Midterm #1

Mark all correct answers in each of the following questions.

1. A person tosses repeatedly three coins – one of 10 shekels, one of 5, and one of 2. The process ends when both the 10 shekel coin and the 5 shekel coin show heads. (For example, if at the first time the results were (H,T,H), at the second – (T,T,H), at the third – (H,T,H), and at fourth – (H,H,T), then the process ends after 4 stages.) Let S_n be the event whereby the process lasts exactly n stages, $n = 1, 2, \ldots$, and H_m the event whereby the 2 shekel coin showed heads altogether m times, $m = 0, 1, , \ldots$. (Thus, in the example above, S_4 and H_3 occurred.)

   (a) $P(S_5) = 1/2^5$.

   (b) $P(H_0) = 1/5$.

   (c) $P(H_m) = \frac{2 \cdot 3^n}{5^{n+1}}$ for $m \geq 1$.

   (d) $P(S_n | H_m) = \frac{5^{m+1} \cdot 3^{n-m} \binom{n}{m}}{8^{n+1}}$ for every $m, n \geq 1$.

   (e) Let X be the number of stages of the process, and Y the number of times the 2 shekel coin shows heads throughout the process. (Thus, in the example above, X assumes the value 4 and Y the value 3.) The distribution functions F_X and F_Y of the random variables X and Y have exactly the same sets of discontinuity points.

2. We draw all cards from a full deck of 52 cards one by one without replacement.

   (a) The probability that all 13 diamond cards will be drawn at even stages is $(1/2)^{13}$.

(b) The probability that all 4 queens will be drawn before all 4 kings is $1/70$.

(c) Let A be the event whereby the first 26 cards to be drawn include at least one ace, and B the event whereby the last 26 cards to be drawn include at least one "7". The events A and B are independent.

(d) Let C be the event whereby the first "8" to be drawn precedes the first "7" to be drawn, and D the event whereby the first "8" to be drawn precedes the first "6" to be drawn. The events C and D are independent.

(e) (Recall that 26 of the cards are red and 26 are black.) Let E be the event whereby, during the process, the number of red cards already drawn is at least as large as the number of black cards already drawn. Then $P(E) = 1/53$.

3. A die is tossed repeatedly until each even number is obtained at least once. Let A be the event whereby the die is tossed exactly 4 times, B the event whereby the upface at the first toss is "3", and X the random variable counting the number of tosses.

(a) $P(A|B) = 1/36$.

(b) $P(\bar{B}) = 1/18$.

(c) $P(A \cup B) = 11/54$.

(d) The random variable X is geometrically distributed.

(e) Suppose we repeat the process 10 times (each time until all even numbers show up at least once). Denote by Y the number of times, out of 10, in which there were exactly 4 tosses. Then Y is binomially distributed.

4. (a) You are required to take at least one out of 2 gambles. In the first, a die is tossed 5 times, and you win 1 shekel if none of the results is either "3" or "6". In the second, a die is tossed 10 times, and you win 1 shekel if none of the results is "6". The two possibilities are equally good.

(b) A, B, C are events in a probability space. If $P(C|A \cap B) = P(C)$, then $P(A \cap B \cap C) = P(A)P(B)P(C)$. However, the above condition does not guarantee that A, B, C are independent.

(c) If X is a random variable defined on a symmetric probability space, then the function F_X assumes only rational values.

(d) There exists a random variable X such that the equality $F_X(x) = 3/7$ holds for exactly 3 points x on the real line.

(e) X and Y are random variables which assume only positive values. It is given that $F_Y(t) = F_X(2t)$ for every $t \in \mathbf{R}$. You may participate at exactly one out of two gambles, in one of which your total winnings are X and in the other Y. You should prefer the gamble with total winnings X.

Solutions

1. (a) The process ends at some stage, if it got to this stage at all, if both the 10 shekel coin and the 5 shekel coin show heads at this stage. The probability for this is $1/4$. Hence the probability for the process to terminate after exactly n steps, for any $n \geq 1$, is

$$P(S_n) = \left(\frac{3}{4}\right)^{n-1} \cdot \frac{1}{4}.$$

In particular:

$$P(S_5) = \left(\frac{3}{4}\right)^{4} \cdot \frac{1}{4} = \frac{3^4}{4^5}.$$

(b) By the law of total probability:

$$\begin{aligned} P(H_0) &= \sum_{n=1}^{\infty} P(S_n)P(H_0|S_n) \\ &= \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^{n-1} \cdot \frac{1}{4} \cdot \frac{1}{2^n} \\ &= \frac{1}{8} \sum_{n=0}^{\infty} \left(\frac{3}{8}\right)^{n} = \frac{1}{5}. \end{aligned}$$

(c) Similarly to the preceding part:

$$\begin{aligned} P(H_m) &= \sum_{n=m}^{\infty} \left(\frac{3}{4}\right)^{n-1} \cdot \frac{1}{4} \cdot \frac{\binom{n}{m}}{2^n} \\ &= \frac{1}{3} \sum_{n=m}^{\infty} \binom{n}{m} \left(\frac{3}{8}\right)^n \\ &= \frac{1}{3} \cdot \frac{(3/8)^m}{(5/8)^{m+1}} \\ &= \frac{8}{15} \cdot \left(\frac{3}{5}\right)^m . \end{aligned}$$

(d)

$$\begin{aligned} P(S_n|H_m) &= \frac{P(S_n \cap H_m)}{P(H_m)} \\ &= \frac{(3/4)^{n-1} \cdot 1/4 \cdot \binom{n}{m}/2^n}{8/15 \cdot (3/5)^m} \\ &= \frac{5^{m+1} \cdot 3^{n-m} \cdot \binom{n}{m}}{8^{n+1}} . \end{aligned}$$

(e) The discontinuity points of the distribution function of a discrete random variable are exactly the values the random variable may assume. Since X assumes only positive integer values, while Y assumes the value 0 also, the point 0 is a discontinuity point of F_Y, but not of F_X.

Thus, (b) and (d) are true.

2. (a) The number of possibilities of placing the 13 diamond cards in the sequence is $\binom{52}{13}$. The number of possibilities for placing them within the 26 even stages is $\binom{26}{13}$. Hence the required probability is $\binom{26}{13}/\binom{52}{13}$.

(b) Consider the 8 drawings of queens and kings. There are $\binom{8}{4} = 70$ possibilities for selecting which of these drawings will be of queens (and thereby which of kings). Out of these, only one satisfies the required condition. Hence the probability is $1/70$.

(c) $\bar{A}$ is the event whereby all aces are among the last 26 cards to be drawn. Hence $P(\bar{A}) = \binom{26}{4}/\binom{52}{4}$. Similarly, $P(\bar{B}) = \binom{26}{4}/\binom{52}{4}$. Now, $\bar{A} \cap \bar{B}$ is the event whereby all "7" cards are among the 26 first cards to be drawn, while all aces are among the last 26. There are altogether $\binom{52}{8}$ possibilities to place these 8 cards in the sequence, out of which $\binom{26}{4}^2$ satisfy the required condition. Hence:

$$P(\bar{A} \cap \bar{B}) = \frac{\binom{26}{4}^2}{\binom{52}{8}}.$$

It follows that

$$P(\bar{A} \cap \bar{B}) \neq P(\bar{A})P(\bar{B}),$$

and therefore A, B are not independent.

Intuitively, the reason the events are dependent is that A gives the information that some card is drawn within the first 26 stages, and therefore makes it more likely for other cards to be drawn during the last 26 stages.

(d) By symmetry, $P(C) = P(D) = 1/2$. Again by symmetry, each of the $3! = 6$ orderings of the first "6", the first "7" and the first "8" is equally likely, which implies that $P(C \cap D) = 2/6 = 1/3$. It follows that C and D are dependent.

Intuitively, the event C hints that the first "8" is drawn relatively early, and therefore it is plausible that it is drawn before the first "6". Hence $P(D|C) > P(D)$.

(e) The event E is analogous to the event whereby, in the ballot problem, the first candidate has never less votes than his opponent, in the special case where each gets 26 votes. Hence $P(E) = 1/27$.

Thus, only (b) is true.

3. (a) The event $A \cap B$ occurs if the upface at the first toss is "3", and in the next 3 tosses each even number is obtained exactly once. Hence

$$P(A \cap B) = \frac{3!}{6^4} = \frac{1}{6^3},$$

and therefore:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/6^3}{1/6} = \frac{1}{6^2} .$$

(b) Clearly, $P(\bar{B}) = 5/6$.

(c) We have $A = A_1 \cup A_2$, where A_1 is the event whereby in one of the first three tosses the upface is odd, and the other 3 out of the 4 first tosses give all even numbers, and A_2 is the event whereby in the course of the first 3 tosses we get only even numbers, but only two distinct numbers, and in the 4-th toss we get the third even number. It is easy to see that $P(A_1) = 3 \cdot 3 \cdot 3!/6^4 = 1/24$ and $P(A_2) = 3 \cdot 3 \cdot 2/6^4 = 1/72$. Since A_1 and A_2 are disjoint, this yields $P(A) = 1/24 + 1/72 = 1/18$. Consequently:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{18} + \frac{1}{6} - \frac{1}{6^3} = \frac{47}{6^3} .$$

(d) A geometric random variable may assume all positive integer values, whereas X is at least 3.

(e) In each of the 10 times we repeat the process, the probability of success (i.e., that the number of stages will be exactly 4) is 1/18. Since the results of these repetitions are independent, $Y \sim B(10, 1/18)$.

Thus, (a) and (e) are true.

4. (a) The probability of winning in the first gamble is $(2/3)^5$ and in the second $(5/6)^{10}$. Since

$$(5/6)^{10} = (25/36)^5 > (2/3)^5,$$

the second gamble yields a better chance of winning.

(b) Let $\Omega = \{1, 2, 3, 4\}$, with each point having a probability of 1/4, and put $A = B = \{1, 2\}$, $C = \{1, 3\}$. Then $P(C|A \cap B) = P(C) = 1/2$, yet $P(A \cap B \cap C) = 1/4$ while $P(A)P(B)P(C) = 1/8$. The events A, B, C are not pairwise independent, and therefore certainly not independent.

(c) Suppose the probability space consists of N points. Then the probability of every point of the space is $1/N$, and that of any event is a/N, where a is the number of points comprising the event. Hence the distribution function of any random variable defined on this space may assume only the values $0, 1/N, 2/N, \ldots, 1$.

(d) Since F_X is monotonic, if it assumes the same value at two points, then it assumes the same value throughout the whole interval between these two points, and thus at infinitely many points.

(e) We have $F_Y(t) = F_X(2t) \geq F_X(t)$ for $t \geq 0$ and $F_Y(t) = F_X(t) = 0$ for $t \leq 0$. Therefore the gamble with total winnings Y is inferior to the gamble with total winnings X.

Thus, (c) and (e) are true.