Midterm

Mark the correct answer in each part of the following questions.

1. Reuven and Shimon play a dreidel game. In each round, each of them rolls the dreidel. If one of them gets a G and the other not, he wins; if both get a G or none gets it, the game continues. They play the game 11 times. Let R_i be the number of rounds in the i-th game, $1 \leq i \leq 11$. (For example, if in the first round of the sixth game Reuven got a P and Shimon an H, in the second both got a G, in the third – Reuven a P and Shimon an N, and in the fourth – Reuven a G and Shimon an H, then $R_6 = 4$.) Let R be the total number of rounds in all games, X the total number of wins of Reuven and A the event whereby at some point during the match Shimon has had more wins than Reuven.

   (a) R_1 is distributed:

      (i) $B(4, 1/4)$.
      (ii) $G(3/4)$.
      (iii) $\overline{B}(4, 1/4)$.
      (iv) approximately $P(1)$.
      (v) None of the above.

   (b) $E(R_1) =$

      (i) 8/3.
      (ii) 4.
      (iii) 25/4.
      (iv) 8.
      (v) None of the above.

   (c) $E(1/R_1) =$

      (i) $\frac{3}{5} \ln \frac{5}{3}$.

(ii) $\frac{3}{8}\ln\frac{8}{3}$.
(iii) $\frac{5}{8}\ln\frac{8}{5}$.
(iv) Does not exist.
(v) None of the above.

(d) X and R are distributed as follows:

(i) $X \sim U[0,11],\ R \sim G(1/11)$.
(ii) $X \sim H(11,11,11),\ R$is distributed approximately P(11).
(iii) $X \sim B(11,1/2),\ R \sim G(10/11)$.
(iv) $X \sim B(11,1/2),\ R \sim \overline{B}(11,3/8)$.
(v) None of the above.

(e) $P(A|X=7) =$

(i) $\frac{1}{12}$.
(ii) $\frac{1}{6}$.
(iii) $\frac{1}{4}$.
(iv) $\frac{1}{3}$.
(v) None of the above.

2. An urn contains cards, marked by the numbers $1, 2, \ldots, n$. The cards are drawn randomly one by one without replacement. Let $X_i,\ 1 \le i \le n$, be the number on the i-th card to be drawn, $Y_i,\ 1 \le i \le n$, the stage at which the card marked by the number i is drawn, $Z_i,\ 1 \le i \le n$, the (first) stage at which all cards $1, 2 \ldots, i$ have already been drawn, and M the maximal number i such that the sequence $X_1, X_2, \ldots, X_i$ is increasing. (For example, if $n = 4$ and the cards have been drawn in the order $2, 4, 1, 3$, then $X_1 = 2, X_2 = 4, X_3 = 1, X_4 = 3, Y_1 = 3, Y_2 = 1, Y_3 = 4, Y_4 = 2, Z_1 = Z_2 = 3, Z_3 = Z_4 = 4, M = 2$.)

(a) For $n = 10$ we have

(i) $P(X_1 > X_2, X_3 > X_4) = 1/8$ and $P(X_1 > X_2 > X_3) = 1/6$.
(ii) $P(X_1 > X_2, X_3 > X_4) = 1/4$ and $P(X_1 > X_2 > X_3) = 1/6$.
(iii) $P(X_1 > X_2, X_3 > X_4) = 1/8$ and $P(X_1 > X_2 > X_3) = 1/3$.
(iv) $P(X_1 > X_2, X_3 > X_4) = 1/4$ and $P(X_1 > X_2 > X_3) = 1/3$.

(v) None of the above.

(b) For $n = 10$ we have $P(X_5 = 4|Y_7 = 9) =$:

(i) 1/100.

(ii) 1/90.

(iii) 1/10.

(iv) 1/9.

(v) None of the above.

(c) Z_1 is distributed:

(i) Uniformly.

(ii) Binomially.

(iii) Hypergeometrically.

(iv) Geometrically.

(v) None of the above.

(d)

(i) $P(Z_i = k) = k(k-1)(k-2)\dots(k-i+1)/n!$ for $i \le k \le n$.

(ii) $P(Z_i = k) = (k-1)(k-2)\dots(k-i)/n!$ for $i \le k \le n$.

(iii) $P(Z_i = k) = k(k-1)(k-2)\dots(k-i)/n!$ for $i \le k \le n$.

(iv) $P(Z_i = k) = i(k-1)(k-2)\dots(k-i+1)/n!$ for $i \le k \le n$.

(v) None of the above.

(e) Suppose $n = 6$ and we repeat the experiment 2160 times. The probability that $M = 6$ in exactly 3 out of the experiments is approximately:

(i) e^{-3}.

(ii) $3e^{-3}$.

(iii) $9e^{-3}/2$.

(iv) $9e^{-3}$.

(v) None of the above.

3. (a) A person draws a random number out of a discrete uniform distribution $U[1, 2]$. If the selected number is not 1, he draws again a random number, this time among $1, 2, 3$. In general, if the number

drawn at the $(n-1)$-st stage is 1, the experiment is stopped; if not – he draws a number from $U[1, n+1]$. Let X be the number of stages until the game stops. (For example, if the number chosen at the first stage is 1, at the second stage 2, at the third stage 4, at the fourth stage 3 and at the fifth stage 1, then $X = 5$.) Then $P(X = 2 | X \text{is even}) =$

(i) $\frac{\operatorname{arctg} 2}{\pi}$.
(ii) e^{-2}.
(iii) $\frac{1}{6 \ln 2}$.
(iv) $\frac{\pi}{12}$.
(v) None of the above.

(b) The Department of Randomization Sciences at Ben-Gurion University has $2n$ students, n of whom are boys and n girls. The department has to send each student a letter regarding his/her academic status. The secretary deals with the letters just as the absent-minded secretary in the problem we considered in class. Denote by A the event whereby each of the boys gets a letter intended to some boy and each girl a letter intended to some girl, by B the event whereby none of the boys gets his letter, and by C the analogous event for girls.

(i) $P(A) = \frac{1}{n!}$, $P(B|A) \underset{n\to\infty}{\longrightarrow} 0, P(B \cap C|A) \underset{n\to\infty}{\longrightarrow} 0$.
(ii) $P(A) = \frac{1}{\binom{2n}{n}}$, $P(B|A) \underset{n\to\infty}{\longrightarrow} \frac{1}{e}, P(B \cap C|A) \underset{n\to\infty}{\longrightarrow} \frac{1}{e^2}$.
(iii) $P(A) = \frac{1}{\binom{2n}{n}}$, $P(B|A) \underset{n\to\infty}{\longrightarrow} 1 - \frac{1}{e}, P(B \cap C|A) \underset{n\to\infty}{\longrightarrow} \left(1 - \frac{1}{e}\right)^2$.
(iv) $P(A) = \frac{1}{n!}$, $P(B|A) \underset{n\to\infty}{\longrightarrow} \frac{1}{e}, P(B \cap C|A) \underset{n\to\infty}{\longrightarrow} \frac{1}{e^2}$.
(v) None of the above.

(c) A discrete random variable X assumes all non-zero integer values according to the probability function

$$P(X = n) = c \cdot \sin^2 \frac{1}{|n|}, \qquad n = \pm 1, \pm 2, \ldots,$$

where c is a suitable constant. Let A be the event whereby X assumes either a positive even value or a negative odd value, and $B = \{X > 0\}$.

(i) $P(A) = P(B) = 1/2$ and $E(X) = 0$.
(ii) $P(A) = P(B) > 1/2$ and $E(X) = 0$.
(iii) $P(A) = P(B) = 1/2$ and $E(X) > 0$.
(iv) $P(A) = P(B) = 1/2$ but X does not have an expectation.
(v) None of the above.

(d) A continuous random variable X has a density function f given by

$$f_X(x) = \begin{cases} (\sin x)^{1/\sin x}, & \frac{\pi}{12} \leq x \leq \frac{\pi}{3}, \\ c \cdot \cos x, & \frac{\pi}{3} < x \leq \frac{\pi}{2}, \\ 0, & \text{otherwise}, \end{cases}$$

where c is a suitable constant. Then

$$P\left(\frac{\pi}{6} - 10^{-6} \leq X \leq \frac{\pi}{6} + 10^{-6}\right) \approx$$

(i) $\frac{1}{4} \cdot 10^{-6}$.
(ii) $\frac{1}{2} \cdot 10^{-6}$.
(iii) 10^{-6}.
(iv) $2 \cdot 10^{-6}$.
(v) None of the above.

(e) The distribution function of a random variable X is given by:

$$F_X(x) = \begin{cases} 0, & x \leq 0, \\ (\operatorname{tg} x)^{1/\sin x}, & 0 < x \leq \frac{\pi}{4}, \\ 1, & \frac{\pi}{4} < x. \end{cases}$$

Then $P\left(\frac{\pi}{6} \leq X \leq \frac{\pi}{4}\right) =$
(i) $\pi/12$.
(ii) $1/3$.
(iii) $1/2$.
(iv) $2/3$.
(v) None of the above.

Solutions

1. (a) The game ends at any given round (assuming it did not end earlier) if exactly one of the players gets a G, which happens with a probability of $2 \cdot 3/4 \cdot 1/4 = 3/8$. Viewing this event as a success, we may say that the game ends upon the first success. Hence the number of rounds in a single game is distributed $G(3/8)$.

   Thus, (v) is true.

   (b) The expected value of a $G(p)$-distributed random variable is $1/p$, and hence

$$E(R_1) = \frac{1}{3/8} = \frac{8}{3}.$$

   Thus, (i) is true.

   (c) Let us find, more generally, $E(1/X)$ for a $G(p)$-distributed random variable X. We have:

$$\begin{aligned} E(1/X) &= \textstyle\sum_{n=1}^{\infty}(1-p)^{n-1}p \cdot \frac{1}{n} \\ &= \frac{p}{1-p}\sum_{n=1}^{\infty}\frac{(1-p)^n}{n} \\ &= -\tfrac{p}{1-p}\ln(1-(1-p)) \\ &= -\tfrac{p}{1-p}\ln p. \end{aligned}$$

   Plugging in $p = 3/8$, we obtain in our case:

$$E(1/R_1) = \frac{3}{5} \cdot \ln\frac{8}{3}.$$

   Thus, (v) is true.

   (d) Due to the symmetry, Reuven wins each game with a probability of $1/2$. Since the games are independent, his total number of wins is distributed $B(11, 1/2)$.

   With the definition of success in (a), we may view R as the number of rounds in a sequence of independent experiments, with possible outcomes of success and failure for each, until 11 successes are obtained. Hence $R \sim \overline{B}(11, 3/8)$.

   Thus, (iv) is true.

(e) The question, regarding the probability of A when the value of X is given, is equivalent to the Ballot Problem. As the event $\{X = 7\}$ means that Reuven wins 7 times and Shimon wins 4, we obtain:

$$P(A|X = 7) = 1 - \frac{7 - 4 + 1}{7 + 1} = \frac{1}{2}.$$

Thus, (v) is true.

2. (a) Due to symmetry, the events $\{X_i < X_j\}$ have the same probability for all pairs i, j with $i \neq j$. Also, such events for mutually disjoint pairs are independent. Thus:

$$P(X_1 > X_2, X_3 > X_4) = P(X_1 > X_2)P(X_3 > X_4) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Similarly, all $3! = 6$ possible orders of the values of X_1, X_2 and X_3 are equi-probable, so that:

$$P(X_1 > X_2 > X_3) = \frac{1}{6}.$$

Thus, (ii) is true.

(b) The event $\{Y_7 = 9\}$ occurs if the card marked by 7 is the ninth to be drawn. If this is the case, then each of the other cards has an equal probability of $1/9$ to be drawn at any of the drawings but the ninth. In particular:

$$P(X_5 = 4|Y_7 = 9) = \frac{1}{9}.$$

Thus, (iv) is true.

(c) Z_1 is the number of the drawing at which the card marked by 1 is drawn. Clearly, this card (as any other) has equal probabilities of being drawn at each of the stages, and therefore $Z_1 \sim U[1, n]$.

Thus, (i) is true.

(d) The event $\{Z_i \le k\}$ occurs if all cards $1, 2, \ldots, i$ are drawn within the first k stages. Consequently

$$P(Z_i \le k) = \frac{k}{n} \cdot \frac{k-1}{n-1} \cdot \ldots \cdot \frac{k-i+1}{n-i+1},$$

which yields

$$\begin{aligned} P(Z_i = k) &= P(Z_i \le k) - P(Z_i \le k-1) \\ &= \frac{k(k-1)\ldots(k-i+1)}{n(n-1)\ldots(n-i+1)} - \frac{(k-1)(k-2)\ldots(k-i)}{n(n-1)\ldots(n-i+1)} \\ &= \frac{i(k-1)\ldots(k-i+1)}{n(n-1)\ldots(n-i+1)}. \end{aligned}$$

Thus, (v) is true.

(e) The event $\{M = n\}$ occurs if all cards are drawn by order, so that $P(M = n) = 1/n!$. For $n = 6$ we have $P(M = n) = 1/720$. The number of times we get $M = 6$ out of 2160 trials is distributed $B(2160, 1/720)$. By the Poissonian approximation of the binomial, this number is distributed approximately $P(3)$. It follows that the required probability is approximately

$$\frac{3^3}{3!} \cdot e^{-3} = \frac{9}{2} \cdot e^{-3}.$$

Thus, (iii) is true.

3. (a) Let us first find the probability function of X. We have $X = n$ if at the first time we do not choose 1 – which happens with probability $1/2$, at the second time we do not choose 1 – which happens with probability $2/3$, ..., at the $(n-1)$-st time we do not choose 1 – which happens with probability $(n-1)/n$, but at the n-th time we do choose 1 – which happens with probability $1/(n+1)$. Hence:

$$P(X = n) = \frac{1}{2} \cdot \frac{2}{3} \cdot \ldots \cdot \frac{n-1}{n} \cdot \frac{1}{n+1} = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

It follows that

$$P(X \text{ is even}) = \sum_{k=1}^{\infty}\left(\frac{1}{2k}-\frac{1}{2k+1}\right)$$
$$= \left(\tfrac{1}{2}-\tfrac{1}{3}\right)+\left(\tfrac{1}{4}-\tfrac{1}{5}\right)+\ldots = 1-\ln 2,$$

and therefore:

$$P(X=2|X \text{ is even}) = \frac{\mathrm{P(X=2)}}{\mathrm{P(X \text{ is even})}} = \frac{1}{6(1-\ln 2)}.$$

Thus, (v) is true.

(b) There are $\binom{2n}{n}$ ways for choosing the set of n students to whom the boys' letters will go, of which only one belongs to A. Hence:

$$P(A) = \frac{1}{\binom{2n}{n}}.$$

Under A, the event B is equivalent to the event studied in the absent-minded secretary problem, and the same holds for C. Moreover, B and C are independent under A, and therefore:

$$P(A) = \frac{1}{\binom{2n}{n}}, \quad P(B|A) \underset{n\to\infty}{\longrightarrow} \frac{1}{e}, P(B\cap C|A) \underset{n\to\infty}{\longrightarrow} \frac{1}{e^2}.$$

Thus, (ii) is true.

(c) Obviously, $P(B) = 1/2$. Now both events A and B contain the event $C = \{X = 2, 4, 6, \ldots\}$. Since the events $A - C$ and $B - C$ occur when X assumes a negative odd value and a positive odd value, respectively, and $P(n) = P(-n)$ for each n, we clearly have $P(A - C) = P(B - C)$, and therefore $P(A) = P(B)$.

We mention, more generally, the following observation. Let D be any set of non-zero integers, which is anti-symmetric with respect to 0 (namely, for each positive integer n, exactly one of the numbers n and $-n$ belongs to D). Then, due to the symmetry of X, we have $P(X \in D) = 1/2$. The events A and B are particular cases, with D consisting of all positive even integers and negative odd integers for A and of all positive integers for B.

Since X is symmetric about 0, one would guess that $E(X) = 0$. However, we need to check whether the series defining $E(X)$ indeed converges absolutely. Namely, we need to check whether the series $\sum_{n=1}^{\infty} n \sin^2(1/n)$ converges. Now, as $\sin x \approx x$ for $x \approx 0$, the series in question behaves as the series $\sum_{n=1}^{\infty} n(1/n)^2 = \sum_{n=1}^{\infty} 1/n$, and therefore diverges.
Thus, (iv) is true.

(d)

$$\begin{aligned} P\left(\tfrac{\pi}{6} - 10^{-6} \le X \le \tfrac{\pi}{6} + 10^{-6}\right) &= F_X\left(\tfrac{\pi}{6} - 10^{-6}\right) - F_X\left(\tfrac{\pi}{6} + 10^{-6}\right) \\ &\approx 2 \cdot 10^{-6} \cdot F'_X\left(\tfrac{\pi}{6}\right) \\ &= 2 \cdot 10^{-6} \cdot f_X\left(\tfrac{\pi}{6}\right) \\ &= 2 \cdot 10^{-6} \cdot \tfrac{1}{4} = \tfrac{1}{2} \cdot 10^{-6}. \end{aligned}$$

Thus, (ii) is true.

(e)

$$\begin{aligned} P\left(\tfrac{\pi}{6} \le X \le \tfrac{\pi}{4}\right) &= F_X\left(\tfrac{\pi}{4}\right) - F_X\left(\tfrac{\pi}{6}\right) \\ &= 1^{\sqrt{2}} - \left(\tfrac{1}{\sqrt{3}}\right)^2 = \tfrac{2}{3}. \end{aligned}$$

Thus, (iv) is true.