Review Questions

Mark the correct answer in each part of the following questions.

1. Consider the absent-minded secretary problem. Suppose that the handwriting of the secretary is so sloppy that, when copying an address to the envelope, it will come out unreadable with probability $1/2$. Moreover, to reduce the workload of the postmen, the post office disposes of $2/3$ of the letters.

   (a) Suppose the letter to some specific addressee, say Reuven, has been placed in the right envelope. The probability that the address was readable, given that Reuven did not receive the letter, is

   (i) $1/6$.
   (ii) $1/3$.
   (iii) $2/5$.
   (iv) $1/2$.
   (v) None of the above.

   (b) Let X be the random variable, counting the letters arriving at their intended destination. Let Y be the random variable, counting the same, but under the additional assumption that all letters have been placed at the right envelopes (with possibly unreadable addresses).

   (i) $Y \sim B(n, 2/3)$ and $X \sim B(n, 1/6n)$.
   (ii) $Y \sim B(n, 1/6)$ and $X \sim B(n, 1/6n)$.
   (iii) $Y \sim B(n, 2/3)$ but X is not binomially distributed.
   (iv) $Y \sim B(n, 1/6)$ but X is not binomially distributed.
   (v) None of the above.

(c) Let p_n denote the probability that none of the letters reaches its intended destination. Then:

(i) $p_n \underset{n\to\infty}{\longrightarrow} e^{-1/6}$.

(ii) $p_n \underset{n\to\infty}{\longrightarrow} e^{-1/3}$.

(iii) $p_n \underset{n\to\infty}{\longrightarrow} e^{-1/2}$.

(iv) $p_n \underset{n\to\infty}{\longrightarrow} e^{-5/6}$.

(v) None of the above.

2. Reuven and Shimon participate in the following game. At the first stage of the game, a coin is tossed again and again, until for the first time, say at the n-th toss, it shows a head. If n is even, the game ends. If n is odd, the game continues as follows. We put in an urn n balls, marked by the numbers $1, 2, \ldots, n$, and then take them out in a random order. One of the players is now given the chance to guess in what order the balls have been taken out of the urn. If he guesses correctly, he receives a prize. If n is one of the numbers $1, 5, 9, 13, \ldots$, then the player who gets the right to guess is Reuven, while if n is the one of the numbers $3, 7, 11, 15, \ldots$, then Shimon gets the right. Denote by p_R the probability of Reuven to win a prize and by p_S that of Shimon.

(a) The probability for the game to continue to the second stage (namely, that n will be odd) is

(i) 1/3.

(ii) 1/2.

(iii) 2/3.

(iv) 8/9.

(v) None of the above.

(b) Reuven's probability of winning is greater than Shimon's. In fact, $p_R - p_S =$

(i) e^{-2}.

(ii) $\sin 1/2$.

(iii) $\cos 1/2$.

(iv) $\ln 2$.

(v) None of the above.

3. Let A, B, C be events with positive probabilities in a probability space.

   (a)

   (i) If A, B are independent, Then so are $A \cap B, A \cup B$.

   (ii) If A, B are independent, and so are A, C, then so are $A, B \cup C$ as well.

   (iii) If A, B are independent, and so are A, C, then so are $A, B \cap C$ as well.

   (iv) If A, B, C are independent, then so are $A \cup \overline{B}, C$.

   (v) None of the above.

4. A coin is tossed $2n$ times.

   (a) The probability that the number of heads received in the first n tosses is the same as that in the last n tosses is

   (i) $1/2^n$.

   (ii) $1/(2n+1)$.

   (iii) $\binom{2n}{n}/2^{2n}$.

   (iv) $1/2$.

   (v) None of the above.

   (b) The probability that, at no point during the experiment, will the difference between the number of heads up to that point and the number of tails up to that point exceed 1 (in absolute value) is

   (i) $1/2^n$.

   (ii) $1/2^{2n}$.

   (iii) $\binom{2n}{n}/2^{2n}$.

   (iv) $\binom{n}{2}/2^{2n}$.

   (v) None of the above.

(c) Suppose it is known that, at the end of the experiment, the difference between the total number of heads and the total number of tails (in absolute value) is at most 2. The probability that the two numbers are actually equal is

(i) 1/5.

(ii) 1/3.

(iii) 1/2.

(iv) $\frac{\binom{2n}{n}}{\binom{2n}{n}+2\binom{2n}{n-1}}$.

(v) None of the above.

(d) Suppose it is known that the total number of heads was only 3. The probability that at the first 2 tosses the coin showed tails is

(i) $\frac{(2n-3)(2n-4)}{2n(2n-1)}$.

(ii) 1/4.

(iii) $\frac{n-1}{n}$.

(iv) $\frac{n-2}{n-1}$.

(v) None of the above.

Solutions

1. (a) Let W be the event whereby Reuven's address is written legibly on the envelope, S – the event whereby the post office does not dispose of the letter, and $R = W \cap S$ – the event whereby Reuven's letter arrives. The probability in question is $P\left(W|\overline{R}\right)$. Now

$$P(R) = P(W)P(S) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6},$$

and therefore:

$$P(W|\overline{R}) = \frac{P(W \cap \overline{R})}{P(\overline{R})} = \frac{P(W)P(\overline{R}|W)}{1 - P(R)} = \frac{\frac{1}{2} \cdot \frac{2}{3}}{1 - \frac{1}{6}} = \frac{2}{5}.$$

Thus, (iii) is true.

(b) Consider Y first. Since we assume all letters to have been correctly placed in the corresponding envelopes, we simply count the envelopes with legible addresses, of which the post office has not disposed. Each envelope has a probability of $1/6$ to survive these obstructions, and therefore $Y \sim B(n, 1/6)$.

Clearly, X assumes all integer values between 0 and n. Since the events whereby distinct letters arrive at their destinations are not independent, one should get the feeling that X is not binomially distributed. We now turn to provide an exact proof of this fact. If X is binomially distributed, then we must have $X \sim B(n, p)$ for some p. Now the event $\{X = n\}$ means that all letters were correctly placed in the corresponding envelopes, which happens with probability $1/n!$, all addresses were legible, which happens with probability $1/2^n$, and the post office did not dispose of any letter, which happens with probability $1/3^n$. Altogether,

$$P(X = n) = \frac{1}{6^n n!}.$$

The event $\{X = n-1\}$ means that all letters were correctly placed in the corresponding envelopes, and that all but one had both legibly written addresses and were not disposed of. Therefore:

$$P(X = n - 1) = \frac{1}{n!} \cdot n \cdot \frac{1}{6^{n-1}} \cdot \frac{5}{6}.$$

Hence

$$p^n = \frac{1}{6^n n!},$$

and

$$np^{n-1}q = \frac{5n}{6^n n!}.$$

Dividing the two equalities by sides, we conclude that $q/p = 5$, which implies that $p = 1/6$. Thus $p^n = 1/6^n$, which yields a contradiction.

Thus, (iv) is true.

(c) This part is solved exactly as in the case presented in class except that, when calculating the probability $P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r})$ of any letters $i_1, i_2, \ldots, i_r$ to reach their destinations, we need to multiply by an extra factor of $1/6^r$. Namely, we need to account for the probability that these r letters have not only been placed in the correct envelopes, but that all corresponding addresses are legible and the post office does not dispose of any of them. Hence, with the notations used in class:

$$P\left(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r}\right) = \frac{1}{n(n-1) \cdot \ldots \cdot (n-r+1)} \cdot \frac{1}{6^r}.$$

Similarly to the case for the original problem,

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \frac{1}{6} - \frac{(1/6)^2}{2!} + \ldots + (-1)^{n-1}\frac{(1/6)^n}{n!},$$

and therefore:

$$P\left(\overline{\bigcup_{i=1}^{n} A_i}\right) = 1 - \frac{1}{6} + \frac{(1/6)^2}{2!} + \ldots + (-1)^{n}\frac{(1/6)^n}{n!}.$$

The right-hand side consists of the first $n+1$ terms in the Maclaurin series of the function $f(x) = e^x$ at the point $-1/6$, and thus

$$p_n \underset{n\to\infty}{\longrightarrow} e^{-1/6}.$$

Thus, (i) is true.

2. (a) Let A_n, $n \geq 1$, be the event whereby the coin shows a head for the first time at the n-th toss. The event that the game continues to the second stage is $\bigcup_{n=1}^{\infty} A_{2n-1}$. Hence the required probability is:

$$P\left(\bigcup_{n=1}^{\infty} A_{2n-1}\right) = \sum_{n=1}^{\infty} \frac{1}{2^{2n-1}}.$$

The series on the right-hand side is a geometric series, and we easily find its sum to be $2/3$.

Thus, (iii) is true.

(b) Let R be the event that Reuven wins a prize, and S the analogous event for Shimon. By the law of total probability:

$$p_R = \sum_{n=0}^{\infty} P\left(A_{4n+1}\right) P\left(R | A_{4n+1}\right).$$

Now the probability of guessing the correct order of a randomly ordered sequence of any length k is $1/k!$, and thus:

$$p_R = \sum_{n=0}^{\infty} \frac{1}{(4n+1)!} \cdot \frac{1}{2^{4n+1}}.$$

Similarly:

$$p_S = \sum_{n=0}^{\infty} \frac{1}{(4n+3)!} \cdot \frac{1}{2^{4n+3}}.$$

Both series are absolutely convergent, and therefore:

$$p_R - p_S = \frac{1}{1!} \cdot \frac{1}{2} - \frac{1}{3!} \cdot \frac{1}{2^3} + \frac{1}{5!} \cdot \frac{1}{2^5} - \frac{1}{7!} \cdot \frac{1}{2^7} + \ldots.$$

The series on the right-hand side is the Maclaurin series of the sine function, calculated at the point $1/2$. Consequently:

$$p_R - p_S = \sin 1/2.$$

Thus, (ii) is true.

3. (a) One easily checks that, if A, B are any events in a probability space, with $A \subseteq B$, then A, B are independent if and only if either $P(A) = 0$ or $P(B) = 1$. Hence, in the case of (i), any independent A, B with $0 < P(A), P(B) < 1$ provide a counter-example.

For both (ii) and (iii), consider the following example. A random family with two children is selected. Consider the following events:

A – the first child is a boy,

B – the second child is a boy,

C – the two children are of the same sex.

A mentioned in class, each pair of events is independent. However,

$$P(A) = P(B) = P(C) = 1/2,$$

and

$$P(B \cup C) = 3/4,\ \ P(B \cap C) = 1/4,\ \ P(A \cap (B \cup C)) = 1/4,$$

which easily implies that neither $A, B \cup C$ nor $A, B \cap C$ are independent.

Under the assumptions of (iv):

$$\begin{aligned} P\left(\left(A \cup \overline{B}\right) \cap C\right) &= P\left((A \cap C) \cup \left(\overline{B} \cap C\right)\right) \\ &= P\left(A \cap C\right) + P\left(\overline{B} \cap C\right) - P\left(A \cap \overline{B} \cap C\right) \\ &= P\left(A\right) P\left(C\right) + P\left(\overline{B}\right) P\left(C\right) - P\left(A\right) P\left(\overline{B}\right) P\left(C\right) \\ &= P\left(C\right) \left(P\left(A\right) + P\left(\overline{B}\right) - P\left(A\right) P\left(\overline{B}\right)\right) \\ &= P\left(C\right) P\left(A \cup \overline{B}\right). \end{aligned}$$

Thus, (iv) is true.

4. (a) Note that, to require that the number of heads in the first n tosses is the same as that in the last n tosses, is tantamount to requiring that the sum of the number of heads in the first n tosses and the number of tails in the last n tosses is n. Referring to a result in the first n tosses as a success if it is a head, and to a result in the last n tosses as a success if it is a tail, we may say that the question is about the probability of n successes in a sequence of $2n$ independent trials, with a probability of success of $1/2$ in each. This probability is $\binom{2n}{n}/2^{2n}$.

Thus, (iii) is true.

(b) The event considered in this part occurs if and only if after the tosses at the odd places, we have each time the opposite result. Namely, the first toss may result in either a head or a tail, but in the second, the coin must show the opposite result. Similarly, the fourth toss must give the opposite of what the third toss has given, and so forth. Thus, at each odd stage we have two options, and at each even stage we have but one. Hence, the event in question comprises 2^n possibilities.

Thus, (i) is true.

(c) Denoting by $A_k,\ 0 \le k \le 2n$, the event whereby the total number of heads is k, the question is about $P(A_n|A_{n-1} \cup A_n \cup A_{n+1})$. We have:

$$\begin{aligned} P(A_n|A_{n-1} \cup A_n \cup A_{n+1}) &= \frac{P(A_n)}{P(A_{n-1} \cup A_n \cup A_{n+1})} \\ &= \frac{\binom{2n}{n}/2^{2n}}{\left(\binom{2n}{n-1} + \binom{2n}{n} + \binom{2n}{n+1}\right)/2^{2n}} \\ &= \frac{\binom{2n}{n}}{\binom{2n}{n} + 2\binom{2n}{n-1}}. \end{aligned}$$

Thus, (iv) is true.

(d) Let A be the event whereby at the first 2 tosses the coin showed tails, and B be the event whereby the total number of heads was only 3. Then $A \cap B$ is the event whereby the first 2 tosses resulted in tails, and out of the following $n-2$ there were exactly 3 heads. Hence:

$$\begin{aligned} P(A|B) &= \frac{1/2^2 \cdot \binom{2n-2}{3}/2^{2n-2}}{\binom{2n}{3}/2^{2n}} \\ &= \frac{\binom{2n-2}{3}}{\binom{2n}{3}} = \frac{(2n-3)(2n-4)}{2n(2n-1)}. \end{aligned}$$

Thus, (i) is true.