Probability Theory for EE Students

Solutions to Selected Exercises

1 Probability Spaces

1.

(c) For the event in question to occur, the first $\lceil n/2 \rceil$ tosses may have any outcomes, and then the other $\lfloor n/2 \rfloor$ tosses are uniquely determined. Hence the required probability is $2^{\lceil n/2 \rceil}/2^n = 1/2^{\lfloor n/2 \rfloor}$.

2. 5/16.

5.

(a) $\frac{1}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)} = \frac{2^n n!}{(2n)!}$.

(b) $\frac{n!}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)} = \frac{2^n}{\binom{2n}{n}}$.

6.

(a)

(i) $\left(1 - \frac{1}{n}\right)^{r-1}$.

(ii) $\frac{n(n-1)\ldots(n-r+1)}{n^r}$.

(b)

(i) $\left(1 - \frac{N}{n}\right)^{r-1}$.

(ii) $\frac{(n)_{Nr}}{((n)_N)^r}$, where we have denoted $(x)_r = x(x-1)\ldots(x-r+1)$.

9. $\frac{9\cdot 10^{k-1}}{10^n} = 9/10^{n-k+1}$.

10. Note that

$$\left\{(X^2 - Y^2) \bmod 2 = 0\right\} = \bigcup_{k=0}^{1} \{X \bmod 2 = Y \bmod 2 = k\}$$

and

$$\left\{(X^2 - Y^2) \bmod 3 = 0\right\} = \{X \bmod 3 = Y \bmod 3 = 0\} \bigcup \{X \bmod 3,\ Y \bmod 3 \neq 0\}.$$

Clearly,

$$p_2 = \frac{1}{n^2}\left(\left[\frac{n}{2}\right]^2 + \left(n - \left[\frac{n}{2}\right]\right)^2\right) = 1 - \frac{2}{n}\left[\frac{n}{2}\right] + \frac{2}{n^2}\left[\frac{n}{2}\right]^2,$$

and

$$p_3 = \frac{1}{n^2}\left(\left[\frac{n}{3}\right]^2 + \left(n - \left[\frac{n}{3}\right]\right)^2\right) = 1 - \frac{2}{n}\left[\frac{n}{3}\right] + \frac{2}{n^2}\left[\frac{n}{3}\right]^2.$$

For $n \geq 4$ we have:

$$p_3 - p_2 = \frac{2}{n}\left(\left[\frac{n}{2}\right] - \left[\frac{n}{3}\right]\right) - \frac{2}{n^2}\left(\left[\frac{n}{2}\right]^2 - \left[\frac{n}{3}\right]^2\right) = \frac{2}{n}\left(\left[\frac{n}{2}\right] - \left[\frac{n}{3}\right]\right)\left(1 - \frac{1}{n}\left(\left[\frac{n}{3}\right] + \left[\frac{n}{2}\right]\right)\right) > 0.$$

11. The sets A_1 and A_2 may be chosen in $2^n \cdot 2^n = 4^n$ ways altogether. To satisfy the condition $A_1 \bigcap A_2 = \emptyset$, we have to require that each $j \in \{1, 2, \ldots, n\}$ belongs to at most one of the sets A_1 and A_2. Thus we have 3 possibilities for each j, namely either $j \in A_1 \bigcap \overline{A}_2$ or $j \in \overline{A}_1 \bigcap A_2$ or $j \in \overline{A}_1 \bigcap \overline{A}_2$. Hence the number of possibilities satisfying the requirement is 3^n. It follows that the probability of the event in question is $(3/4)^n$.

13. Due to symmetry, all $3! = 6$ possible orderings of $X_1,\ X_2$ and X_3 are equi-probable, whence each has probability $1/6$.

14.

(a) $2/3$

(b) $\frac{1}{2^{k+1}}$

(c) $\frac{1}{2^k}$

15.

(a) $\frac{15}{16}$

(b) $\frac{2}{3}$

21.

(a) Since all the events in the union are disjoint, the probability is the sum of probabilities. Consequently:

$$P\left(\bigcup_{i=1}^{\infty}\left[\frac{1}{2i+1},\frac{1}{2i}\right]\right)=\sum_{i=1}^{\infty}\left(\frac{1}{2i}-\frac{1}{2i+1}\right)=1-\ln 2$$

.

(b) For any n, the set in question is contained in the set of numbers whose infinite decimal expansion does not contain the digit 7 in any of the first n places. The latter set is clearly of probability $(9/10)^n$. Thus the probability of our set is less than $(9/10)^n$ for each n, and therefore it vanishes.

(c) As in the preceding part, the probability is 0.

2 Basic Combinatorics

24. There are two points to consider: how many arrangements are there for 5 digits and 5 letters, and how many choices of digits and letters are there for any arrangement. Consider the second point: there are 10 different digits, and repetitions are allowed, so there are 10^5 possibilities. There are 26 letters, so altogether there are $10^5 \cdot 26^5$ possibilities. The only difference between the parts of the question is the number of arrangements. Denote a place for a digit by d, and a place for a letter by l.

(a) The single arrangement is *dddddlllll*. The number of possibilities is $10^5 \cdot 26^5$.

(b) In general the digits and letters should alternate. However, a single pair of adjacent digits is still possible. Observe that each letter (except perhaps for the last) is followed by a digit, so we have five objects: four ld pairs, and a single l. The remaining d may be anywhere in between, or to the left of, or to the right of these five objects – altogether 6 possibilities.

ldldldldld *ldldldlddl* *ldldlddldl*
ldlddldldl *lddldldldl* *dldldldldl* .

The overall number is $6 \cdot 10^5 \cdot 26^5$.

(c) The number of arrangements is $\binom{10}{5}$, so the overall number is $\binom{10}{5} \cdot 10^5 \cdot 26^5$.

26. k^n.

27.

(a) r^n.

(b) The first letter of the word may be any of the letters in Σ. In each of the other $n-1$ places we may put any of the $r-1$ letters distinct from the one in the preceding place. Hence there are $r(r-1)^{n-1}$ possibilities in all.

(c) $\binom{n}{n_1, n_2, \ldots, n_r}$.

(d) $r^{\left[\frac{n+1}{2}\right]}$.

28. Since $n! = e^{\sum_{i=1}^{n} \ln i}$ the inequality

$$e\left(\frac{n}{e}\right)^n \leq n! \leq e\left(\frac{n+1}{e}\right)^{n+1}$$

is equivalent to

$$n \ln n - n + 1 \leq \sum_{i=1}^{n} \ln i \leq (n+1)\ln(n+1) - n.$$

Since $\int \ln x dx = x \ln x - x + c$ and the function $\ln x$ is increasing, we have

$$\int_1^n \ln x dx \leq \sum_{i=1}^{n} \ln i \leq \int_1^{n+1} \ln x dx,$$

which gives the required result.

29.

(a)

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} \approx \frac{\sqrt{2\pi 2n}\left(\frac{2n}{e}\right)^{2n}}{\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)^2} = \frac{2^{2n}}{\sqrt{\pi n}}.$$

(b) Consider the $2n$th row of Pascal's triangle. The sum of all entries is 2^{2n}, and therefore each of them, in particular the middle entry $\binom{2n}{n}$, is less than 2^{2n}. On the other hand, it is easy to check that the binomial coefficients $\binom{2n}{j}$ increase as j increases from 0 to n, and decrease from that place on. In particular, $\binom{2n}{n}$ is the maximal entry in the row. Consequently:

$$\frac{2^{2n}}{2n+1} \leq \binom{2n}{n} \leq 2^{2n}.$$

35. The first row may be any of the 2^n vectors of length n, consisting of 0's and 1's. After it has been chosen, we have $2^n - 1$

possibilities for choosing the second row, then $2^n - 2$ possibilities for choosing the third row, and so forth. Altogether, the matrix may be chosen in

$$2^n\,(2^n - 1) \cdot \ldots \cdot (2^n - m + 1)$$

different ways.

3 Elementary Probability Calculations

42. The number of possibilities for choosing the cards is $\binom{52}{13}$ (order does not matter). This constitutes the denominator for all parts.

(a) There are 4 possible full hands, so the probability is $4/\binom{52}{13}$.

(b) All the 13 cards should be chosen from the 48 non-ace cards: $\binom{48}{13}/\binom{52}{13} = \frac{39\cdot 38\cdot 37\cdot 36}{52\cdot 51\cdot 50\cdot 49}$.

(c) There are $\binom{4}{1}$ possibilities for choosing one of the four kings, and the same for choosing one of the four queens. The remaining 11 cards should be chosen from the remaining 40 cards (52 in the deck, excluding the ace, king and queen cards). As the total number of possibilities for choosing the cards is $\binom{52}{13}$, the required probability is:

$$\frac{\binom{4}{1}\binom{4}{1}\binom{52-4-4-4}{11}}{\binom{52}{13}} = \frac{\binom{4}{1}^2\binom{40}{11}}{\binom{52}{13}}.$$

(d) There are $\binom{4}{1}$ ways to choose each card, and hence the probability is $\binom{4}{1}^{13}/\binom{52}{13}$.

46.

(a) We have

$$\ln(1-x) = -x - \frac{x^2}{2} + O(x^3)),$$

and therefore

$$-x - x^2 \le \ln(1-x) \le -x$$

in a sufficiently small neighborhood of 0. The required result follows by exponentiation.

(b) If $\alpha_n \not\to 0$, then clearly $\prod_{n=1}^{\infty}(1-\alpha_n) = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Suppose therefore that $\alpha_n \to 0$. Passing to logarithms, we see that $\prod_{n=1}^{\infty}(1-\alpha_n) = 0$ if and only if $\sum_{n=1}^{\infty}\ln(1-\alpha_n) = -\infty$. By part (a), for sufficiently large n we have

$$-2\alpha_n \le -\alpha_n - \alpha_n^2 \le \ln(1-\alpha_n) \le -\alpha_n.$$

Hence the series $\sum_{n=1}^{\infty} - \ln(1-\alpha_n)$ and $\sum_{n=1}^{\infty} \alpha_n$ diverge to ∞ together.

47.

(a) The required probability is the ratio between the number of those choices for which all variables assume distinct values and the number of all choices, that is $\frac{n(n-1)\cdot\ldots\cdot(n-k+1)}{n^k}$

(b) Write:

$$\frac{n(n-1)\cdot\ldots\cdot(n-k+1)}{n^k} = \prod_{i=1}^{k-1}\left(1-\frac{i}{n}\right).$$

By the preceding exercise:

$$e^{-x-x^2} \le 1-x \le e^{-x}.$$

Thus we have

$$\prod_{i=1}^{k-1} e^{-\frac{i}{n}-\left(\frac{i}{n}\right)^2} \le \prod_{i=1}^{k-1}\left(1-\frac{i}{n}\right) \le \prod_{i=1}^{k-1} e^{-\frac{i}{n}}.$$

Now

$$\begin{aligned}\sum_{i=1}^{k-1}\left(\frac{i}{n}+\left(\frac{i}{n}\right)^2\right) &= \frac{k(k-1)}{2n}+\frac{k(k-1)(2k-1)}{6n^2}\\ &= \frac{(3n+2k-1)k(k-1)}{6n^2}.\end{aligned}$$

Thus

$$\prod_{i=1}^{k-1} e^{-\frac{i}{n}-\left(\frac{i}{n}\right)^2} = e^{-\frac{(3n+2k-1)k(k-1)}{6n^2}}.$$

Hence as $n \longrightarrow \infty$ and $\frac{k}{\sqrt{n}} \longrightarrow \theta$, the power in the exponent tends to

$$\frac{\sqrt{n}\theta(\sqrt{n}\theta-1)}{2n}+\frac{\sqrt{n}\theta(\sqrt{n}\theta-1)(2\sqrt{n}\theta-1)}{6n^2} \longrightarrow \frac{\theta^2}{2}.$$

The right-hand side of the inequality may be evaluated in a similar way, so the limit is $e^{-\frac{\theta^2}{2}}$.

49.

(a) Let us show that:

$$\limsup_{n\to\infty} A_n = [0,2], \qquad \liminf_{n\to\infty} A_n = [1/2,1].$$

Indeed, if $x \in [0,1]$, then $x \in A_n$ for each even n, while if $x \in [1,2]$, then $x \in A_n$ for each odd n, so that $\limsup_{n\to\infty} A_n \supseteq [0,2]$. On the other hand, if $x < 0$, then $x \notin A_n$ for any n, while if $x > 2$ then $x \notin A_n$ for $n > \frac{1}{x-2}$. This gives the inverse inclusion $\limsup_{n\to\infty} A_n \subseteq [0,2]$.
If $x \in [1/2,1]$, then $x \in A_n$ for each n, and in particular $\liminf_{n\to\infty} \supseteq [1/2,1]$. If $x < 1/2$, then $x \notin A_n$ for any odd n, while if $x > 1$ then $x \notin A_n$ for odd $n > \frac{1}{x-1}$. Therefore $\liminf_{n\to\infty} \subseteq [1/2,1]$.

(b) A point belongs to $\limsup_{n\to\infty} A_n$ if it belongs to infinitely many of the events A_n, which happens if and only if it belongs to the union $\cup_{i=k}^{\infty} A_i$ for each k. It follows that

$$\limsup_{n\to\infty} A_n = \cap_{k=1}^{\infty} \cup_{i=k}^{\infty} A_i,$$

which representation proves that $\limsup_{n\to\infty} A_n$ is an event. Similarly

$$\liminf_{n\to\infty} A_n = \cup_{k=1}^{\infty} \cap_{i=k}^{\infty} A_i,$$

which proves that $\liminf_{n\to\infty} A_n$ is an event.

50.

(a) The number of all subsets of A is of size 2^n. Thus, equivalently, we have to calculate the sum of those binomial coefficients $\binom{n}{k}$ with even k. Since the expression $\frac{1+(-1)^k}{2}$ takes the value 1 for even k and vanishes for odd k, we have:

$$\begin{aligned}\sum_{2|k} \binom{n}{k} &= \sum_{k=0}^{n} \frac{1+(-1)^k}{2} \binom{n}{k} \\ &= \frac{1}{2}\sum_{k=0}^{n} \binom{n}{k} + \frac{1}{2}\sum_{k=0}^{n} (-1)^k \binom{n}{k} \\ &= \frac{1}{2} \cdot 2^n + \frac{1}{2} \cdot (1-1)^n = 2^{n-1}.\end{aligned}$$

Consequently the required probability is $\frac{1}{2}$.

(b) A simple calculation yields:

$$1 + \omega^k + \omega^{2k} = \begin{cases} 3, & 3|k, \\ 0, & 3 \nmid k. \end{cases}$$

Consequently:

$$\begin{aligned}\sum_{3|k}\binom{n}{k} &= \sum_{k=0}^{n}\frac{1+\omega^k+\omega^{2k}}{3}\binom{n}{k}\\ &= \frac{1}{3}\left[2^n+(1+\omega)^n+(1+\omega^2)^n\right]\\ &= \frac{2^n+(-\omega^2)^n+(-\omega)^n}{3}\ .\end{aligned}$$

Hence the probability for $|R|$ to be divisible by 3 is

$$\frac{1}{3}\left[1+\frac{(-\omega^2)^n+(-\omega)^n}{2^n}\right]\ .$$

Similarly, to find the probability for $|R|$ to be 1 modulo 3, we calculate:

$$1+\omega^2\omega^k+\omega\omega^{2k}=\begin{cases}3, & k\equiv 1 (\operatorname{mod} 3),\\ 0, & \text{otherwise.}\end{cases}$$

Hence

$$\begin{aligned}\sum_{k\equiv 1(\operatorname{mod} 3)}\binom{n}{k} &= \sum_{k=0}^{n}\frac{1+\omega^2\omega^k+\omega\omega^{2k}}{3}\binom{n}{k}\\ &= \frac{1}{3}\left[2^n+\omega^2(1+\omega)^n+\omega(1+\omega^2)^n\right]\\ &= \frac{2^n+\omega^2(-\omega^2)^n+\omega(-\omega)^n}{3}\ ,\end{aligned}$$

and the probability is $\frac{1}{3}\left[1-\frac{(-\omega^2)^{n+1}+(-\omega)^{n+1}}{2^n}\right]$.

(c)

$$P(|R|\equiv i(\text{mod }\ 4))=\begin{cases}\dfrac{2^n+(1+i)^n+(1-i)^n}{2^{n+2}}, & i=0,\\[2ex] \dfrac{2^n-i(1+i)^n+i(1-i)^n}{2^{n+2}}, & i=1,\\[2ex] \dfrac{2^n-(1+i)^n-(1-i)^n}{2^{n+2}}, & i=2,\\[2ex] \dfrac{2^n+i(1+i)^n-i(1-i)^n}{2^{n+2}}, & i=3.\end{cases}$$

4 Conditional Probability

53.

(a) The required probability is $P(B \mid A) = P(B \cap A)/P(A)$, where

$$P(A) = \frac{1^n + 2^n + \ldots + N^n}{N^n(N+1)}$$

and

$$P(B \cap A) = \frac{1^{n+1} + 2^{n+1} + \ldots + N^{n+1}}{N^{n+1}(N+1)}.$$

(b) The quantity

$$P(A) = \sum_{k=0}^{N} \frac{1}{N+1} \cdot \left(\frac{k}{N}\right)^n = \frac{1^n + 2^n + \ldots + N^n}{N^n(N+1)}$$

is a Darboux sum of the function x^n in the interval $(0,1)$, and therefore it converges to the integral $\int_0^1 x^n dx = \dfrac{1}{n+1}$ as $N \to \infty$. Similarly, the expression $\frac{1^{n+1}+2^{n+1}+\ldots+N^{n+1}}{N^{n+1}(N+1)}$ converges to $\int_0^1 x^{n+1} dx = \dfrac{1}{n+2}$ as $N \to \infty$. Thus, the limit of the quotient as $N \to \infty$ is approximately $\frac{n+1}{n+2}$.

57.

(a) Let A_n be the event that the n-th shot hits the target. We have $P(A_n) = \dfrac{1}{(2n)^2}$. The trials are independent, and thus the probability of not passing the test is

$$P\left(\bigcap_{n=1}^{\infty} \overline{A_n}\right) = \prod_{n=1}^{\infty} \left(1 - \frac{1}{(2n)^2}\right) = \prod_{n=1}^{\infty} \frac{4n^2-1}{4n^2}.$$

Now:

$$\prod_{n} \frac{4n^2-1}{4n^2} = \frac{3}{4} \cdot \frac{15}{16} \cdot \frac{35}{36} \cdot \ldots = \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{3 \cdot 5}{4 \cdot 4} \cdot \frac{5 \cdot 7}{6 \cdot 6} \cdot \ldots,$$

and according to the hint in the question:

$$P\left(\bigcap_{n=1}^{\infty} \overline{A_n}\right) = \frac{2}{\pi}.$$

(b) Let A_n be defined analogously to the previous part. We have $P(A_n) = \frac{n}{n+1}$. The trials are independent, and thus the probability of passing the first k trials is

$$P\left(\bigcap_{n=1}^{k} A_n\right) = \prod_{n=1}^{k} \frac{n}{n+1} = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{k}{k+1} = \frac{1}{k+1}.$$

The probability of passing the test is obtained by taking the limit as $k \to \infty$:

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{k\to\infty} P\left(\bigcap_{n=1}^{k} A_n\right) = 0.$$

58.

(a) Let A_n be the event that the first round has ended after k tosses. There are 6^3 possible outcomes in each toss of the dice, 6 of which will yield the same result. Thus

$$P(A_n) = \left(\frac{35}{36}\right)^{n-1} \cdot \frac{1}{36}.$$

Let B_k be the event that k distinct results were obtained in the second round. In each toss of the second round, the probability of all three receiving distinct results is $1 \cdot \frac{5}{6} \cdot \frac{4}{6} = \frac{5}{9}$. Hence:

$$P\left(B_k | A_n\right) = \binom{n}{k} \left(\frac{5}{9}\right)^{k} \left(\frac{4}{9}\right)^{n-k}.$$

For $k > 0$:

$$P\left(B_k\right) = \sum_{n=k}^{\infty} P\left(A_n\right) P\left(B_k | A_n\right) = \sum_{n=k}^{\infty} \left(\frac{35}{36}\right)^{n-1} \cdot \frac{1}{36} \cdot \binom{n}{k} \left(\frac{5}{9}\right)^{k} \left(\frac{4}{9}\right)^{n-k}$$

$$= \frac{1}{36}\left(\frac{5}{9}\right)^{k}\left(\frac{9}{4}\right)^{k} \cdot \frac{36}{35} \sum_{n=k}^{\infty} \binom{n}{k} \left(\frac{4}{9}\right)^{n} \left(\frac{35}{36}\right)^{n}$$

$$= \frac{1}{35} \cdot \left(\frac{5}{4}\right)^{k} \sum_{n=k}^{\infty} \binom{n}{k} \left(\frac{35}{81}\right)^{n} = \frac{1}{35} \cdot \left(\frac{5}{4}\right)^{k} \frac{\left(\frac{35}{81}\right)^{k}}{\left(1 - \frac{35}{81}\right)^{k+1}}$$

$$= \frac{1}{35}\left(\frac{5}{4}\right)^{k} \frac{81 \cdot 35^k}{46^{k+1}} = \frac{81}{35 \cdot 46}\left(\frac{175}{176}\right)^{k}.$$

For $k=0$:

$$P(B_0)=\sum_{n=1}^{\infty}P(A_n)\cdot\left(\frac{4}{9}\right)^n=\sum_{n=1}^{\infty}\left(\frac{35}{36}\right)^{n-1}\frac{1}{36}\cdot\left(\frac{4}{9}\right)^n$$

$$=\frac{1}{36}\cdot\frac{4}{9}\sum_{\ell=0}^{\infty}\left(\frac{4}{9}\right)^{\ell}\left(\frac{35}{36}\right)^{\ell}=\frac{1}{81}\cdot\frac{1}{1-\dfrac{35}{81}}=\frac{1}{46}.$$

(b) For $k>0$ the required probability is:

$$P(A_n|B_k)=\frac{P(A_n\cap B_k)}{P(B_k)}=\frac{P(B_k|A_n)\,P(A_n)}{P(B_k)}$$

$$=\binom{n}{k}\frac{\left(\dfrac{5}{9}\right)^k\left(\dfrac{4}{9}\right)^{n-k}\left(\dfrac{35}{36}\right)^{n-1}\dfrac{1}{36}}{\dfrac{81}{35\cdot 46}\left(\dfrac{175}{176}\right)^k}$$

$$=\binom{n}{k}\frac{\left(\dfrac{5}{4}\right)^k\left(\dfrac{4}{9}\right)^{n}\left(\dfrac{35}{36}\right)^{n}\dfrac{1}{35}}{\dfrac{81}{35\cdot 46}\left(\dfrac{5\cdot 35}{4\cdot 46}\right)^k}$$

$$=\binom{n}{k}\frac{\left(\dfrac{35}{81}\right)^n}{\dfrac{81}{46}\left(\dfrac{35}{46}\right)^k}=\binom{n}{k}\frac{35^{n-k}\cdot 46^{k+1}}{81^{n+1}}.$$

For $k=0$:

$$P(A_n|B_0)=\frac{P(B_0|A_n)\,P(A_n)}{P(B_0)}=\frac{\left(\dfrac{4}{9}\right)^n\cdot\left(\dfrac{35}{36}\right)^{n-1}\dfrac{1}{36}}{\dfrac{1}{46}}$$

$$=\left(\frac{4}{9}\right)^n\cdot\left(\frac{35}{36}\right)^{n-1}\cdot\frac{23}{18}=\left(\frac{35}{81}\right)^{n-1}\cdot\frac{46}{81}.$$

59.

(a) $P(X_2 > 0) = 1 - P(X_2 = 0)$. We have

$$\begin{aligned} P(X_2 = 0) &= P(X_1 = 0) + P(X_1 = 1)\, P(X_2 = 0 | X_1 = 1) \\ &\quad + P(X_1 = 2)\, P(X_2 = 0 | X_1 = 2) \\ &= \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{16} = \frac{25}{64}. \end{aligned}$$

Thus

$$P(X_2 > 0) = 1 - P(X_2 = 0) = \frac{39}{64}.$$

One may also find the probability directly:

$$\begin{aligned} P(X_2 > 0) &= P(X_1 = 1)\, P(X_2 > 0 | X_1 = 1) \\ &\quad + P(X_1 = 2)\, P(X_2 > 0 | X_1 = 2) \\ &= \frac{1}{2} \cdot \frac{3}{4} + \frac{1}{4}\left(1 - \frac{1}{16}\right) = \frac{39}{64}. \end{aligned}$$

(b) We have:

$$\begin{aligned} P(X_1 = 2 | X_2 = 1) &= \frac{P(X_1 = 2 \cap X_2 = 1)}{P(X_2 = 1)} \\ &= \frac{P(X_1 = 2)\, P(X_2 = 1 | X_1 = 2)}{P(X_2 = 1)}. \end{aligned}$$

First consider $P(X_2 = 1)$:

$$\begin{aligned} P(X_2 = 1) &= P(X_2 = 1 | X_1 = 0)\, P(X_1 = 0) \\ &\quad + P(X_2 = 1 | X_1 = 1)\, P(X_1 = 1) \\ &\quad + P(X_2 = 1 | X_1 = 2)\, P(X_1 = 2) \\ &= 0 + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{4} = \frac{5}{16}. \end{aligned}$$

Thus

$$P(X_1 = 2 | X_2 = 1) = \frac{\frac{1}{4} \cdot \frac{1}{4}}{\frac{5}{16}} = \frac{1}{5}.$$

(c) We have:

$$\begin{aligned} P(X_1 = 1 | X_3 > 0) &= \frac{P(X_1 = 1 \cap X_3 > 0)}{P(X_3 > 0)} \\ &= \frac{P(X_1 = 1)\, P(X_3 > 0 | X_1 = 1)}{P(X_3 > 0)}. \end{aligned}$$

First consider $P(X_3 > 0)$:

$$\begin{aligned} P(X_2 = 1) &= P(X_3 > 0 | X_1 = 0)\, P(X_1 = 0) \\ &\quad + P(X_3 > 0 | X_1 = 1)\, P(X_1 = 1) \\ &\quad + P(X_3 > 0 | X_1 = 2)\, P(X_1 = 2) \\ &= 0 + \frac{1}{2} \cdot P(X_2 > 0) + \frac{1}{4} \cdot (1 - P(X_3 = 0 | X_1 = 2)). \end{aligned}$$

The reproduction of parent particles is independent. Thus, the probability of having 2 parent particles at stage one and 0 at stage three is the same as the probability of having 0 particles at stage two in two identical experiments, starting with one parent particle in each. Thus,

$$P\left(X_3 = 0 | X_1 = 2\right) = P^2\left(X_2 = 0\right) = \left(\frac{25}{4^3}\right) = \frac{625}{4^6}.$$

Altogether

$$P\left(X_1 = 1 | X_3 > 0\right) = \frac{\frac{1}{2} \cdot \frac{39}{64}}{\frac{1}{2} \cdot \frac{39}{64} + \frac{1}{4} \cdot \left(1 - \frac{625}{4^6}\right)} = \frac{128}{217}.$$

66.

(f) By the principle of inclusion and exclusion:

$$1 \geq P(A \cup B) = P\left(A\right) + P\left(B\right) - P\left(B \cap A\right) = a + b - P\left(B \cap A\right).$$

(j) As $P\left(B \cap A\right) \geq 0$

$$\begin{aligned} P(\overline{A} \cap \overline{B}) &= 1 - P(A \cup B) = 1 - \left(P\left(A\right) + P\left(B\right) - P\left(B \cap A\right)\right) \\ &= 1 - a - b + P\left(B \cap A\right) \geq 1 - a - b. \end{aligned}$$

68. Let E_A be the event that A is telling the truth, and E_{DCBA} the event that D says that C says that B says that A is telling the truth. Let E_{DCB} be the event that D says that C says that B is telling the truth. The probability in question is

$$P\left(E_A | E_{DCBA}\right) = \frac{P\left(E_A \cap E_{DCBA}\right)}{P\left(E_{DCBA}\right)} = \frac{P\left(E_A \cap E_{DCB}\right)}{P\left(E_{DCBA}\right)}.$$

Now E_{DCBA} occurs if either all four tell the truth, or all four do not tell the truth, or exactly two of the people tell the truth. In the last of the three cases, there are $\binom{4}{2}$ possibilities to choose the two people telling the truth. Thus

$$P\left(E_{DCBA}\right) = \left(\frac{1}{3}\right)^4 + \left(\frac{2}{3}\right)^4 + \binom{4}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2 = \frac{41}{81}.$$

Similarly, E_{DCB} occurs if either all three tell the truth, or exactly one of them does. Thus

$$P\left(E_{DCB}\right) = \left(\frac{1}{3}\right)^3 + \binom{3}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^2 = \frac{13}{27}.$$

Altogether,

$$P\left(E_A | E_{DCBA}\right) = \frac{\frac{1}{3} \cdot \frac{13}{27}}{\frac{41}{81}} = \frac{13}{41}.$$

5 Discrete Distributions

72.

(b) Let X_i be the number of the ball drawn in the i draw, for $i = 1, \ldots, 5$. Let $X = \min\{X_1, \ldots, X_5\}$. For $k = 1, \ldots, 9$:

$$P(X = k) = P(X \leq k) - P(X \leq k - 1).$$

In the case the balls are chosen with replacement.

$$\begin{aligned} P(X \leq k) &= 1 - P(X > k) = 1 - P(X_1 > k) \cdot \ldots \cdot P(X_5 > k) \\ &= 1 - (1 - P(X_1 \leq k)) \cdot \ldots \cdot (1 - P(X_5 \leq k)) \\ &= 1 - \left(1 - \frac{k}{13}\right) \cdot \ldots \cdot \left(1 - \frac{k}{13}\right) = 1 - \left(\frac{13-k}{13}\right)^5 . \end{aligned}$$

Thus

$$\begin{aligned} P(X = k) = P(X \leq k) - P(X \leq k-1). \\ &= 1 - \left(\frac{13-k}{13}\right)^5 - \left(1 - \left(\frac{13-(k-1)}{13}\right)^5\right) \\ &= \left(\frac{14-k}{13}\right)^5 - \left(\frac{13-k}{13}\right)^5 . \end{aligned}$$

On the other hand, if the balls are chosen without replacement. Let $1 \leq k \leq 9$. The event $X = k$ occurs if the number on one of the five balls drawn is k, and the numbers on all the other four balls is larger than k, for which there are $\binom{13-k}{4}$ possibilities. As there are $\binom{13}{5}$ possibilities for choosing the 5 numbers

$$P(X = k) = \frac{\binom{13-k}{4}}{\binom{13}{5}}.$$

76.

(c) For any $c > 0$ the values assumed by $p(x)$ are non-negative. The value of c is determined by the requirement that their sum be 1. First let us decompose the given rational function. Namely, we are looking for constants a, b and d for which:

$$\frac{1}{x(x+1)(x+2)} = \frac{a}{x} + \frac{b}{x+1} + \frac{d}{x+2} .$$

This gives:

$$a(x+1)(x+2)+bx(x+2)+dx(x+1)=1.$$

Making the substitutions $x=0$, $x=-1$ and $x=-2$ we obtain:

$$2a=1, \qquad -b=1, \qquad 2d=1,$$

and therefore

$$a=\frac{1}{2}, \qquad b=-1, \qquad d=\frac{1}{2}\,.$$

Hence:

$$\sum_{x=1}^{\infty}\frac{1}{x(x+1)(x+2)}=\sum_{x=1}^{\infty}\left(\frac{1/2}{x}-\frac{1}{x+1}+\frac{1/2}{x+2}\right)=\frac{1/2}{1}-\frac{1}{2}+\frac{1/2}{2}=\frac{1}{4}\,.$$

Thus $c=4$.

(d) We have:

$$1=\sum_{x=1}^{\infty}\frac{c}{x(x+1)(x+3)}=c\sum_{x=1}^{\infty}\left(\frac{\frac{1}{3}}{x}-\frac{\frac{1}{2}}{x+1}+\frac{\frac{1}{6}}{x+3}\right)=$$
$$=c\left(\frac{1}{3}(1+\frac{1}{2}+\frac{1}{3})-\frac{1}{2}(\frac{1}{2}+\frac{1}{3})+0\right)=c\left(\frac{1}{3}\cdot\frac{11}{6}-\frac{1}{2}\cdot\frac{5}{6}\right)=c\cdot\frac{7}{36}.$$

Thus:

$$c=\frac{36}{7}.$$

For $k=1,2,\ldots$ the distribution function is given by

$$\begin{aligned}
F(k) &= P(X\le k)=\frac{36}{7}\sum_{x\le k}\frac{c}{x(x+1)(x+3)}\\
&= \frac{36}{7}\sum_{x\le k}(\frac{\frac{1}{3}}{x}-\frac{\frac{1}{2}}{x+1}+\frac{\frac{1}{6}}{x+3})\\
&= \frac{36}{7}\left(\frac{1}{3}+\frac{1}{6}+\frac{1}{9}-\frac{1}{4}-\frac{1}{6}+\sum_{x\le k}(\frac{\frac{1}{3}}{x+3}-\frac{\frac{1}{2}}{x+3}+\frac{\frac{1}{6}}{x+3})\right.\\
&\qquad \left.-\frac{\frac{1}{3}}{k+1}-\frac{\frac{1}{3}}{k+2}-\frac{\frac{1}{3}}{k+3}+\frac{\frac{1}{2}}{k+2}+\frac{\frac{1}{2}}{k+3}\right)\\
&= \frac{36}{7}\left(\frac{7}{36}-\frac{\frac{1}{3}}{k+1}+\frac{\frac{1}{6}}{k+2}+\frac{\frac{1}{6}}{k+3}\right)=1-\frac{6}{7}\left(\frac{2}{k+1}+\frac{1}{k+2}+\frac{1}{k+3}\right).
\end{aligned}$$

Thus for $t\in\mathbf{R}$:

$$F(t)=\begin{cases}1-\frac{6}{7}\left(\frac{2}{\lfloor t\rfloor+1}+\frac{1}{\lfloor t\rfloor+2}+\frac{1}{\lfloor t\rfloor+3}\right), & t\ge 1,\\ 0, & \text{otherwise.}\end{cases}$$

(e) We have:

$$1=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{c}{2^k3^l}=c\sum_{k=0}^{\infty}\frac{1}{2^k}\sum_{l=0}^{\infty}\frac{1}{3^l}=c\cdot\frac{1}{1-\frac{1}{2}}\cdot\frac{1}{1-\frac{1}{3}}=c\cdot 2\cdot\frac{3}{2}=3c.$$

Thus:

$$c=\frac{1}{3}.$$

Let $S=\left\{x=2^k3^l:\ k,l=0,1,\ldots\right\}$. For $k=1,2,\ldots$ the distribution function is given by

$$\begin{aligned}
F(k) &= P(X\le k)=\frac{1}{3}\sum_{x\in S,\,x\le k}\frac{1}{x}=\frac{1}{3}\sum_{j=0}^{\lfloor \lg_2(k)\rfloor}\sum_{i=0}^{\left\lfloor \lg_3(\frac{k}{2^j})\right\rfloor}\frac{1}{2^j}\frac{1}{3^i}\\
&= \frac{1}{3}\sum_{j=0}^{\lfloor \lg_2(k)\rfloor}\frac{1}{2^j}\cdot\frac{1-\left(\frac{1}{3}\right)^{1+\left\lfloor \lg_3(\frac{k}{2^j})\right\rfloor}}{1-\frac{1}{3}}=\frac{1}{3}\sum_{j=0}^{\lfloor \lg_2(k)\rfloor}\frac{1}{2^j}\cdot\frac{3}{2}\cdot\left(1-\left(\frac{1}{3}\right)^{1+\left\lfloor \lg_3(\frac{k}{2^j})\right\rfloor}\right)\\
&= \frac{1}{2}\sum_{j=0}^{\lfloor \lg_2(k)\rfloor}\frac{1}{2^j}-\frac{1}{2}\sum_{j=0}^{\lfloor \lg_2(k)\rfloor}\left(\frac{1}{3}\right)^{1+\left\lfloor \lg_3(\frac{k}{2^j})\right\rfloor}\\
&= 1-\left(\frac{1}{2}\right)^{1+\lfloor \lg_2(k)\rfloor}-\frac{1}{2}\sum_{j=0}^{\lfloor \lg_2(k)\rfloor}\left(\frac{1}{3}\right)^{1+\left\lfloor \lg_3(\frac{k}{2^j})\right\rfloor}.
\end{aligned}$$

Thus for $t\in\mathbf{R}$:

$$F(t)=\begin{cases}1-\left(\frac{1}{2}\right)^{1+\lfloor \lg_2(k)\rfloor}-\frac{1}{2}\sum_{j=0}^{\lfloor \lg_2(\lfloor t\rfloor)\rfloor}\left(\frac{1}{3}\right)^{1+\left\lfloor \lg_3(\frac{\lfloor t\rfloor}{2^j})\right\rfloor}, & t\ge 1,\\ 0, & \text{otherwise.}\end{cases}$$

77.

$$F_X(x)=P(X\le x)=\sum_{r_k\le x}\frac{1}{2^k}$$

At all the rational points, $F_X(x)$ is discontinuous. Indeed, let $x=r_k$. Then for every $\delta>0$

$$F_X(r_k)-F_X(r_k-\delta)\ge P(X=r_k)=\frac{1}{2^k}.$$

On the other hand, the function is continuous at each irrational point x_0. Being a distribution function, it is continuous from the right. Thus, given any $\varepsilon>0$, we need to show that there exists some

$\delta > 0$ such that, if $x \in (x_0 - \delta, x_0)$, then $|F_X(x) - F_X(x_0)| < \varepsilon$. In fact there is only a finite number of k's such that $\frac{1}{2^k} \geq \varepsilon$. Denote the largest such k by K. For $0 < \delta < \min\{|x_0 - r_j| : 1 \leq j \leq K+1\}$, and for every $x \in (x_0 - \delta, x_0)$ we have:

$$|F_X(x) - F_X(x_0)| = P(\{r_j : r_j \in (x, x_0]\}) \leq \sum_{j>K+1} \frac{1}{2^j} = \frac{1}{2^{K+1}} < \varepsilon.$$

6 Expectation

84.

(c) We have

$$\begin{aligned}
E(X^2) &= \sum_{k=0}^{n} \frac{\binom{a}{k}\binom{b}{n-k}}{\binom{a+b}{n}} \cdot k^2 = a \sum_{k=1}^{n} \frac{\binom{a-1}{k-1}\binom{b}{n-k}}{\frac{a+b}{n} \cdot \binom{a+b-1}{n-1}} \cdot (k-1+1) \\
&= \frac{an}{a+b} \sum_{\ell=0}^{n-1} \frac{\binom{a-1}{\ell}\binom{b}{n-1-\ell}}{\binom{a+b-1}{n-1}} \cdot (\ell+1) \\
&= \frac{an}{a+b} \sum_{\ell=0}^{n-1} \frac{\binom{a-1}{\ell}\binom{b}{n-1-\ell}}{\binom{a+b-1}{n-1}} \cdot \ell + \frac{an}{a+b} \sum_{\ell=0}^{n-1} \frac{\binom{a-1}{\ell}\binom{b}{n-1-\ell}}{\binom{a+b-1}{n-1}}. \\
&= \frac{an}{a+b} E(L) + \frac{an}{a+b} \sum_{\ell=0}^{n-1} P(L = \ell),
\end{aligned}$$

where $L \sim H(n-1, a-1, b)$. Thus

$$\begin{aligned}
E(X^2) &= \frac{an}{a+b} \cdot \frac{(a-1) \cdot (n-1)}{a+b-1} + \frac{an}{a+b} \cdot 1 \qquad (2) \\
&= \frac{an}{a+b} \left(1 + \frac{(a-1) \cdot (n-1)}{a+b-1}\right).
\end{aligned}$$

For $E(X^3)$ we have

$$\begin{aligned}
E(X^3) &= \sum_{k=0}^{n} \frac{\binom{a}{k}\binom{b}{n-k}}{\binom{a+b}{n}} \cdot k^3 = a \sum_{k=1}^{n} \frac{\binom{a-1}{k-1}\binom{b}{n-k}}{\frac{a+b}{n} \cdot \binom{a+b-1}{n-1}} \cdot k^2 \\
&= \frac{an}{a+b} \sum_{\ell=0}^{n-1} \frac{\binom{a-1}{\ell}\binom{b}{n-1-\ell}}{\binom{a+b-1}{n-1}} \cdot (\ell+1)^2 = \frac{an}{a+b} E((L+1)^2).
\end{aligned}$$

Here $L \sim H(n-1, a-1, b)$. By the linearity of the expectation

$$E(X^3) = \frac{an}{a+b} \cdot \left(E(L^2) + 2E(L) + 1\right).$$

By (2)

$$E(L^2) = \frac{(a-1)(n-1)}{a+b-1}\left(1 + \frac{(a-2)\cdot(n-2)}{a+b-2}\right).$$

Thus:

$$\begin{aligned}E(X^3) &= \frac{an}{a+b}\Bigg(\frac{(a-1)(n-1)}{a+b-1}\left(1 + \frac{(a-2)\cdot(n-2)}{a+b-2}\right) \\ &\qquad + 2\frac{(a-1)(n-1)}{a+b-1} + 1\Bigg) \\ &= \frac{an}{a+b}\left(\frac{(a-1)(n-1)}{a+b-1}\left(3 + \frac{(a-2)\cdot(n-2)}{a+b-2}\right) + 1\right).\end{aligned}$$

(e)

$$\begin{aligned}E(X^2) &= \sum_{k=r}^{\infty}\binom{k-1}{r-1}p^r(1-p)^{k-r}\cdot k^2 = \frac{r}{p}\sum_{k=r}^{\infty}\binom{k}{r}p^r(1-p)^{k-r}\cdot k \\ &= \frac{r}{p}\sum_{k'=r+1}^{\infty}\binom{k'-1}{(r+1)-1}p^{r+1}(1-p)^{k'-r(r+1)}\cdot\left(k'-1\right) \\ &= \frac{r}{p}E(R-1),\end{aligned}$$

here $R \sim \overline{B}(r+1,p)$. Thus

$$E(X^2) = \frac{r}{p}\left(\frac{r+1}{p} - 1\right) = \frac{r(r+1-p)}{p^2}.$$

Similarly, for $E(X^3)$ we have

$$\begin{aligned}E(X^3) &= \frac{r}{p}\sum_{k'=r+1}^{\infty}\binom{k'-1}{(r+1)-1}p^{r+1}(1-p)^{k'-r(r+1)}\cdot\left(k'-1\right)^2 \\ &= \frac{r}{p}E\left((R-1)^2\right) = \frac{r}{p}\left(E\left(R^2\right) - 2E(R) + 1\right) \\ &= \frac{r}{p}\left(\frac{(r+1)(r+2-p)}{p^2} - 2\cdot\frac{r+1}{p} + 1\right).\end{aligned}$$

(f) We have

$$\begin{aligned}E(X^2) &= \sum_{k=0}^{\infty}\frac{e^{-\lambda}\lambda^k}{k!}\cdot k^2 = \lambda\sum_{k=1}^{\infty}\frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!}\cdot(k-1+1) \\ &= \lambda\sum_{k=1}^{\infty}\frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!}\cdot(k-1) + \lambda\sum_{k=1}^{\infty}\frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!} \\ &= \lambda\sum_{\ell=0}^{\infty}\frac{e^{-\lambda}\lambda^{\ell}}{\ell!}\cdot\ell + \lambda\sum_{\ell=0}^{\infty}\frac{e^{-\lambda}\lambda^{\ell}}{\ell!} = \lambda^2 + \lambda. \\ &= \lambda\cdot E(X) + \lambda\sum_{\ell=0}^{\infty}P(X=\ell) = \lambda^2 + \lambda.\end{aligned}$$

For $E\left(X^3\right)$:

$$E(X^3) = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} \cdot k^3 = \lambda \sum_{k=1}^{\infty} \frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!} \cdot k^2$$

$$= \lambda \sum_{\ell=0}^{\infty} \frac{e^{-\lambda}\lambda^{\ell}}{\ell!} \cdot (\ell+1)^2 = \lambda E((L+1)^2),$$

where $L \sim P(\lambda)$. By the linearity of the expectation

$$E(X^3) = \lambda \cdot \big(E(L^2) + 2E(L) + 1\big)$$

$$= \lambda \left(\lambda^2 + \lambda + 2\lambda + 1\right) = \lambda \left(\lambda^2 + 3\lambda + 1\right).$$

85.

(b)

$$E(X) = \sum_{k=0}^{n} 2^k \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (2p)^k (1-p)^{n-k}$$

$$= (2p + (1-p))^n = (1+p)^n.$$

(c)

$$E(X) = \textstyle\sum_{k=0}^{n} \sin k \cdot \displaystyle\binom{n}{k} \cdot p^k(1-p)^{n-k} = \textstyle\sum_{k=0}^{n} \displaystyle\frac{e^{ik} - e^{-ik}}{2i} \cdot \binom{n}{k} \cdot p^k(1-p)^{n-k}$$

$$= \frac{1}{2i} \sum_{k=0}^{n} \binom{n}{k} \cdot (pe^{i})^k (1-p)^{n-k} - \frac{1}{2i} \sum_{k=0}^{n} \binom{n}{k} \cdot (pe^{-i})^k (1-p)^{n-k}$$

$$= \frac{1}{2i} \left((pe^{i} + 1 - p)^n - (pe^{-i} + 1 - p)^n\right).$$

88.

(a) Let $A_k,\ 1 \le k \le n$, denote the event whereby k is the largest number in the sample. Then

$$P(A_k) = \frac{k^n - (k-1)^n}{N^n},$$

and therefore

$$E(X) = \frac{1}{N^n} \sum_{k=1}^{N} k(k^n - (k-1)^n).$$

It follows that:

$$E(X) = \frac{1}{N^n} \left(N^{n+1} - \sum_{k=1}^{N-1} k^n \right) = N - \sum_{k=1}^{N-1} \left(\frac{k}{N}\right)^n.$$

(b) Write $E(X)$ in the form:

$$E(X) = N - N \sum_{k=0}^{N-1} \frac{1}{N} \left(\frac{k}{N} \right)^n .$$

The sum on the right-hand side is a Darboux sum corresponding to the integral $\int_0^1 x^n dx$. Hence $E(X)$ behaves asymptotically as $N - \frac{N}{n+1}$.

(c) As n becomes large, all the terms in the sum on the right hand side of the expression for $E(X)$ tend to 0, and therefore $E(X)$ tends to N.

91. The series yielding the expectation is:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^n}{n} \cdot \frac{1}{2^n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}.$$

However, while the series converges, it only converges conditionally. Hence, $E(X)$ does not exist.

94. Denote by D the distance between v_1 and v_2. Since $P(D = 1) = p$:

$$E(D) \geq 1 \cdot P(D = 1) + 2 \cdot P(D \geq 2) = p + 2(1-p) = 2 - p.$$

On the other hand, by the solution of Problem 50 we have

$$P(D \geq 3) \leq (n-2) \cdot \left(1 - p^2\right)^{n-2},$$

and consequently:

$$\begin{aligned} E(D) &\leq 1 \cdot P(D = 1) + 2 \cdot P(D \geq 2) + n \cdot P(D \geq 3) \\ &\leq p + 2(1-p) + n(n-2) \cdot \left(1 - p^2\right)^{n-2} \underset{n \to \infty}{\longrightarrow} 2 - p. \end{aligned}$$

Thus $E(D) \underset{n \to \infty}{\longrightarrow} 2 - p$.

97.

(Algorithm a) Denote by "success" the event that an n-tuple forms a permutation. The probability for this event is $p = \frac{n!}{n^n}$. The number of selections of n-tuples is distributed $G(p)$. Hence the expected number of selections is

$$\frac{1}{p} = \frac{n^n}{n!} .$$

Since each selection consists of n integers, the expected number of random integers by this algorithm is $\frac{n^{n+1}}{n!}$. , which is approximately $\sqrt{\frac{2\pi}{n}} e^n$.

(Algorithm b) Denote by X_i the number of steps required to obtain the i-th digit, $i = 1, \ldots, n$. Clearly, $X_i \sim G(1-(i-1)/n)$, and therefore

$$E(X_i) = \frac{n-i+1}{n}, \qquad i = 1, \ldots, n.$$

The total number of selections is $X = \sum_{i=1}^{n} X_i$, and hence:

$$E(X) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \frac{n-i+1}{n} = n\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right).$$

This time the expected number of selections is only about $n \log n$.

7 Continuous Distributions

99.

(c) The variable assumes values in $(0,1)$, and therefore for $t \in (0,\infty)$ we have:

$$\begin{aligned} F_X(t) = {} & P(-\ln U \leq t) = P\left(U \geq e^{-t}\right) = 1 - P\left(U < e^{-t}\right) \\ & = 1 - F_U\left(e^{-\mathrm{t}}\right) = 1 - e^{-\mathrm{t}}. \end{aligned}$$

Thus:

$$F_X(t) = \begin{cases} 0, & t < 0, \\ 1 - e^{-t}, & t \geq 0. \end{cases} \qquad f_X(t) = \begin{cases} 0, & t < 0, \\ e^{-t}, & t \geq 0. \end{cases}$$

Thus $X \sim \mathrm{Exp}(1)$.

100.

(a) Sufficient conditions on θ_1 and θ_2 are: (i) $\theta_1 + \theta_2 = 1$, (ii) $\theta_1,\ \theta_2 \geq 0$. Suppose θ_1 and θ_2 satisfy conditions (i) and (ii). Clearly $\theta_1 f_1(x) + \theta_2 f_2(x) \geq 0$ for every x, and

$$\begin{aligned} \int_{-\infty}^{\infty} (\theta_1 f_1(x) + \theta_2 f_2(x))dx &= \theta_1 \int_{-\infty}^{\infty} f_1(x)dx + \theta_2 \int_{-\infty}^{\infty} f_2(x)dx \\ &= \theta_1 + \theta_2 = 1. \end{aligned} \tag{4}$$

Hence, $\theta_1 f_1(x) + \theta_2 f_2(x)$ is a density function

(b) Condition (i) is necessary. Suppose $\theta_1 + \theta_2 \neq 1$. Then as in (4)

$$\int_{-\infty}^{\infty} (\theta_1 f_1(x) + \theta_2 f_2(x))dx \neq 1.$$

On the other hand, condition (ii) is not necessary. For example, let $X \sim U(0,1)$ and $Y \sim U(0,2)$. Then $-\frac{1}{2} f_X(x) + \frac{3}{2} f_Y(x)$ is a density function.

(c) Let f_1, f_2 be the density function of the distributions $U(0,1)$ and $U(1,2)$, respectively. If $\theta_1 < 0$ or $\theta_2 < 0$, then $\theta_1 f_1(x) + \theta_2 f_2(x)$ is negative on $(0,1)$ or on $(1,2)$, respectively, and hence is not a density function.

102.

(c)

$$f(x) = c(e^{-2|x|} + e^{-3|x|}), \qquad -\infty < x < \infty.$$

Find c: Let $X \sim \text{Exp}(2), Y \sim \text{Exp}(3)$. We have

$$\begin{aligned} 1 &= c\int_{-\infty}^{\infty} (e^{-2|x|} + e^{-3|x|})dx = 2c\int_{0}^{\infty} (e^{-2x} + e^{-3x})dx \\ &= c\int_{0}^{\infty} 2e^{-2x}dx + \frac{2}{3}c\int_{0}^{\infty} 3e^{-3x}dx \\ &= c\int_{0}^{\infty} f_X(x)dx + \frac{2}{3}c\int_{0}^{\infty} f_Y(x)dx. \\ &= c \cdot 1 + \frac{2}{3}c \cdot 1 = c \cdot \frac{5}{3}, \end{aligned}$$

so that

$$c = \frac{3}{5}.$$

The distribution function F: For $t < 0$

$$\begin{aligned} F(t) &= \frac{3}{5}\int_{-\infty}^{t} (e^{-2|x|} + e^{-3|x|})dx = \frac{3}{5}\int_{|t|}^{\infty} e^{-2x}dx + \frac{3}{5}\int_{|t|}^{\infty} e^{-3x}dx \\ &= \frac{3}{5} \cdot \frac{1}{2}\int_{|t|}^{\infty} 2e^{-2x}dx + \frac{3}{5} \cdot \frac{1}{3}\int_{|t|}^{\infty} 3e^{-3x}dx \\ &= \frac{3}{10}\int_{|t|}^{\infty} f_X(x)dx + \frac{1}{5}\int_{|t|}^{\infty} f_Y(x)dx \\ &= \frac{3}{10}(1 - F_X(|t|)) + \frac{1}{5}(1 - F_Y(|t|)) \\ &= \frac{3}{10}e^{-2|t|} + \frac{1}{5}e^{-3|t|}. \end{aligned}$$

For $t \geq 0$

$$\begin{aligned}
F(t) &= \frac{3}{5}\int_{-\infty}^{t}(e^{-2|x|}+e^{-3|x|})dx \\
&= \frac{3}{5}\int_{-\infty}^{0}(e^{-2|x|}+e^{-3|x|})dx + \frac{3}{5}\int_{0}^{t}(e^{-2x}+e^{-3x})dx \\
&= \frac{3}{10}+\frac{1}{5}+\frac{3}{10}F_X(t)+\frac{1}{5}F_Y(t) \\
&= \frac{1}{2}+\frac{3}{10}(1-e^{-2t})+\frac{1}{5}(1-e^{-3t}).
\end{aligned}$$

Altogether,

$$F(t) = \begin{cases} \dfrac{3}{10}e^{-2|t|}+\dfrac{1}{5}e^{-3|t|}, & t < 0, \\[2ex] \dfrac{1}{2}+\dfrac{3}{10}(1-e^{-2t})+\dfrac{1}{5}(1-e^{-3t}), & t \geq 0. \end{cases}$$

105. Obviously, X is distributed Cauchy.

106.

(e) We have:

$$\begin{aligned}
E\left(X\right) &= \frac{\pi}{4}\int_{-1}^{1} x\cdot\cos(\pi x/2)dx \\
&= \frac{\pi}{4}\left(\left[x\frac{2}{\pi}\sin(\pi x/2)\right]_{-1}^{1} - \frac{2}{\pi}\int_{-1}^{1}\sin(\pi x/2)dx\right) \\
&= \frac{\pi}{4}\left(\frac{2}{\pi}\left[1\cdot 1-\left(-1\right)\left(-1\right)\right] - \frac{2}{\pi}\cdot\frac{2}{\pi}\left[\cos(\pi x/2)\right]_{-1}^{1}\right) = 0.
\end{aligned}$$

Note that the density function is even, so that the random variable is symmetric around 0. Thus, the result was expected.

108.

(d)$E(\max(U_1, U_2))$:

$$\begin{aligned}
\int_0^1\int_0^1 \max\left(x,y\right)dxdy &= \int_0^1\left(\int_0^y ydx+\int_y^1 xdx\right)dy = \int_0^1\left(y^2+\left[\frac{x^2}{2}\right]_y^1\right)dy \\
&= \int_0^1\left(\frac{y^2}{2}+\frac{1}{2}\right)dy = \left[\frac{y^3}{6}+\frac{y}{2}\right]_0^1 = \frac{2}{3}.
\end{aligned}$$

8 Variance and Covariance

118. We have

$$V(e^U + e^{1-U}) = V\left(e^U\right) + V\left(e^{1-U}\right) + 2\mathrm{Cov}\left(e^U, e^{1-U}\right).$$

As $U \sim U(0,1)$, the variable $1-U$ is also $U(0,1)$-distributed. Thus:

$$\begin{aligned} E\left(e^{1-U}\right) &= E\left(e^U\right) = \int_0^1 e^x dx = e - 1, \\ E\left(e^{2-2U}\right) &= E\left(e^{2U}\right) = \int_0^1 e^{2x} dx = \frac{1}{2}\left(e^2 - 1\right), \\ V\left(e^{1-U}\right) &= V\left(e^U\right) = \frac{1}{2}\left(e^2-1\right) - (e-1)^2 = -\frac{e^2}{2} + 2e - \frac{3}{2}. \end{aligned}$$

We also have

$$\begin{aligned} \mathrm{Cov}\left(e^U, e^{1-U}\right) &= E\left(e^U \cdot e^{1-U}\right) - E\left(e^U\right) E\left(e^{1-U}\right) \\ &= E(e) - (e-1)(e-1) = e - (e-1)^2. \end{aligned}$$

Altogether

$$V(e^U + e^{1-U}) = 2\left(-\frac{e^2}{2} + 2e - \frac{3}{2}\right) + 2\left(e - (e-1)^2\right) = -3e^2 + 10e - 5.$$

119.

(a) Let X denote the number of ones. Then $X = \sum_{i=1}^n X_i$, where $X_i = 1$ if the outcome of the ith roll is 1 and $X_i = 0$ otherwise. Let Y and Y_i, $1 \le i \le n$, be defined similarly for the sixes. Obviously, $X, Y \sim B(n, 1/6)$, so that:

$$E(X) = E(Y) = \frac{n}{6}.$$

Now

$$\begin{aligned} E(XY) &= E\left(\sum_{i=1}^n X_i \cdot \sum_{j=1}^n Y_j\right) \\ &= \sum_{i,j=1}^n E(X_iY_j) = \sum_{i\neq j} E(X_i)E(Y_j) + \sum_{i=1}^n E(X_iY_i) \\ &= n(n-1)\cdot\frac{1}{6}\cdot\frac{1}{6} + \sum_{i=1}^n 0 = \frac{n(n-1)}{6}, \end{aligned}$$

and therefore

$$\mathrm{Cov}(X,Y) = \frac{n(n-1)}{36} - \frac{n}{6}\cdot\frac{n}{6} = -\frac{n}{36}.$$

(b) Let X denote the number of ones. Then $X = \sum_{i=1}^{n} X_i$, where $X_i = 1$ if the outcome of the ith roll is 1 and $X_i = 0$ otherwise. Let Y denote the sum of all outcomes and Y_i, the outcome of the ith roll, $1 \le i \le n$. Obviously, $X_i \sim B(1, 1/6)$, and $Y_i \sim U[1, 6]$, so that:

$$E(X_i) = \frac{1}{6}, \qquad E(Y_i) = \frac{7}{2}.$$

Now

$$\begin{aligned}\mathrm{Cov}(X_i, Y_i) &= E(X_i \cdot Y_i) - E(X_i)\, E(Y_i) \\ &= P(X_i = Y_i = 1) \cdot 1 \cdot 1 - \frac{1}{6} \cdot \frac{7}{2} = \frac{1}{6} - \frac{7}{12} = -\frac{5}{12}.\end{aligned}$$

By the bilinearity of the covariance

$$\mathrm{Cov}(X, Y) = \mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{Cov}(X_i, Y_j).$$

From independence

$$\mathrm{Cov}(X, Y) = \sum_{i=1}^{n} \mathrm{Cov}(X_i, Y_i) = -\frac{5n}{12}.$$

120.

(a) Obviously, $X \sim H(m, a, b)$, $Y \sim H(n, a, b)$, and therefore

$$E(X) = \frac{ma}{a+b}, \qquad V(X) = \frac{mab}{(a+b)^2}\left(1 - \frac{m-1}{a+b-1}\right),$$

and

$$E(Y) = \frac{na}{a+b}, \qquad V(Y) = \frac{nab}{(a+b)^2}\left(1 - \frac{n-1}{a+b-1}\right).$$

(b) Write $X = \sum_{i=1}^{n} X_i$, where $X_i = 1$ if the ith ball is white and $X_i = 0$ otherwise. Write $Y = \sum_{i=1}^{n} Y_i$, analogously for the second batch. Then

$$\begin{aligned}E(XY) &= E\left(\sum_{i=1}^{m} X_i \cdot \sum_{j=1}^{n} Y_j\right) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} E(X_i Y_j) = mn\frac{a(a-1)}{(a+b)(a+b-1)},\end{aligned}$$

so that

$$\begin{aligned}\mathrm{Cov}(X, Y) &= mn\frac{a(a-1)}{(a+b)(a+b-1)} - \frac{ma}{a+b} \cdot \frac{na}{a+b} \\ &= -\frac{mnab}{(a+b)^2(a+b-1)}.\end{aligned}$$

(c) The covariance is negative since the more white balls there are in the first batch the less we should expect to have in the second.

9 Multi-Dimensional Distributions

126.

(a) The probability function of (X,Y) is

$$P(X=x,Y=y)=\frac{\binom{4}{x}\binom{4}{y}\binom{44}{2-x-y}}{\binom{52}{2}}, \qquad x,y\in\{0,1,2\}.$$

As displayed in the following table:

x\y	0	1	2
0	$\frac{\binom{44}{2}}{\binom{52}{2}}$	$\frac{\binom{4}{1}\binom{44}{1}}{\binom{52}{2}}$	$\frac{\binom{4}{2}}{\binom{52}{2}}$
1	$\frac{\binom{4}{1}\binom{44}{1}}{\binom{52}{2}}$	$\frac{\binom{4}{1}\binom{4}{1}}{\binom{52}{2}}$	0
2	$\frac{\binom{4}{2}}{\binom{52}{2}}$	0	0

(b) We have

$$\begin{aligned}P(X\geq Y)&=P(X=0,Y=0)+P(X=1,Y=0)+P(X=2,Y=0)+P(X=1,Y=1)\\&=1-P(X<Y)=1-P(X=0,Y=1)-P(X=0,Y=2)\\&=1-\frac{\binom{4}{1}\binom{44}{1}+\binom{4}{2}}{\binom{52}{2}}=\frac{44}{51}.\end{aligned}$$

(c) We have $X,Y\sim H(2,4,48)$, and thus

$$E(X)=E(Y)=\frac{2\cdot 4}{52}=\frac{2}{13}.$$

We also have

$$E(XY)=\sum_{i=0}^{2}\sum_{j=0}^{2}i\cdot j\cdot P(X=i,Y=j)=1\cdot 1\cdot\frac{\binom{4}{1}\binom{4}{1}}{\binom{52}{2}}=\frac{8}{663}.$$

Altogether

$$\mathrm{Cov}(X,Y) = \frac{8}{663} - (\frac{2}{13})^2 = -\frac{100}{8619}.$$

We now present an alternative solution. Let S_i be the number of cards of type i that were drawn, $i = 1, 2, \ldots, 13$. Consider $V(\sum_{i=1}^{13} S_i)$. As we draw two cards exactly, the sum is constant, so that $V(\sum_{i=1}^{13} S_i) = 0$. On the other hand, calculating the variance of the sum yields:

$$0 = V(\sum_{i=1}^{13} S_i) = \sum_{i=1}^{13} V(S_i) + 2 \sum_{1 \le i < j \le 13} \mathrm{Cov}(S_i, S_j).$$

By symmetry $\mathrm{Cov}(S_i, S_j) = \mathrm{Cov}(X, Y)$ for all i, j. Thus:

$$13{\cdot}12{\cdot}\mathrm{Cov}(X,Y) = -\sum_{i=1}^{13} V(S_i) = -13{\cdot}\frac{2 \cdot 4 \cdot 48}{52^2}\left(1 - \frac{2-1}{52-1}\right),$$

and

$$\mathrm{Cov}(X,Y) = -\frac{100}{8619}.$$

130.

(a)

$$U_1 \sim U(0,1).$$

(b)

$$U_2 \sim U(0,2).$$

(c) For $0 \le t \le 1$ we have:

$$F_X(t) = P(X \le t) = P(\sqrt{U_2/2} \le t) = P(U_2 \le 2t^2) = \frac{2t^2}{2} = t^2.$$

Since X assumes values only in $[0,1]$ this yields:

$$F_X(t) = \begin{cases} 0, & t < 0, \\ t^2, & 0 \le t \le 1, \\ 1, & t > 1. \end{cases}$$

Therefore:

$$f_X(t) = \begin{cases} 2t, & 0 \le t \le 1, \\ 0, & \text{Otherwise.} \end{cases}$$

(d) By the independence of U_1 and U_2

$$\begin{aligned} F_V(t) &= P(V \le t) = P(\max(U_1, U_2) \le t) = P(U_1 \le t, U_2 \le t) \\ &= P(U_1 \le t) \cdot P(U_2 \le t) = F_{U_1}(t) \cdot F_{U_2}(t), \end{aligned}$$

for $t \in \mathbf{R}$. Thus:

$$F_V(t) = \begin{cases} 0, & t < 0, \\ \frac{t^2}{2}, & 0 \le t < 1, \\ \frac{t}{2}, & 1 \le t < 2, \\ 1, & t \ge 2. \end{cases}$$

Consequently

$$f_V(t) = \begin{cases} t, & 0 \le t < 1, \\ \frac{1}{2}, & 1 \le t \le 2 \\ 0, & \text{Otherwise.} \end{cases}$$

(e) By the independence of U_1 and U_2

$$\begin{aligned} F_Y(t) &= P(Y \le t) = P(\min(U_1, U_2) \le t) = 1 - P(U_1 \ge t, U_2 \ge t) \\ &= 1 - P(U_1 \ge t) \cdot P(U_2 \ge t) = 1 - (1 - F_{U_1}(t))(1 - F_{U_2}(t)), \end{aligned}$$

for $t \in \mathbf{R}$. Thus:

$$F_Y(t) = \begin{cases} 0, & t < 0, \\ 1 - (1-t)(2-t)\frac{1}{2} = \frac{3t}{2} - \frac{t^2}{2}, & 0 \le t < 1, \\ 1, & t \ge 1, \end{cases}$$

and

$$f_Y(t) = \begin{cases} \frac{3}{2} - t, & 0 \le t < 1, \\ 0, & \text{Otherwise.} \end{cases}$$

(f) We have

$$F_W(t) = P(W \le t) = P(U_1 \cdot U_2 \le t).$$

For $0 \le t \le 2$, this gives:

$$F_W(t) = \frac{1}{2} \cdot 2 \cdot \frac{1}{2} + \int_{\frac{t}{2}}^{1} \int_0^{\frac{t}{x}} \frac{1}{2} dy dx = \frac{t}{2} + \int_{\frac{t}{2}}^{1} \frac{t}{2x} dx = \frac{t}{2}\left(1 - \ln\frac{t}{2}\right).$$

Hence

$$F_W(t) = \begin{cases} 0, & t < 0, \\ \frac{t}{2}\left(1 - \ln\frac{t}{2}\right), & 0 \le t < 2, \\ 1, & t \ge 2, \end{cases}$$

and

$$f_W(t) = \begin{cases} \frac{1}{2}\ln\frac{2}{t}, & 0 \le t < 2, \\ 0, & \text{Otherwise.} \end{cases}$$

134.

(a) $c = \frac{4}{3\pi}$.

(b) $E(X^2) = \frac{7}{18}$.

(c) $\rho(X, Y) = 0$.

136.

(a) Let $S = \{(x, y) : 0 \le x \le y \le 1\}$. Then:

$$1 = \iint_S cxydxdy = c\int_0^1 \int_0^y xydxdy = c\int_0^1 y \cdot \frac{y^2}{2}dy = \frac{c}{8}.$$

Thus $c = 8$.

(b) We have

$$E(X) = \iint_S 8xy \cdot xdxdy = 8\int_0^1 \int_0^y x^2ydxdy = 8\int_0^1 y \cdot \frac{y^3}{3}dy = \frac{8}{15},$$
$$E(Y) = \iint_S 8xy \cdot ydxdy = 8\int_0^1 \int_0^y xy^2dxdy = 8\int_0^1 y^2 \cdot \frac{y^2}{2}dy = \frac{4}{5},$$
$$E(XY) = \iint_S 8xy \cdot xydxdy = 8\int_0^1 \int_0^y x^2y^2dxdy = 8\int_0^1 y^2 \cdot \frac{y^3}{3}dy = \frac{4}{9}.$$

Thus:

$$\text{Cov}(X, Y) = \frac{4}{9} - \frac{8}{15} \cdot \frac{4}{5} = \frac{4}{225}.$$

(c) By bilinearity:

$$\begin{aligned}\text{Cov}(X+Y, X-Y) &= \text{Cov}(X, X) - \text{Cov}(X, Y) + \text{Cov}(Y, X) - \text{Cov}(Y, Y) \\ &= V(X) - V(Y).\end{aligned}$$

Now

$$E(X^2) = \iint_S 8xy \cdot x^2dxdy = 8\int_0^1 \int_0^y x^3ydxdy = 8\int_0^1 y \cdot \frac{y^4}{4}dy = \frac{1}{3},$$
$$E(Y^2) = \iint_S 8xy \cdot y^2dxdy = 8\int_0^1 \int_0^y xy^3dxdy = 8\int_0^1 y^3 \cdot \frac{y^2}{2}dy = \frac{2}{3}.$$

Thus

$$V(X) = E(X^2) - E^2(X) = \frac{1}{3} - \left(\frac{8}{15}\right)^2 = \frac{11}{225},$$
$$V(Y) = E(Y^2) - E^2(Y) = \frac{2}{3} - \left(\frac{4}{5}\right)^2 = \frac{2}{75},$$

and

$$\text{Cov}(X+Y, X-Y) = \frac{11}{225} - \frac{2}{75} = \frac{1}{45}.$$

(d) We have

$$\begin{aligned} V(X+Y) &= V(X)+V(Y)+2\text{Cov}(X,Y) \\ &= \frac{11}{225}+\frac{2}{75}+2\cdot\frac{4}{225}=\frac{1}{9}. \end{aligned}$$

137.

(a) Let $S=\{(x,y):\ x,y\geq 0,\ 1\leq x^2+y^2\leq 4\}$. Then:

$$1=\iint_S \frac{c}{\sqrt{x^2+y^2}}dxdy=c\int_1^2\int_0^{\frac{\pi}{2}}\frac{1}{r}\cdot rd\theta dr=c\int_1^2 1dr\cdot\int_0^{\frac{\pi}{2}}1d\theta=c\cdot\frac{\pi}{2}.$$

Thus $c=\frac{2}{\pi}$.

(b) We have:

$$\begin{aligned} E(X) &= \iint_S \frac{\frac{2}{\pi}}{\sqrt{x^2+y^2}}\cdot xdxdy=\frac{2}{\pi}\int_1^2\int_0^{\frac{\pi}{2}} r\cdot\cos\theta d\theta dr \\ &= \frac{2}{\pi}\int_1^2 rdr\cdot\int_0^{\frac{\pi}{2}}\cos\theta d\theta=\frac{2}{\pi}\cdot\frac{3}{2}=\frac{3}{\pi}. \end{aligned}$$

(c) We have:

$$\begin{aligned} E\left(X^2\right) &= \iint_S \frac{\frac{2}{\pi}}{\sqrt{x^2+y^2}}\cdot x^2dxdy=\frac{2}{\pi}\int_1^2\int_0^{\frac{\pi}{2}} r^2\cdot\cos^2\theta d\theta dr \\ &= \frac{2}{\pi}\int_1^2 r^2\int_0^{\frac{\pi}{2}}\frac{\cos 2\theta+1}{2}d\theta dr=\frac{2}{\pi}\cdot\frac{7}{3}\cdot\frac{\pi}{4}=\frac{7}{6}. \end{aligned}$$

Thus:

$$V(X)=\frac{7}{6}-\frac{9}{\pi^2}.$$

(d)We have

$$\rho(X,Y)=\frac{\text{Cov}(X,Y)}{\sigma_X\cdot\sigma_Y}=\frac{E(XY)-E(X)E(Y)}{\sigma_X\cdot\sigma_Y}.$$

By symmetry,

$$E(X)=E(Y)=\frac{3}{\pi}$$

and

$$\sigma_X=\sigma_Y=\sqrt{V(X)}=\sqrt{\frac{7}{6}-\frac{9}{\pi^2}}.$$

Now:

$$E(XY) = \iint_S \frac{\frac{2}{\pi}}{\sqrt{x^2+y^2}} \cdot xydxdy = \frac{2}{\pi}\int_1^2 \int_0^{\frac{\pi}{2}} r \cdot \cos\theta \cdot r \cdot \sin\theta d\theta dr$$
$$= \frac{2}{\pi}\int_1^2 r^2 dr \int_0^{\frac{\pi}{2}} \cos\theta\sin\theta d\theta = \frac{2}{\pi}\cdot\frac{7}{3}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2\theta d\theta = \frac{7}{3\pi}.$$

Thus

$$\rho(X,Y) = \frac{\frac{7}{3\pi} - \frac{9}{\pi^2}}{\frac{7}{6} - \frac{9}{\pi^2}} = \frac{14\pi - 54}{7\pi^2 - 54} \approx -0.6639.$$

10 Independence

145.

(a) Note that $X_i \sim U[1,6]$ for $i = 1,2$. We have:

$$E(S) = E(X_1 + X_2) = E(X_1) + E(X_2) = \frac{7}{2} + \frac{7}{2} = 7.$$

The random variable $|X_1 - X_2|$ receives the values $0,\ldots,5$ with probabilities:

d	0	1	2	3	4	5
p	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

Thus

$$E(D) = E(|X_1 - X_2|) = \sum_{i=0}^{5} i \cdot P(|X_1 - X_2| = i) = \frac{35}{18}.$$

As X_1 and X_2 have the same distribution

$$E(SD) = E((X_1 + X_2)|X_1 - X_2|) = E(X_1 \cdot |X_1 - X_2|) + E(X_2 \cdot |X_1 - X_2|)$$
$$= 2E(X_1 \cdot |X_1 - X_2|).$$

The probability function of the variable $(X_1, |X_1 - X_2|)$ is given in the following table:

$x_1 \setminus d$	0	1	2	3	4	5
1	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
2	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	0
3	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{36}$	0	0
4	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{36}$	0	0
5	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	0
6	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$

Thus,

$$E(SD) = 2\sum_{i=1}^{6}\sum_{j=0}^{5} i \cdot j \cdot P(X_1 = i, |X_1 - X_2| = j) = \frac{245}{18}.$$

Therefore

$$E(SD) = \frac{245}{18} = 7 \cdot \frac{35}{18} = E(S)\,E(D).$$

(b) We have

$$P(X_1 + X_2 = 7, |X_1 - X_2| = 0) = 0 \neq \frac{2}{36} \cdot \frac{6}{36} = P(X_1 + X_2 = 7)\,P(|X_1 - X_2| = 0).$$

Thus, S and D are not independent.

152.

(a)

$$P(X = k, Y = m) = \frac{e^{-\lambda}\lambda^k}{k!}\frac{1}{2^k}\binom{k}{m}.$$

(b) It is easy to see thay $Y \sim P(\frac{\lambda}{2})$, and in particular $E(Y) = \frac{\lambda}{2}$ and $V(Y) = \frac{\lambda}{2}$.

(c) We have

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \cdot \sigma_Y} = \frac{E(XY) - E(X)\,E(Y)}{\sigma_X \cdot \sigma_Y}.$$

By (a) and (b):

$$E(X) = \lambda, \qquad E(Y) = \frac{\lambda}{2},$$
$$\sigma_X = \sqrt{\lambda}, \qquad \sigma_Y = \sqrt{\frac{\lambda}{2}}.$$

Also:

$$E(XY) = \sum_{i=0}^{\infty}\sum_{j=0}^{i} i \cdot j \cdot P(X = i, Y = j) = \sum_{i=0}^{\infty} i \cdot e^{-\lambda}\frac{\lambda^i}{i!}\sum_{j=0}^{i}\binom{i}{j}\left(\frac{1}{2}\right)^i \cdot j$$
$$= \sum_{i=0}^{\infty} i \cdot e^{-\lambda}\frac{\lambda^i}{i!}E(Y_i),$$

where $Y_i \sim B\left(i, \frac{1}{2}\right)$. Hence:

$$E(XY) = \frac{1}{2}\sum_{i=0}^{\infty} i^2 \cdot e^{-\lambda}\frac{\lambda^i}{i!} = \frac{1}{2}E(X^2) = \frac{1}{2}\left(\lambda + \lambda^2\right).$$

Altogether:

$$\rho(X,Y) = \frac{\frac{1}{2}(\lambda+\lambda^2) - \lambda\cdot\frac{\lambda}{2}}{\sqrt{\lambda}\cdot\sqrt{\frac{\lambda}{2}}} = \frac{1}{\sqrt{2}} \approx 0.707.$$

155.

(c) We have

$$\begin{aligned}E(\sin 2\pi X) &= \int_0^1 \sin 2\pi x dx = 0,\\ E(\cos 2\pi X) &= \int_0^1 \cos 2\pi x dx = 0,\\ E(\sin 2\pi X\cdot\cos 2\pi X) &= \int_0^1 \sin 2\pi x\cdot\cos 2\pi x dx = 0,\end{aligned}$$

Thus $\rho(\sin 2\pi X, \cos 2\pi X) = 0$. Note that $Y_1 = \sin(2\pi X)$ and $Y_2 = \cos(2\pi X)$ are not independent but $\rho(Y_1, Y_2) = 0$.

11 Normal Distribution

160.

(a) We have:

$$\begin{aligned}1 &= c\int_{-\infty}^{\infty}\left(2e^{-x^2}+3e^{-2x^2}\right)dx = 2c\int_{-\infty}^{\infty}e^{-x^2}dx + 3c\int_{-\infty}^{\infty}e^{-2x^2}dx\\ &= 2c\int_{-\infty}^{\infty}e^{-\frac{t^2}{2}}\cdot\frac{1}{\sqrt{2}}dt + 3c\int_{-\infty}^{\infty}e^{-\frac{t^2}{2}}\cdot\frac{1}{2}dt = c\left(\frac{2}{\sqrt{2}}\cdot\sqrt{2\pi}+\frac{3}{2}\cdot\sqrt{2\pi}\right)\\ &= c\sqrt{\pi}\left(2+\frac{3}{\sqrt{2}}\right).\end{aligned}$$

Alternatively, we note that

$$\begin{aligned}\int_{-\infty}^{\infty}e^{-x^2}dx &= \sqrt{2\pi\left(1/\sqrt{2}\right)^2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi(1/\sqrt{2})^2}}e^{-\frac{x^2}{2(1/\sqrt{2})^2}}dx\\ &= \sqrt{2\pi\left(1/\sqrt{2}\right)^2}\int_{-\infty}^{\infty}f_{Y_1}(x)dx = \sqrt{\pi},\end{aligned}$$

where $Y_1 \sim N(0, \frac{1}{2})$. Similarly

$$\int_{-\infty}^{\infty} e^{-2x^2} dx = \sqrt{2\pi\,(1/2)^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(1/2)^2}} e^{-\frac{x^2}{2(1/2)^2}} dx$$
$$= \sqrt{2\pi\,(1/2)^2} \int_{-\infty}^{\infty} f_{Y_2}(x) dx = \sqrt{\frac{\pi}{2}},$$

where $Y_2 \sim N(0, \frac{1}{4})$. Thus

$$1 = 2c \int_{-\infty}^{\infty} e^{-x^2} dx + 3c \int_{-\infty}^{\infty} e^{-2x^2} dx$$
$$= 2c\sqrt{\pi} \int_{-\infty}^{\infty} f_{Y_1}(x) dx + 3c\frac{\sqrt{\pi}}{2} \int_{-\infty}^{\infty} f_{Y_2}(x) dx = c\sqrt{\pi}\left(2 + \frac{3}{\sqrt{2}}\right).$$

Thus

$$c = \left(\sqrt{\pi}\left(2 + \frac{3}{\sqrt{2}}\right)\right)^{-1} = \frac{1}{\sqrt{\pi}} \cdot \frac{1 \cdot \left(2 - \frac{3}{\sqrt{2}}\right)}{\left(2 + \frac{3}{\sqrt{2}}\right)\left(2 - \frac{3}{\sqrt{2}}\right)} = \frac{1}{\sqrt{\pi}} \cdot \frac{4 - 3\sqrt{2}}{2 \cdot \left(4 - \frac{9}{2}\right)}$$
$$= \frac{1}{\sqrt{\pi}} \cdot \left(3\sqrt{2} - 4\right).$$

(b) We have:

$$E(X) = c \int_{-\infty}^{\infty} x \left(2e^{-x^2} + 3e^{-2x^2}\right) dx = 0,$$

and

$$V(X) = E(X^2) = c \int_{-\infty}^{\infty} x^2 \left(2e^{-x^2} + 3e^{-2x^2}\right) dx.$$
$$= 2c\sqrt{\pi} \int_{-\infty}^{\infty} x^2 f_{Y_1}(x) dx + 3c\frac{\sqrt{\pi}}{2} \int_{-\infty}^{\infty} x^2 f_{Y_1}(x) dx$$
$$= 2c\sqrt{\pi} E(Y_1^2) + 3c\frac{\sqrt{\pi}}{2} E(Y_2^2)$$
$$= 2c\sqrt{\pi} V(Y_1) + 3c\frac{\sqrt{\pi}}{2} V(Y_2) = 2c\sqrt{\pi} \cdot \frac{1}{2} + 3c\sqrt{\frac{\pi}{2}} \cdot \frac{1}{4}$$
$$= c\left(\sqrt{\pi} + \frac{3}{4} \cdot \sqrt{\frac{\pi}{2}}\right) = \left(\sqrt{\pi} + \frac{3}{4} \cdot \sqrt{\frac{\pi}{2}}\right) \cdot \frac{1}{\sqrt{\pi}} \cdot \left(3\sqrt{2} - 4\right)$$
$$= 3\sqrt{2} - \frac{3}{\sqrt{2}} - \frac{7}{4} = \frac{6\sqrt{2} - 7}{4}.$$

161. By induction. For $k = 1$

$$E\left(Z^2\right) = V(Z) + E^2(Z) = 1 = (2 \cdot 1 - 1)!!.$$

Now assume that $E\left(Z^{2k}\right) = (2k-1)!!$. For $k+1$ we integrate by parts:

$$\begin{aligned} E\left(Z^{2k+2}\right) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^{2k+2}e^{-\frac{x^2}{2}}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^{2k+1}\cdot xe^{-\frac{x^2}{2}}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\left[-x^{2k+1}e^{-\frac{x^2}{2}}\right]_{-\infty}^{\infty} + \frac{2k+1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^{2k}e^{-\frac{x^2}{2}}\,dx \\ &= (2k+1)\cdot(2k-1)!! = (2k+1)!!. \end{aligned}$$

162. For $E(|X|)$ we have:

$$\begin{aligned} E\left(|X|\right) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} |x|e^{-\frac{x^2}{2}}\,dx = \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} xe^{-\frac{x^2}{2}}\,dx. \\ &= \frac{2}{\sqrt{2\pi}}\left[-e^{-\frac{x^2}{2}}\right]_{0}^{\infty} = \frac{2}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}. \end{aligned}$$

For $E(|X|^3)$ we have:

$$\begin{aligned} E\left(|X|^3\right) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} |x|^3e^{-\frac{x^2}{2}}\,dx = \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} x^2\cdot xe^{-\frac{x^2}{2}}\,dx. \\ &= \frac{2}{\sqrt{2\pi}}\left[-x^2e^{-\frac{x^2}{2}}\right]_{0}^{\infty} + \frac{4}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-\frac{x^2}{2}}\,dx = 2\sqrt{\frac{2}{\pi}}. \end{aligned}$$

163. Let us first calculate the third moment for $Z \sim N(0,1)$:

$$\begin{aligned} E\left(Z^3\right) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^3e^{-\frac{x^2}{2}}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^2\cdot xe^{-\frac{x^2}{2}}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\left[-x^2e^{-\frac{x^2}{2}}\right]_{-\infty}^{\infty} + \frac{2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-\frac{x^2}{2}}\,dx = 0. \end{aligned}$$

For $X \sim N(\mu,\sigma^2)$, let $Z = \dfrac{X-\mu}{\sigma}$. Recall that $Z \sim N(0,1)$. Now $X = \sigma Z + \mu$, and therefore:

$$\begin{aligned} E(X^3) &= E\left((\sigma Z+\mu)^3\right) \\ &= E\left(Z^3 + 3\sigma^2\mu Z^2 + 3\sigma\mu^2 Z + \mu^3\right) \\ &= \sigma^3\cdot E\left(Z^3\right) + 3\sigma^2\mu\cdot E\left(Z^2\right) + 3\sigma\mu^2\cdot E\left(Z\right) + \mu^3. \end{aligned}$$

Now, $E\left(Z\right) = 0$, $E(Z^2) = V(Z) = 1$, Altogether

$$E(X^3) = 3\sigma^2\mu + \mu^3.$$

12 Limit Theorems

164. As g is increasing, $P(|X| \geq t) = P(g(|X|) \geq g(t))$. As g is positive, $g(t) > 0$, and by Markov's inequality

$$P(|X| \geq t) = P(g(|X|) \geq g(t)) \leq \frac{E(g(|X|))}{g(t)}.$$

165. The variables are independent, and thus $Y_1X_1, \ldots, Y_nX_n$ are independent. We have $E(X_i) = \mu$, $V(X_i) = \sigma^2$, $E(Y_i) = 1 \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2} = 0$ and $V(Y_i) = 1^2 \cdot \frac{1}{2} + (-1)^2 \cdot \frac{1}{2} = 1$. Thus:

$$E(X_iY_i) = E(X_i)E(Y_i) = 0,$$

and

$$V(X_iY_i) = E((X_iY_i)^2) = E(X_i^2)E(Y_i^2) = \sigma^2 + \mu^2.$$

By the central limit theorem

$$\lim_{n\to\infty} P\left(\frac{\frac{1}{n}S_n}{\frac{\sqrt{\sigma^2+\mu^2}}{\sqrt{n}}} \leq x\right) = \Phi(x), \qquad -\infty < x < \infty.$$

Thus, as $n \to \infty$, the distribution function of $\dfrac{\frac{1}{\sqrt{n}}S_n}{\sqrt{\sigma^2+\mu^2}}$ tends to that of a standard normal variable. Hence, the distribution function of $\frac{1}{\sqrt{n}}S_n$ tends to that of $\sqrt{\sigma^2+\mu^2}Z$, where $Z \sim N(0,1)$, which is the distribution function of an $N(0, \sigma^2+\mu^2)$-distributed variable.

166. The mean and the variance of the population are

$$\mu = \frac{1}{3} \cdot 3 + \frac{1}{3} \cdot 5 + \frac{1}{3} \cdot 8 = \frac{16}{3},$$

and

$$\sigma^2 = \frac{1}{3}(9 + 25 + 64) - \left(\frac{16}{3}\right)^2 = \frac{38}{9}.$$

Let $X_1, \ldots, X_{54}$ be the sample. The mean and the variance of the population are

$$\mu = \frac{1}{3} \cdot 3 + \frac{1}{3} \cdot 5 + \frac{1}{3} \cdot 8 = \frac{16}{3},$$

and

$$\sigma^2 = \frac{1}{3}(9 + 25 + 64) - \left(\frac{16}{3}\right)^2 = \frac{38}{9}.$$

Let $X_1, \ldots, X_{54}$ be the sample. By the central limit theorem

$$P\left(5 \leq \overline{X}_{54} \leq 5.2\right) = P\left(\frac{5-\mu}{\sigma/\sqrt{54}} \leq \frac{\overline{X}_{54}-\mu}{\sigma/\sqrt{54}} \leq \frac{5.2-\mu}{\sigma/\sqrt{54}}\right)$$

$$= P\left(\frac{5-\frac{16}{3}}{\sqrt{38}/3\sqrt{54}} \leq \frac{\overline{X}_{54}-\frac{16}{3}}{\sqrt{38}/3\sqrt{54}} \leq \frac{5.2-\frac{16}{3}}{\sqrt{38}/3\sqrt{54}}\right)$$

$$\approx \Phi\left(\frac{15.6-16}{\sqrt{38}/\sqrt{54}}\right) - \Phi\left(\frac{15-16}{\sqrt{38}/\sqrt{54}}\right)$$

$$= \Phi\left(\frac{-0.4}{0.8388}\right) - \Phi\left(\frac{-1}{0.8388}\right)$$

$$= 1 - \Phi\left(\frac{0.4}{0.8388}\right) - 1 + \Phi\left(\frac{1}{0.8388}\right)$$

$$= \Phi\left(1.1921\right) - \Phi\left(0.4768\right) = 0.88 - 0.68 = 0.20.$$

167. The mean and the variance of the population are

$$\mu = E(X_i) = \frac{1}{2},$$
$$\sigma^2 = V(X_i) = \frac{1}{4}.$$

Let $X_1, \ldots, X_{200}$ be the sample. For $d > 0$, the central limit theorem gives

$$P\left(|\overline{X}_{200} - \mu| > d\right) = 1 - P\left(|\overline{X}_{200} - \mu| \leq d\right)$$

$$= 1 - P\left(\frac{-d}{1/2\sqrt{200}} \leq \frac{\overline{X}_{200}-\frac{1}{2}}{1/2\sqrt{200}} \leq \frac{d}{1/2\sqrt{200}}\right)$$

$$\approx 1 - \left(\Phi\left(20\sqrt{2}\cdot d\right) - \Phi\left(-20\sqrt{2}\cdot d\right)\right)$$

$$= 1 - \left(2\Phi\left(20\sqrt{2}\cdot d\right) - 1\right) = 2 - 2\Phi(20\sqrt{2}\cdot d).$$

We want to choose d so that the right-hand side will be approximately 0.05. Thus $\Phi(20\sqrt{2}\cdot d) \approx 0.975$, so that $20\sqrt{2}\cdot d \approx 1.96$, and therefore $d \approx 0.069$.

13 The Moment Generating Function

168.

(b) We have:

$$\Psi(t) = E(e^{tX}) = \sum_{i=0}^{\infty} e^{t\cdot i} e^{-\lambda} \frac{\lambda^i}{i!} = e^{-\lambda} \sum_{i=0}^{\infty} \frac{(e^t\lambda)^i}{i!}$$
$$= e^{-\lambda} e^{e^t\lambda} = e^{\lambda\left(e^t-1\right)}.$$

(c) We have:

$$\Psi(t) = E(e^{tX}) = \sum_{i=1}^{\infty} e^{t\cdot i}(1-p)^{i-1}p = pe^t \sum_{i=1}^{\infty} e^{t(i-1)}(1-p)^{i-1}$$
$$= pe^t \sum_{i=1}^{\infty} \left(e^t(1-p)\right)^{i-1} = pe^t \sum_{j=0}^{\infty} \left(e^t(1-p)\right)^j.$$

The expectation will exist if $|e^t(1-p)| < 1$. In that case

$$\Psi(t) = \frac{pe^t}{1-e^t(1-p)} = \frac{p}{e^{-t}+p-1}.$$

(d) We have:

$$\Psi(t) = E(e^{tU}) = \int_a^b e^{tx} \cdot \frac{1}{b-a} dx = \frac{1}{b-a}\left[\frac{1}{t}e^{tx}\right]_a^b = \frac{e^{tb}-e^{ta}}{t(b-a)}.$$

169. The variable $X+Y$ assumes the values $n = 0, 1, 2, \ldots$ with probabilities

$$P(X+Y=n) = \sum_{k=0}^{\infty} P(Y=k)\cdot P(X+Y=n|Y=k)$$
$$= \sum_{k=0}^{n} P(Y=k)\cdot P(X=n-k|Y=k)$$
$$= \sum_{k=0}^{n} P(Y=k)\cdot P(X=n-k)$$
$$= \sum_{k=0}^{n} e^{-\lambda_2}\frac{\lambda_2^k}{k!}\cdot e^{-\lambda_1}\frac{\lambda_1^{n-k}}{(n-k)!} = \frac{1}{n!}\cdot e^{-(\lambda_1+\lambda_2)} \sum_{n=0}^{k} \frac{n!\lambda_2^k\lambda_1^{n-k}}{k!(n-k)!}$$
$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} \binom{n}{k} \lambda_2^k \lambda_1^{n-k} = \frac{(\lambda_1+\lambda_2)^n}{n!} e^{-(\lambda_1+\lambda_2)}.$$

Thus $X+Y \sim P(\lambda_1+\lambda_2)$. By the solution of Problem 168.(c)

$$\Psi(t) = e^{(\lambda_1+\lambda_2)\left(e^t-1\right)}.$$

170. We have:

$$\Psi(t) = E(e^{tU}) = \int_0^1 e^{tx}dx = \left[\frac{1}{t}e^{tx}\right]_0^1 = \frac{1}{t}\left(e^t-1\right).$$

Using the Maclaurin expansion of e^t we get

$$\Psi(t) = \frac{1}{t}\left(-1+1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\dots\right) = 1+\frac{t}{2!}+\frac{t^2}{3!}+\dots.$$

Now $E(X^n) = \Psi^{(n)}(0)$ is the coefficient of $\frac{t^n}{n!}$ in the series. Thus, $E(X^n) = \Psi^{(n)}(0) = \frac{1}{n+1}$. Consequently

$$V(X^n) = E(X^{2n}) - E^2(X^n) = \frac{1}{2n+1} - \left(\frac{1}{n+1}\right)^2.$$

171.

(c) Let $S = |X-Y|$. We have:

$$F_S(s) = P(|X-Y| \le s) = P(-s \le X-Y \le s).$$

By independence, the joint density function is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{a^2}, & (x,y) \in [0,a]^2, \\ 0, & \text{otherwise.} \end{cases}$$

As the density function is constant in the region where it is non-zero, we have

$$F_S(s) = \frac{\left|\left\{(x,y) : (x,y) \in [0,a]^2,\ -s \le x-y \le s\right\}\right|}{\left|[0,a]^2\right|},$$

where we have denoted here by $|A|$ the area of a set $A \subseteq \mathbf{R}^2$. For $0 \le s \le a^2$ we have

$$F_S(s) = \left(a^2 - 2\cdot\frac{(a-s)^2}{2}\right)\cdot\frac{1}{a^2} = 1-\left(1-\frac{s}{a}\right)^2 = \frac{2s}{a}-\frac{s^2}{a^2}.$$

Thus

$$F_S(s) = \begin{cases} 0, & s<0, \\ \dfrac{2s}{a}-\dfrac{s^2}{a^2}, & 0 \le s < a, \\ 1, & s \ge a, \end{cases} \qquad f_S(s) = \begin{cases} \dfrac{2}{a}-\dfrac{2s}{a^2}, & 0 \le s \le a, \\ 0, & \text{otherwise.} \end{cases}$$

The moment generating function of S is:

$$\begin{aligned}\Psi(t) = E(e^{tX}) &= \int_0^a e^{tx}\left(\frac{2}{a} - \frac{2x}{a^2}\right) dx = \left[\frac{2}{a}\cdot\frac{1}{t}e^{tx}\right]_0^a - \frac{2}{a^2}\int_0^a xe^{tx}dx \\ &= \frac{2}{at}\left(e^{ta}-1\right) - \frac{2}{a^2}\left(\left[\frac{1}{t}\cdot xe^{tx}\right]_0^a - \int_0^a \frac{1}{t}e^{tx}dx\right) \\ &= \frac{2}{at}\left(e^{ta}-1\right) - \frac{2}{at}e^{ta} + \frac{2}{a^2}\cdot\frac{1}{t^2}\left[e^{tx}\right]_0^a \\ &= \frac{2}{a^2t^2}\left(e^{ta}-1\right) - \frac{2}{at}.\end{aligned}$$

(d) Let $U_1, U_2 \sim (0,1)$, and $W = U_1U_2$. For $0 \le t \le 1$ we have:

$$F_W(t) = t + \int_t^1 \int_0^{\frac{t}{x}} dydx = t + \int_t^1 \frac{t}{x}dx = t\left(1 - \ln t\right).$$

We have:

$$F_W(t) = \begin{cases} 0, & t < 0, \\ t\left(1-\ln t\right), & 0 \le t < 1, \\ 1, & t \ge 1, \end{cases}$$

and

$$f_W(t) = \begin{cases} -\ln t, & 0 \le t \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

By definition, the moment generating function of W is:

$$\Psi_W(t) = E(e^{tX}) = -\int_0^1 e^{tx}\ln x dx.$$

If the moment generating function exists, then $\Psi_W^{(n)}(0) = E(W^n)$.
By the independence of U_1, U_2, the n-th moment of W is:

$$E\left(W^n\right) = E\left(U_1^nU_2^n\right) = E\left(U_1^n\right)E\left(U_2^n\right) = \frac{1}{n+1}\cdot\frac{1}{n+1} = \frac{1}{\left(n+1\right)^2}.$$

Thus:

$$\Psi_W(t) = \sum_{n=0}^{\infty}\frac{t^n}{(n+1)^2\cdot n!}.$$

Alternatively, by definition:

$$\begin{aligned}
\Psi_W(t) &= \lim_{\delta\to 0} -\int_\delta^1 e^{tx}\ln x dx \\
&= \lim_{\delta\to 0} -\left(\left[\frac{e^{tx}\cdot \ln x}{t}\right]_\delta^1 - \int_\delta^1 \frac{e^{tx}}{tx}dx\right) \\
&= \lim_{\delta\to 0} -\left(\left[\frac{e^{tx}\cdot \ln x}{t}\right]_\delta^1 - \int_\delta^1 \frac{1}{tx}\sum_{i=0}^\infty \frac{(xt)^i}{i!}dx\right) \\
&= \lim_{\delta\to 0} -\left(\left[\frac{e^{tx}\cdot \ln x}{t}\right]_\delta^1 - \int_\delta^1 \frac{1}{xt}dx - \int_\delta^1 \sum_{i=1}^\infty \frac{(xt)^{i-1}}{i!}dx\right) \\
&= \lim_{\delta\to 0} -\left(\left[\frac{e^{tx}\cdot \ln x}{t}\right]_\delta^1 - \left[\frac{\ln x}{t}\right]_\delta^1 - \sum_{i=1}^\infty \int_\delta^1 \frac{(xt)^{i-1}}{i!}dx\right) \\
&= \lim_{\delta\to 0} -\left(\frac{\ln\delta\left(1-e^{t\delta}\right)}{t} - \sum_{i=1}^\infty \left[\frac{x^i\cdot t^{i-1}}{i\cdot i!}\right]_\delta^1\right).
\end{aligned}$$

Now

$$\begin{aligned}
\lim_{\delta\to 0}\sum_{i=1}^\infty \left[\frac{x^i\cdot t^{i-1}}{i\cdot i!}\right]_\delta^1 &= \lim_{\delta\to 0}\left(\sum_{i=1}^\infty \frac{t^{i-1}}{i\cdot i!} - \sum_{i=1}^\infty \frac{\delta^i\cdot t^{i-1}}{i\cdot i!}\right) \\
&= \sum_{i=0}^\infty \frac{t^i}{(i+1)\cdot(i+1)!} = \sum_{i=0}^\infty \frac{t^i}{(i+1)^2\cdot i!},
\end{aligned}$$

and by L'hopital's rule

$$\begin{aligned}
\lim_{\delta\to 0}\frac{\ln\delta(1-e^{t\delta})}{t} &= \lim_{\delta\to 0}\frac{\ln\delta}{t\cdot\frac{1}{1-e^{t\delta}}} = \lim_{\delta\to 0}\frac{1/\delta}{t\cdot\frac{-1}{(1-e^{t\delta})^2}\cdot(-e^{t\delta})\cdot t} \\
&= \lim_{\delta\to 0}\frac{(1-e^{t\delta})^2}{t^2\cdot\delta\cdot e^{t\delta}} = \lim_{\delta\to 0}\frac{2(1-e^{t\delta})(-te^{t\delta})}{t^2\cdot e^{t\delta}+t^2\cdot\delta\cdot e^{t\delta}\cdot t} = 0.
\end{aligned}$$

Thus:

$$\Psi_W(t) = \sum_{n=0}^\infty \frac{t^n}{(n+1)^2\cdot n!}.$$

Now XY is distributed as a^2W, and therefore the moment generating function of XY is given by

$$\Psi_{XY}(t) = \Psi_W(a^2t) = \sum_{n=0}^\infty \frac{(a^2t)^n}{(n+1)^2\cdot n!}.$$