Final #2

Mark the correct answer in each part of the following questions.

1. The participants of a sport event need to organize themselves in teams. The first person to arrive starts a new team. The second person joins the team of the first with probability $1/2$ and starts a new team with probability $1/2$. In general, when any person arrives, if by now there are k teams, he joins each of these teams with probability $1/(k+1)$ and starts a new team with probability $1/(k+1)$. For $n \geq 1$, denote by X_n the number of people who arrived until the n-th team has been started. (For example, if the first three people started new teams, the fourth joined the first, the fifth joined the third, the sixth and seventh joined the first, and the eighth started a new team, then $X_1 = 1, X_2 = 2, X_3 = 3, X_4 = 8$.)

   (a) $P(X_n = n) =$

   (i) $1/n!$.
   (ii) $1/2^{n-1}$.
   (iii) $1/\binom{n+1}{2}$.
   (iv) $1/n$.
   (v) None of the above.

   (b) For $m \geq 1$ we have $P(X_2 > m) =$

   (i) $1/(2m-1)!$.
   (ii) $1/m!$.
   (iii) $1/2^{m-1}$.
   (iv) $m/2^m$.
   (v) None of the above.

   (c) $E(X_n) =$

(i) $2n-1$.

(ii) $\binom{n}{2}+1$.

(iii) $\binom{n+1}{2}$.

(iv) n^2.

(v) None of the above.

(d) $\rho(X_n, X_{2n}) \underset{n\to\infty}{\longrightarrow}$

(i) $-\frac{1}{2\sqrt{2}}$.

(ii) 0.

(iii) $\frac{1}{2\sqrt{2}}$.

(iv) 1.

(v) None of the above.

(e) The process is repeated 1320 times, each time until the tenth team is started. Let S be the total number of people required in all experiments. Then $P(71280 \leq S \leq 73920) \approx$

(i) 0.05.

(ii) 0.32.

(iii) 0.68.

(iv) 0.95.

(v) None of the above.

2. From a full deck of 52 cards we draw 26 cards without replacement. Let W_2 be the total number of 2's drawn, W_3 – total number of 3's, ..., W_{ace} – total number of aces. For $0 \leq i \leq 26$, denote by X_i the number of hearts drawn in the first i drawings, and by Y_i the analogue for diamonds.

(a) By Stirling's Formula ($n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$) we have $P(W_2 = W_3 = \ldots = W_{\text{ace}} = 2) \approx$

(i) $\frac{3^{13}}{2^{39}\cdot\sqrt{26\pi}}$.

(ii) $\frac{1}{6^{13}\sqrt{\pi}}$.

(iii) $\frac{3^{13}\cdot\sqrt{26\pi}}{2^{39}}$.

(iv) $\frac{2^{13}\cdot 3^{39}\sqrt{\pi}}{26^{26}}$.

(v) None of the above.

(b) $P(X_i \geq Y_i + 4 \ \forall\ 4 \leq i \leq 22\ | X_4 = 4, X_{22} = 13, Y_{26} = 13) =$

(i) 1/27.

(ii) 1/14.

(iii) 1/10.

(iv) 1/7.

(v) None of the above.

(c) $\rho(X_{26}, Y_{26}) =$.

(i) -1.

(ii) $-1/3$.

(iii) 1/4.

(iv) 1/2.

(v) None of the above.

(d) A direct application of Markov's Inequality yields:

(i) $P(X_{26} \geq 10) \leq 0.35$.

(ii) $P(X_{26} \geq 10) \leq 0.65$.

(iii) $P(X_{26} \geq 10) \leq 0.85$.

(iv) $P(X_{26} \geq 10) \leq 0.95$.

(v) None of the above.

Remark: We mean here the best bound that be reached. For example, if Markov's Inequality implies that the above probability is at most 0.35, hence it is also at most 0.65, and (0.85 and 0.95), but only (i) should be marked as the correct answer.

(e) Suppose now that all 52 cards are drawn from the deck without replacement. Let A be the event whereby out of the first 4 cards to be drawn there is exactly one heart and one diamond, out of the following 4 cards – again exactly one heart and one diamond, and so forth. The conditional probability that in each such quadruple the heart card and the diamond card are of different values (i.e., not both are 2's, not both are 3's, ..., not both are aces) is approximately

(i) $\frac{1}{2e^2}$.

(ii) $\frac{1}{e^2}$.
(iii) $\frac{2}{e^2}$.
(iv) $\frac{1}{e}$.
(v) None of the above.

3. We define a random one-to-one function h from the set $\mathbf{N} = \{1, 2, 3, \ldots\}$ to itself as follows. To define $h(1)$, we toss a coin repeatedly until a head shows up; if this happens at the k-th toss, we let $h(1) = k$. In general, suppose $h(1), h(2), \ldots, h(n-1)$ have already been defined. We toss a coin repeatedly until a head shows up; if this happens at the k-th toss, we let $h(n)$ be the k-th number not equal to any of the numbers $h(1), h(2), \ldots, h(n-1)$. For each $n \geq 1$, let $X_n = h(n)$, let Y_n be the number of the stage at which the number n has been selected as the value of h at some point (namely, $Y_n = h^{-1}(n)$), and Z_n be the number of tosses at the n-th stage. (For example, if we toss the coin 4 times until the first time a head shows up, 1 time until the second, 20 times until the third and 3 times until the fourth, then $X_1 = 4, X_2 = 1, X_3 = 22, X_4 = 5, Y_1 = 2, Y_4 = 1, Y_5 = 4, Y_{22} = 3, Z_1 = 4, Z_2 = 1, Z_3 = 20, Z_4 = 3$.)

(a) The probability that the function h is onto $\mathbf{N}$ is:

(i) 0.
(ii) 1/4.
(iii) 1/2.
(iv) 1.
(v) None of the above.

(b)

(i) The random variables $X_1, X_2, \ldots$ are both independent and identically distributed, the random variable Y_1 is geometrically distributed.
(ii) The random variables $X_1, X_2, \ldots$ are independent but not identically distributed, the random variable Y_1 is geometrically distributed.

(iii) The random variables $X_1, X_2, \ldots$ are non-independent but identically distributed, the random variable Y_1 is geometrically distributed.

(iv) The random variables $X_1, X_2, \ldots$ are neither independent nor identically distributed, the random variable Y_1 is geometrically distributed.

(v) None of the above.

(c) For $k \geq 1$, we have:

(i) $P(X_2 = k) = 1/2^k - 1/2^{2k}$.

(ii) $P(X_2 = k) = 1/2^{k+1}$.

(iii) $P(X_2 = k) = 1/2^{k-1} - 3/2^{2k}$.

(iv) $P(X_2 = k) = (k+1)/2^{k+2}$.

(v) None of the above.

(d) $P(X_1 = 1 | X_2 = 2) =$

(i) 1/5.

(ii) 2/5.

(iii) 3/5.

(iv) 4/5.

(v) None of the above.

(e)

(i) None of the three sequences $(Z_n)_{n=1}^{\infty}, (Z_n^2)_{n=1}^{\infty}, (2^{Z_n})_{n=1}^{\infty}$ satisfies the weak law of large numbers.

(ii) The sequence $(Z_n)_{n=1}^{\infty}$ satisfies the weak law of large numbers, but the two sequences $(Z_n^2)_{n=1}^{\infty}, (2^{Z_n})_{n=1}^{\infty}$ do not.

(iii) The sequences $(Z_n)_{n=1}^{\infty}, (Z_n^2)_{n=1}^{\infty}$ satisfy the weak law of large numbers, but the sequence $(2^{Z_n})_{n=1}^{\infty}$ does not.

(iv) All three sequences $(Z_n)_{n=1}^{\infty}, (Z_n^2)_{n=1}^{\infty}, (2^{Z_n})_{n=1}^{\infty}$ satisfy the weak law of large numbers.

(v) None of the above.

4. The number X (in millions) of senecios growing annually in a certain region is (approximately) distributed according to the density function f, defined by

$$f(x) = \begin{cases} cxe^{-x^2}, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

for an appropriate constant c.

(a) $c =$

(i) $\frac{1}{2\sqrt{\pi}}$.

(ii) $\frac{1}{\sqrt{\pi}}$.

(iii) 1.

(iv) 2.

(v) None of the above.

(b) Let ψ be the moment generating function of X. Then $\psi(0.001) \approx$)

(i) $1 + \frac{1}{8000}$.

(ii) $1 + \frac{\sqrt{\pi}}{2000}$.

(iii) $1 + \frac{1}{1000e}$.

(iv) $1 + \frac{\ln 2}{500}$.

(v) None of the above.

(c) $E\left(e^{X^2}/X(1+X)^2\right) =$

(i)$2/e$.

(ii) $2\ln 2$.

(iii) 2.

(iv) $2\sqrt{\pi}$.

(v) None of the above.

(d) For $t_1, t_2 > 0$ we have $P(X \geq t_1 + t_2 | X \geq t_1) =$

(i) $e^{-2t_2^2 - 2t_1 t_2}$.

(ii) $e^{-t_2^2 - t_1 t_2}$.

(iii) $e^{-2t_2^2 - t_1 t_2}$.

(iv) $e^{-t_2^2 - 2t_1 t_2}$.

(v) None of the above.

(e) There are two other regions in which the numbers Y, Z of senecios growing annually is distributed as X. The three random variables X, Y, Z are pairwise independent, but not independent. Set $S = X + Y + Z$. Then $\mathrm{Cov}(X, S)$

(i) is the same as $-V(X)$.

(ii) is the same as $V(X)$.
(iii) is the same as $3V(X)$.
(iv) cannot be determined by the given data.
(v) None of the above.

Solutions

1. (a) The event $\{X_n = n\}$ occurs if each of the first n people starts his own team. Hence:

$$P(X_n = n) = \frac{1}{1} \cdot \frac{1}{2} \cdot \ldots \cdot \frac{1}{n} = \frac{1}{n!}.$$

Thus, (i) is true.

(b) We have $X_2 > m$ when all the people from the second up to the m-th joined the first person's team. Each of them, given that all his predecessors from the second on have not started new teams, joins the existing team with probability $1/2$. Thus:

$$P(X_2 > m) = \frac{1}{1} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \ldots \cdot \frac{1}{2} = \frac{1}{2^{m-1}}.$$

One can also reach the same conclusion by noting that X_2 is distributed as $X + 1$, where $X \sim G\left(\frac{1}{2}\right)$. Hence:

$$\begin{aligned} P(X_2 > m) &= P(X + 1 > m) = P(X \geq m) \\ &= \sum_{k=m}^{\infty} \frac{1}{2^k} = \frac{1}{2^{m-1}}. \end{aligned}$$

Thus, (iii) is true.

(c) For $1 \leq i \leq n$ denote by Y_i the number of people who arrive from the time the $(i-1)$-st team is started until the i-th is. Namely, $Y_i = X_i - X_{i-1}$, where $X_0 = 0$. Clearly, the variables Y_i are independent and $G\left(\frac{1}{i}\right)$-distributed. In these terms:

$$X_n = Y_1 + Y_2 + \ldots + Y_n. \tag{1}$$

Hence:

$$\begin{aligned} E(X_n) &= E(Y_1 + Y_2 + \ldots + Y_n) \\ &= \sum_{i=1}^{n} E(Y_i) = \sum_{i=1}^{n} i \\ &= \frac{n+1}{2} \cdot n = \binom{n+1}{2}. \end{aligned}$$

Thus, (iii) is true.

(d) By (1) and the independence of the Y_i's:

$$\begin{aligned} V(X_n) &= \sum_{i=1}^{n} V(Y_i) = \sum_{i=1}^{n} \frac{(i-1)i^2}{i} \\ &= \sum_{i=1}^{n} (i-1)i = 2 \cdot \sum_{i=1}^{n} \binom{i}{2} \qquad (2) \\ &= 2 \cdot \binom{n+1}{3} = \frac{(n-1)n(n+1)}{3}. \end{aligned}$$

Thus:

$$V(X_{2n}) = \frac{2(2n-1)n(2n+1)}{3}. \qquad (3)$$

Since

$$X_{2n} = X_n + Y_{n+1} + \ldots + Y_{2n}, \qquad (4)$$

we have

$$\mathrm{Cov}(X_n, X_{2n}) = \mathrm{Cov}(X_n, X_n) + \sum_{i=n+1}^{2n} \mathrm{Cov}(X_n, Y_i).$$

Since X_n and Y_i, $n+1 \leq i \leq 2n$, are independent, $\mathrm{Cov}(X_n, Y_i) = 0$, and we have

$$\mathrm{Cov}(X_n, X_{2n}) = V(X_n) = \frac{(n-1)n(n+1)}{3}. \qquad (5)$$

By (2), (3) and (5) we obtain:

$$\begin{aligned} \rho(X_n, X_{2n}) &= \sqrt{\frac{(n-1)n(n+1)/3}{2(2n-1)n(2n+1)/3}} \\ &= \frac{1}{\sqrt{2}} \cdot \sqrt{\frac{(n-1)(n+1)}{(2n-1)(2n+1)}}. \end{aligned}$$

Hence $\rho(X_n, X_{2n}) \underset{n\to\infty}{\longrightarrow} \frac{1}{2\sqrt{2}}$.

Thus, (iii) is true.

(e) For $1 \le i \le 1320$, denote by $X_{10,i}$ the number of people who arrive until the tenth team was started at the i-th experiment. Obviously, the variables $X_{10,i}$, $\ 1 \le i \le 1320$, are independent and identically distributed with $E(X_{10,i}) = E(X_{10}) = \binom{10+1}{2} = 55$ and $V(X_{10,i}) = V(X_{10}) = \frac{(10-1)10(10+1)}{3} = 330$. Now, $S = \sum_{i=1}^{1320} X_{10,i}$, and hence

$$
\begin{aligned}
&P(71280 \le S \le 73920) \\
&\qquad = P\left(\tfrac{71280-1320\cdot 55}{\sqrt{1320\cdot 330}} \le \tfrac{S-1320\cdot 55}{\sqrt{1320\cdot 330}} \le \tfrac{73920-1320\cdot 55}{\sqrt{1320\cdot 330}}\right) \\
&\qquad \approx P(-2 \le Z \le 2),
\end{aligned}
$$

where Z is a standard normal variable. Therefore:

$$P(71280 \le S \le 73920) \quad \approx \quad 2\Phi(2) - 1 \approx 0.95.$$

Thus, (iv) is true.

2. (a) Obviously,

$$P(W_2 = W_3 = \ldots = W_{\text{ace}} = 2) = \frac{\binom{4}{2}^{13}}{\binom{52}{26}} = \frac{6^{13}}{\binom{52}{26}}. \tag{6}$$

Stirling's Formula implies

$$\binom{2n}{n} = \frac{2n!}{n! \cdot n!} \approx \frac{\sqrt{4n\pi}\left(\frac{2n}{e}\right)^{2n}}{2n\pi\left(\frac{n}{e}\right)^{2n}} = \frac{2^{2n}}{\sqrt{\pi n}}, \tag{7}$$

and in particular, for $n = 26$,

$$\binom{52}{26} \approx \frac{2^{52}}{\sqrt{26\pi}}. \tag{8}$$

Substituting (8) in the right hand-side of (6), we obtain:

$$P(W_2 = W_3 = \ldots = W_{\text{ace}} = 2) \approx \frac{6^{13}\sqrt{26\pi}}{2^{52}} = \frac{3^{13}\sqrt{26\pi}}{2^{39}}.$$

Thus, (iii) is true.

(b) The equalities $X_4 = 4, X_{22} = 13, Y_{26} = 13$ imply that the 26 cards drawn are exactly all hearts and all diamonds; moreover, the first 4 cards are hearts and the last 4 are diamonds. Given this, to require that $X_i \geq Y_i + 4$ for $4 \leq i \leq 22$ amounts to requiring that the 9 hearts and 9 diamonds in between the first 4 hearts and the last 4 diamonds are drawn in such a way that the number of hearts is never below that of diamonds. This is equivalent to the requirement that the first candidate never trails in the Ballot Problem. Hence the probability in question is $1/(9+1) = 1/10$.

Thus, (iii) is true.

(c) Obviously, $X_{26} \sim H(26, 13, 39)$ and $Y_{26} \sim H(26, 13, 39)$. Thus,

$$V(X_{26}) = V(Y_{26}) = \frac{26 \cdot 13 \cdot 39}{52^2}\left(1 - \frac{25}{51}\right). \tag{9}$$

Denote $Z = X_{26} + Y_{26}$. Clearly, $Z \sim H(26, 26, 26)$, and therefore

$$V(Z) = \frac{26^3}{52^2}\left(1 - \frac{25}{51}\right). \tag{10}$$

Hence, by (9):

$$\begin{aligned}
\rho(X_{26}, Y_{26}) &= \frac{\text{Cov}(X_{26}, Y_{26})}{\sqrt{V(X_{26}) \cdot V(Y_{26})}} \\
&= \frac{V(Z) - (V(X_{26}) + V(Y_{26}))}{2 \cdot \sqrt{V(X_{26}) \cdot V(Y_{26})}} \\
&= \frac{V(Z) - 2 \cdot V(X_{26})}{2 \cdot V(X_{26})} \\
&= \frac{1}{2} \cdot \frac{V(Z)}{V(X_{26})} - 1.
\end{aligned}$$

By (9) and (10) we have $\frac{V(Z)}{V(X_{26})} = \frac{4}{3}$. Thus,

$$\rho(X_{26}, Y_{26}) = \frac{1}{2} \cdot \frac{4}{3} - 1 = -\frac{1}{3}.$$

One can also reach the same conclusion almost without computations. In fact, similarly to X_{26} and Y_{26}, we may consider the variables Z_{26} and V_{26}, denoting the number of spades and of clubs among the 26 drawn cards. Due to symmetry, all four variables have the same variance, and all pairs of distinct variables have the same covariance. Since $X_{26} + Y_{26} + Z_{26} + V_{26} = 26$, we have

$$V(X_{26} + Y_{26} + Z_{26} + V_{26}) = 0.$$

On the other hand, due to the symmetry mentioned above,

$$V(X_{26} + Y_{26} + Z_{26} + V_{26}) = 4V(X_{26}) + 12\,\mathrm{Cov}(X_{26}, Y_{26}).$$

It follows that

$$\mathrm{Cov}(X_{26}, Y_{26}) = -\frac{1}{3}V(X_{26}),$$

and consequently

$$\rho(X_{26}, Y_{26}) = -\frac{1}{3}.$$

Thus, (ii) is true.

(d) As mentioned, in part (c), $X_{26} \sim H(26, 13, 39)$, and therefore

$$E(X_{26}) = \frac{26 \cdot 13}{13 + 39} = \frac{13}{2}.$$

Markov's Inequality implies:

$$P(X_{26} \geq 10) \leq \frac{13/2}{10} = 0.65.$$

Thus, (ii) is true.

(e) Here the spades and clubs do not count. Consider the permutation defining the inner order among the hearts. The question is about the probability that the permutation defining the inner order among the diamonds does not agree with that among the hearts for any of the 13 pairs. This is equivalent to the question in the Absent-Minded Secretary Problem. Thus the required probability is

$$1 - \frac{1}{1!} + \frac{1}{2!} - \ldots - \frac{1}{13!} \approx \frac{1}{e}.$$

Thus, (iv) is true.

3. (a) For any positive integer m and n, the probability that m will not be chosen as $h(n)$, given that it has not been chosen as any of the values of h at the points $1, 2, \ldots, n-1$, is at most $1 - 1/2^{m+1}$. By the product formula, the probability that m will be chosen as $h(n)$ for no n is 0. The event whereby h is not onto is the union, over all positive integers m, of the events whereby m is not in the image of h. Since each of the events in the union is of probability 0, so is the probability of the complementary event in question.

   Thus, (iv) is true.

   (b) Clearly, $X_1, X_2, \ldots$ are neither independent nor identically distributed random variables. Indeed, for example, $P(X_1 = 1) = \frac{1}{2} > 0$ and $P(X_2 = 1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} > 0$, while

$$0 = P(X_1 = 1, X_2 = 1) \neq P(X_1 = 1) \cdot P(X_2 = 1).$$

   Moreover, since Y_1 counts the number of experiments held until a head has been obtained at the first toss, we have $Y_1 \sim G(1/2)$.

   Thus, (iv) is true.

(c) For $k \geq 1$ we have:

$$\begin{aligned}
P(X_2 = k) &= \sum_{i=1}^{k-1} P(Z_1 = i, Z_2 = k-1) + \sum_{i=k+1}^{\infty} P(Z_1 = i, Z_2 = k) \\
&= \sum_{i=1}^{k-1} \frac{1}{2^i} \cdot \frac{1}{2^{k-1}} + \sum_{i=k+1}^{\infty} \frac{1}{2^i} \cdot \frac{1}{2^k} \\
&= \frac{1}{2^{k-1}} \left(1 - \frac{1}{2^{k-1}}\right) + \frac{1}{2^k} \cdot \frac{2}{2^{k+1}} \\
&= \frac{1}{2^{k-1}} - \frac{3}{2^{2k}}.
\end{aligned}$$

Thus, (iii) is true.

(d) By part (c) we have:

$$\begin{aligned}
P(X_1 = 1 | X_2 = 2) &= \frac{P(X_1 = 1, X_2 = 2)}{P(X_2 = 2)} \\
&= \frac{P(Z_1 = 1, Z_2 = 1)}{\frac{1}{2^{2-1}} - \frac{3}{2^4}} \\
&= \frac{P(Z_1 = 1) \cdot P(Z_2 = 1)}{5/16} \\
&= \frac{1/2 \cdot 1/2}{5/16} = \frac{4}{5}.
\end{aligned}$$

Thus, (iv) is true.

(e) Obviously, the variables Z_n are independent and $G(1/2)$-distributed. Thus, $E(Z_n)$ and $V(Z_n)$ are finite. Hence, the sequence $(Z_n)_{n=1}^{\infty}$ satisfies the weak law of large numbers. The situation with $(Z_n^2)_{n=1}^{\infty}$ is similar. Namely, the variables Z_n^2 are independent and identically distributed, with final expectation and variance. Thus, $(Z_n^2)_{n=1}^{\infty}$ also satisfies the weak law of large numbers. However,

$$E(2^{Z_n}) = \sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k} = \infty,$$

as in the St. Petersburg Paradox. Hence, the sequence $(2^{Z_n})_{n=1}^{\infty}$ does not satisfy the weak law.

Thus, (iii) is true.

4. (a) Since f is the density function, we obtain

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f(x)dx \\ &= c \cdot \int_0^{\infty} xe^{-x^2}dx = \frac{c}{2} \cdot \int_0^{\infty} e^{-x^2}dx^2 \\ &= \frac{c}{2} \cdot \int_0^{\infty} e^{-t}dt = \frac{c}{2}, \end{aligned}$$

which yields $c = 2$.

Thus, (iv) is true.

(b) A linear approximation near 0 yields:

$$\psi(0.001) \approx \psi(0) + \psi'(0) \cdot \frac{1}{1000} = 1 + E(X) \cdot \frac{1}{1000}, \quad (11)$$

where $E(X) = \int_0^{\infty} 2 \cdot x^2 e^{-x^2}dx$. Now, if $Y \sim N\left(0, \frac{1}{2}\right)$, then

$$\frac{1}{2} = V(Y) = E(Y^2) = \int_{-\infty}^{\infty} \frac{x^2 e^{-x^2}}{\sqrt{\pi}}dx = \int_0^{\infty} \frac{2x^2 e^{-x^2}}{\sqrt{\pi}}dx = \frac{E(X)}{\sqrt{\pi}},$$

which implies

$$E(X) = \frac{\sqrt{\pi}}{2}. \quad (12)$$

Substituting (12) into (11) we obtain

$$\psi(0.001) \approx 1 + \frac{\sqrt{\pi}}{2} \cdot \frac{1}{1000} = 1 + \frac{\sqrt{\pi}}{2000}.$$

Thus, (ii) is true.

(c)

$$\begin{aligned}
E\left(\frac{e^{X^2}}{X(1+X)^2}\right) &= \int_0^\infty \frac{e^{x^2}}{x(1+x)^2}\cdot 2xe^{-x^2}dx \\
&= \int_0^\infty \frac{2}{(1+x)^2}dx \\
&= -2/(1+x)\Big|_0^\infty \\
&= 2.
\end{aligned}$$

Thus, (iii) is true.

(d)

$$\begin{aligned}
P(X \geq t_1 + t_2 | X \geq t_1) &= \frac{\int_{t_1+t_2}^\infty 2xe^{-x^2}dx}{\int_{t_1}^\infty 2xe^{-x^2}dx} \\
&= \frac{\int_{(t_1+t_2)^2}^\infty e^{-u}du}{\int_{t_1^2}^\infty e^{-u}du} \\
&= \frac{e^{-(t_1+t_2)^2}}{e^{-t_1^2}} \\
&= e^{-t_2^2-2t_1t_2}.
\end{aligned}$$

Thus, (iv) is true.

(e)

$$\begin{aligned}
\mathrm{Cov}(X,S) &= \mathrm{Cov}(X, X+Y+Z) \\
&= \mathrm{Cov}(X,X) + \mathrm{Cov}(X,Y) + \mathrm{Cov}(X,Z) \\
&= \mathrm{Cov}(X,X) \\
&= V(X).
\end{aligned}$$

Thus, (ii) is true.