T_k = \begin{cases} 2^k, & x_k < x_{k-1} < \ldots < x_1, \\ 0, & \text{otherwise.} \end{cases} \qquad 2 \leq k \leq n.$$

Clearly,$P(T_k = 2^k) = \frac{1}{k!}$ for $1 \leq k \leq n$, . In these terms,

$$S_n = T_1 + T_2 + \ldots + T_n.$$

Therefore

$$E(T_n) = \sum_{k=1}^{n} 2^k \cdot \frac{1}{k!} \underset{n\to\infty}{\longrightarrow} \sum_{k=1}^{\infty} 2^k \cdot \frac{1}{k!} = e^2 - 1.$$

Thus, (iii) is true.