Review Questions on 2-Dimensional Distributions

Mark the correct answer in each part of the following questions.

1. Suppose that (X, Y) is a two-dimensional continuous random variable with joint density function, defined by

$$f_{X,Y}(x,y) = \begin{cases} cxye^{-(x^2+y^2)^2}, & x, y \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

for an appropriate constant c.

(a) $c =$

(i) $\sqrt{2\pi}$.

(ii) $2\sqrt{2}$.

(iii) 2π.

(iv) 8.

(v) None of the above.

(b) Let $T = X^2 + Y^2$. The density function f_T of T is given by:

(i) $f_T(t) = \begin{cases} 2te^{-t^2}, & t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$

(ii) $f_T(t) = \begin{cases} \frac{2}{\sqrt{\pi}} e^{-t^2}, & t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$

(iii) $f_T(t) = \begin{cases} \sqrt{\frac{2}{\pi}} e^{-t^2/2}, & t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$

(iv) $f_T(t) = \begin{cases} \frac{4}{\sqrt{\pi}} t^2 e^{-t^2}, & t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$

(v) None of the above.

(c) $E(X^2) =$

(i) $\frac{\sqrt{\pi}}{8}$.

(ii) $\frac{\sqrt{2\pi}}{8}$.

(iii) $\frac{\sqrt{\pi}}{4}$.

(iv) $\frac{\sqrt{2\pi}}{4}$.

(v) None of the above.

(d) Let $S = \operatorname{arctg} \frac{Y}{X}$. Then $E(S) =$

(i) $\frac{\pi}{8}$.

(ii) $\frac{\pi}{6}$.

(iii) $\frac{\pi}{4}$.

(iv) $\frac{\pi}{3}$.

(v) None of the above.

(e) $\rho(X/Y, Y/X)$

(i) belongs to the interval $(-1, 0)$.
(ii) belongs to the interval $(0, 1)$.
(iii) is 0.
(iv) is undefined.
(v) None of the above.

2. Suppose that (X, Y) is a two-dimensional continuous random variable with joint density function, defined by

$$f_{X,Y}(x,y) = \begin{cases} c\sin(x+y), & 0 \le x, y \le \pi/2, \\ 0, & \text{otherwise}, \end{cases}$$

for an appropriate constant c.

(a) $c =$

(i) $\frac{4}{\pi^2}$.

(ii) $\frac{1}{2}$.

(iii) $\frac{8}{\pi^2}$.

(iv) 1.

(v) None of the above.

(b) $E(X) =$

(i) $\frac{\pi}{6}$.

(ii) $\frac{\pi}{4}$.

(iii) $\frac{\pi}{3}$.

(iv) $\frac{\pi\sqrt{2}}{4}$.

(v) None of the above.

(c)

(i) $\rho(X, Y) = -1$.

(ii) $-1 < \rho(X, Y) < 0$.

(iii) $\rho(X, Y) = 0$.

(iv) $\rho(X, Y)$ is undefined.

(v) None of the above.

(d) Let $S = X + Y$. The density function f_S is given by:

(i) $$f_S(s) = \begin{cases} \frac{1}{2} s \sin s, & 0 \le s \le \pi/2, \\ \frac{1}{2}(\pi - s) \sin s, & \pi/2 < s \le \pi, \\ 0, & \text{otherwise.} \end{cases}$$

(ii) $$f_S(s) = \begin{cases} \frac{1}{2} s \sin^2 s, & 0 \le s \le \pi/2, \\ \frac{1}{2}(\pi - s) \sin^2 s, & \pi/2 < s \le \pi, \\ 0, & \text{otherwise.} \end{cases}$$

(iii) $$f_S(s) = \begin{cases} \frac{1}{2} s |\cos s|, & 0 \le s \le \pi/2, \\ \frac{1}{2}(\pi - s) |\cos s|, & \pi/2 < s \le \pi, \\ 0, & \text{otherwise.} \end{cases}$$

(iv) $$f_S(s) = \begin{cases} \frac{1}{2} s \cos^2 s, & 0 \le s \le \pi/2, \\ \frac{1}{2}(\pi - s) \cos^2 s, & \pi/2 < s \le \pi, \\ 0, & \text{otherwise.} \end{cases}$$

(v) None of the above.

3. Suppose that (X, Y) is a two-dimensional continuous random variable with joint density function, defined by

$$f_{X,Y}(x, y) = \begin{cases} cxe^{-xy}, & x, y \ge 1, \\ 0, & \text{otherwise,} \end{cases}$$

for an appropriate constant c.

(a) $c =$

(i) $\frac{1}{e^2}$.

(ii) $\frac{1}{e}$.

(iii) 1.

(iv) e.

(v) None of the above.

(b) The following random variable is Exp(1)-distributed:

(i) $X - 1$.
(ii) $(e-1)(X-1)$.
(iii) $2(X-1)$.
(iv) $e(X-1)$.
(v) None of the above.

(c) $E(XY) =$

(i) 1.

(ii) 2.

(iii) 3.

(iv) 4.

(v) None of the above.

(d) Let $T = XY$. The distribution function F_T is given by:

(i) $f_T(t) = \begin{cases} 1 - te^{-(t-1)}, & t \geq 1, \\ 0, & \text{otherwise.} \end{cases}$

(ii) $f_T(t) = \begin{cases} 1 - (2t-1)e^{-(t-1)}, & t \geq 1, \\ 0, & \text{otherwise.} \end{cases}$

(iii) $f_T(t) = \begin{cases} 1 - (3t-2)e^{-(t-1)}, & t \geq 1, \\ 0, & \text{otherwise.} \end{cases}$

(iv) $f_T(t) = \begin{cases} 1-(4t-3)e^{-(t-1)}, & t \geq 1, \\ 0, & \text{otherwise.} \end{cases}$

(v) None of the above.

4. Suppose that (X, Y) is a two-dimensional continuous random variable with joint density function, defined by

$$f_{X,Y}(x,y) = \begin{cases} c\left(x+y^2\right), & 0 \leq x, y \leq 1,\ x^2+y^2 \geq 1, \\ \\ 0, & \text{otherwise,} \end{cases}$$

for an appropriate constant c.

(a) $c =$

(i) $\dfrac{12}{8-\pi}$.

(ii) $\dfrac{15}{8-\pi}$.

(iii) $\dfrac{16}{8-\pi}$.

(iv) $\dfrac{18}{8-\pi}$.

(v) None of the above.

(b) $\mathrm{Cov}(X, Y) =$

(i) $\dfrac{-(752-210\pi)}{225(8-\pi)}$.

(ii) $\dfrac{-(754-210\pi)}{225(8-\pi)}$.

(iii) $\dfrac{-(758-210\pi)}{225(8-\pi)}$.

(iv) $\dfrac{-(764-210\pi)}{225(8-\pi)}$.

(v) None of the above.

(c) $P(X > Y) =$

(i) $\dfrac{44-16\sqrt{2}-3\pi}{6(8-\pi)}$.

(ii) $\dfrac{45 - 16\sqrt{2} - 3\pi}{6(8 - \pi)}$.

(iii) $\dfrac{46 - 16\sqrt{2} - 3\pi}{6(8 - \pi)}$.

(iv) $\dfrac{47 - 16\sqrt{2} - 3\pi}{6(8 - \pi)}$.

(v) None of the above.

Review Solutions

1. **(a)** We have

$$1 = c\int_0^\infty \int_0^\infty xye^{(x^2+y^2)^2} dxdy = c\int_0^\infty r^3 e^{-r^4} dr \int_0^{\pi/2} \sin\theta\cos\theta d\theta$$
$$= c\cdot\left[-\frac{1}{4}e^{-r^4}\right]_0^\infty \cdot \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = c\cdot\frac{1}{4}\cdot\frac{1}{2} = \frac{c}{8}.$$

Hence $c = 8$.

Thus, (iv) is true.

(b) Clearly $F_T(t) = 0$ for $t < 0$. For $t > 0$ we have:

$$F_T(t) = P(X^2 + Y^2 \le t) = P(\sqrt{X^2+Y^2} \le \sqrt{t})$$
$$= 8\int_0^{\sqrt{t}} r^3 e^{-r^4} dr \int_0^{\pi/2} \sin\theta\cos\theta d\theta$$
$$= 8\cdot\left[-\frac{1}{4}e^{-r^4}\right]_0^{\sqrt{t}} \cdot \frac{1}{2} = 8\cdot\frac{1}{2}\cdot\frac{1}{4}\left(1 - e^{-t^2}\right) = 1 - e^{-t^2}.$$

Hence

$$F_T(t) = \begin{cases} 1 - e^{-t^2}, & t \ge 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$f_T(t) = \begin{cases} 2te^{-t^2}, & t \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, (i) is true.

(c) We have:

$$E(X^2) = 8\int_0^\infty \int_0^{\pi/2} r^3 e^{-r^4} \sin\theta\cos\theta \cdot r^2\cos^2\theta d\theta dr$$
$$= 8\int_0^\infty r^2\cdot r^3 e^{-r^4} dr \int_0^{\pi/2} \sin\theta\cos^3\theta d\theta$$
$$= 8\cdot\left(\left[-\frac{r^2}{4}e^{-r^4}\right]_0^\infty + \frac{1}{4}\int_0^\infty e^{-r^4}\cdot 2rdr\right)\cdot\left[-\frac{\cos^4\theta}{4}\right]_0^{\pi/2}$$
$$= \frac{1}{2}\int_0^\infty e^{-r^4}\cdot 2rdr = \frac{1}{2}\int_0^\infty e^{-t^2} dt = \frac{1}{2}\cdot\frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{4}.$$

Thus, (iii) is true.

(d) We have:

$$
\begin{aligned}
E\left(\arctan(Y/X)\right) &= 8\int_0^\infty \int_0^{\pi/2} r^3 e^{-r^4} \sin\theta\cos\theta \cdot \theta d\theta dr \\
&= 8\int_0^\infty r^3 e^{-r^4} dr \int_0^{\pi/2} \sin\theta\cos\theta \cdot \theta d\theta \\
&= 8 \cdot \frac{1}{4} \cdot \left(\left[\frac{\sin^2\theta}{2} \cdot \theta \right]_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin^2\theta}{2} d\theta \right) \\
&= 2 \cdot \left(\frac{\pi}{4} - \frac{1}{2}\int_0^{\pi/2} \frac{1-\cos 2\theta}{2} d\theta \right) \\
&= 2 \cdot \left(\frac{\pi}{4} - \frac{1}{2} \cdot \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_0^{\pi/2} \right) = 2 \cdot \frac{\pi}{8} = \frac{\pi}{4}.
\end{aligned}
$$

An alternative to this calculation is to note that the variables x and y play a symmetric role in $f_{X,Y}$, and hence the distribution is symmetric with respect to the diagonal $x = y$. It follows that the random variable $\Theta = \arctan(Y/X)$ is symmetric with respect to $\frac{\pi}{4}$, and in particular its expectation is $\frac{\pi}{4}$.

Thus, (iii) is true.

(e) We have:

$$
\begin{aligned}
E(Y/X) &= 8\int_0^\infty \int_0^{\pi/2} r^3 e^{-r^4} \sin\theta\cos\theta \cdot (r\sin\theta / r\cos\theta) d\theta dr \\
&= 8\int_0^\infty r^3 e^{-r^4} dr \int_0^{\pi/2} \sin^2\theta d\theta = 8 \cdot \frac{1}{4} \cdot \frac{\pi}{4} = \frac{\pi}{2}.
\end{aligned}
$$

By symmetry

$$E(Y/X) = E(X/Y).$$

Now

$$E(Y/X \cdot X/Y) = E(1) = 1,$$

and

$$\mathrm{Cov}(Y/X, X/Y) = 1 - \frac{\pi^2}{4} \approx -1.46.$$

Thus, (i) is true.

2. **(a)** We have

$$
\begin{aligned}
1 &= c\int_0^{\pi/2}\int_0^{\pi/2} \sin(x+y)dxdy = c\int_0^{\pi/2}\int_0^{\pi/2} \sin x\cos y + \cos x \sin y dxdy \\
&= c\int_0^{\pi/2} \sin x dx \int_0^{\pi/2} \cos y dy + c\int_0^{\pi/2} \cos x dx \int_0^{\pi/2} \sin y dy \\
&= 2c\int_0^{\pi/2} \sin x dx \int_0^{\pi/2} \cos y dy \\
&= 2c\cdot [-\cos x]_0^{\pi/2}\cdot [\sin y]_0^{\pi/2} = 2c.
\end{aligned}
$$

Hence $c = \frac{1}{2}$.
Thus, (ii) is true.

(b) We have

$$
\begin{aligned}
E(X) &= \frac{1}{2}\cdot\int_0^{\pi/2}\int_0^{\pi/2} x\sin(x+y)dxdy \\
&= \frac{1}{2}\cdot\int_0^{\pi/2}\int_0^{\pi/2} x\left(\sin x\cos y + \cos x\sin y\right)dxdy \\
&= \frac{1}{2}\cdot\int_0^{\pi/2} x\cdot\sin x dx\int_0^{\pi/2}\cos y dy + c\int_0^{\pi/2} x\cdot\cos x dx\int_0^{\pi/2}\sin y dy \\
&= \frac{1}{2}\cdot\int_0^{\pi/2} x\cdot\sin x dx + c\int_0^{\pi/2} x\cdot\cos x dx \\
&= \frac{1}{2}\cdot\left([-x\cdot\cos x]_0^{\pi/2} + \int_0^{\pi/2}\cos x dx + [x\cdot\sin x]_0^{\pi/2} - \int_0^{\pi/2}\sin x dx\right) \\
&= \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}.
\end{aligned}
$$

Thus, (ii) is true.

(c) By symmetry $E(Y) = E(X) = \dfrac{\pi}{4}$. We have:

$$
\begin{aligned}
E(XY) &= \frac{1}{2}\cdot\int_0^{\pi/2}\int_0^{\pi/2} xy\sin(x+y)dxdy \\
&= \frac{1}{2}\cdot\int_0^{\pi/2}\int_0^{\pi/2} xy\left(\sin x\cos y + \cos x\sin y\right)dxdy \\
&= \frac{1}{2}\cdot\int_0^{\pi/2} x\cdot\sin x dx\int_0^{\pi/2} y\cdot\cos y dy + \frac{1}{2}\cdot\int_0^{\pi/2} x\cdot\cos x dx\int_0^{\pi/2} y\cdot\sin y dy \\
&= \int_0^{\pi/2} x\cdot\sin x dx\int_0^{\pi/2} y\cdot\cos y dy \\
&= (0+1)\left(\frac{\pi}{2} - 1\right) = \frac{\pi}{2} - 1.
\end{aligned}
$$

Thus $\text{Cov}\,(X,Y) = \frac{\pi}{2} - 1 - \frac{\pi^2}{16} \approx -0.046$, and thus $\rho(X,Y) < 0$. As X is not a linear function of Y, we have $\rho(X,Y) \neq -1$.

It will be instructive to compute $\rho(X,Y)$ exactly. We have:

$$
\begin{aligned}
E(X^2) &= \frac{1}{2} \cdot \int_0^{\pi/2} \int_0^{\pi/2} x^2 \sin(x+y) dx dy \\
&= \frac{1}{2} \cdot \int_0^{\pi/2} x^2 \cdot \sin x dx \int_0^{\pi/2} \cos y dy + \frac{1}{2} \cdot \int_0^{\pi/2} x^2 \cdot \cos x dx \int_0^{\pi/2} \sin y dy \\
&= \frac{1}{2} \cdot \int_0^{\pi/2} x^2 \cdot \sin x dx + \frac{1}{2} \cdot \int_0^{\pi/2} x^2 \cdot \cos x dx \\
&= \frac{1}{2} \cdot \left(\left[-x^2 \cdot \cos x\right]_0^{\pi/2} + \int_0^{\pi/2} 2x \cdot \cos x dx + \left[x^2 \cdot \sin x\right]_0^{\pi/2} - \int_0^{\pi/2} 2x \cdot \sin x dx \right) \\
&= \frac{1}{2} \cdot \left(0 + \int_0^{\pi/2} 2x \cdot \cos x dx + \frac{\pi^2}{4} - \int_0^{\pi/2} 2x \cdot \sin x dx \right) \\
&= \frac{\pi^2}{8} + \frac{1}{2} \cdot \left([2x \cdot \sin x]_0^{\pi/2} - 2 \int_0^{\pi/2} \sin x dx - [-2x \cdot \cos x]_0^{\pi/2} - 2 \int_0^{\pi/2} \cos x dx \right) \\
&= \frac{\pi^2}{8} + \frac{1}{2} (\pi - 2 - 2) = \frac{\pi^2}{8} + \frac{\pi}{2} - 2.
\end{aligned}
$$

Hence, and by symmetry:

$$
V(Y) = V(X) = \frac{\pi^2}{8} + \frac{\pi}{2} - 2 - \left(\frac{\pi}{4}\right)^2 = \frac{\pi^2}{16} + \frac{\pi}{2} - 2.
$$

Thus:

$$
\rho(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{V(X)}\sqrt{V(Y)}} = \frac{\frac{\pi}{2} - \frac{\pi^2}{16} - 1}{\frac{\pi}{2} + \frac{\pi^2}{16} - 2} \approx -0.245.
$$

Thus , (ii) is true.

(d) Clearly, $F_S(t) = 0$, for $t < 0$ and $F_S(t) = 1$ for $t > \pi$.

For $0 < t < \frac{\pi}{2}$ we have:

$$
\begin{aligned}
F_S(t) = P(X+Y \le t) &= \frac{1}{2} \cdot \int_0^t \int_0^{t-x} (\sin x \cos y + \cos x \sin y)\, dydx \\
&= \frac{1}{2} \cdot \int_0^t \sin x \int_0^{t-x} \cos y dyx + \frac{1}{2} \cdot \int_0^t \cos x \int_0^{t-x} \sin y dydx \\
&= \frac{1}{2} \cdot \int_0^t \sin x \,[\sin y]_0^{t-x}\, dx + \frac{1}{2} \cdot \int_0^t \cos x \,[-\cos y]_0^{t-x}\, dx \\
&= \frac{1}{2} \cdot \int_0^t \sin x \cdot \sin (t-x)\, dx + \frac{1}{2} \cdot \int_0^t \cos x \,(1 - \cos (t-x))\, dx \\
&= \frac{1}{2} \cdot \int_0^t (\sin x \cdot \sin (t-x)\, dx - \cos x \cdot \cos (t-x))\, dx + \frac{1}{2} \cdot \int_0^t \cos x dx \\
&= \frac{1}{2} \cdot \int_0^t (\cos x - \cos t)\, dx \\
&= \frac{1}{2} \cdot [\sin x - x \cdot \cos t]_0^t = \frac{1}{2} \cdot (\sin t - t \cdot \cos t)\,.
\end{aligned}
$$

Thus, $f_S(t) = \frac{1}{2} t \sin t$ for $0 < t < \frac{\pi}{2}$. Similarly, for $\frac{\pi}{2} < t < \pi$ we have $f_S(t) = \frac{1}{2}(\pi - t)\sin t$.

Thus , (i) is true.

3. **(a)** We have

$$
\begin{aligned}
1 &= c \int_1^\infty \int_1^\infty x e^{-xy} dydx = c \int_1^\infty \left[-e^{-xy}\right]_1^\infty dx \\
&= c \int_1^\infty e^{-x} dx = c\left[-e^{-x}\right]_1^\infty = c \cdot e^{-1}.
\end{aligned}
$$

Hence $c = e$.
Thus, (iv) is true.

(b) Let $Y = X - 1$. Clearly, $F_Y(t) = 0$ for $t < 0$. For $t \ge 0$ we have:

$$
\begin{aligned}
F_Y(t) &= P(X - 1 \le t) = P(X \le t+1) = 1 - P(X > t+1) \\
&= 1 - e \int_{t+1}^\infty \int_1^\infty x e^{-xy} dydx = 1 - e \int_{t+1}^\infty x \cdot \left[-\frac{1}{x} e^{-xy}\right]_1^\infty dx \\
&= 1 - e \int_{t+1}^\infty e^{-x} dx = 1 - e\left[-e^{-x}\right]_{t+1}^\infty = 1 - e \cdot e^{-t-1} = 1 - e^{-t}.
\end{aligned}
$$

Hence, $Y \sim \text{Exp}(1)$.
Thus, (i) is true.

(c) We have:

$$\begin{aligned}
E(XY) &= e\int_1^\infty \int_1^\infty xy\cdot xe^{-xy}dydx \\
&= e\int_1^\infty x\left(\left[-ye^{-xy}\right]_1^\infty + \int_1^\infty e^{-xy}dy\right)dx \\
&= e\int_1^\infty x\left(e^{-x} + \left[-\frac{1}{x}e^{-xy}\right]_1^\infty\right)dx \\
&= e\int_1^\infty x\left(e^{-x} + \frac{1}{x}e^{-x}\right)dx \\
&= e\int_1^\infty \left(xe^{-x} + e^{-x}\right)dx \\
&= e\left(\left[-xe^{-x}\right]_1^\infty + \int_1^\infty e^{-x}dx + \int_1^\infty e^{-x}dx\right) \\
&= e\left(e^{-1} + 2\cdot e^{-1}\right) = 3.
\end{aligned}$$

Thus, (iii) is true.

(d) Clearly, $F_T(t) = 0$ for $t < 1$. For $t \geq 1$:

$$\begin{aligned}
F_T(t) = P(XY \leq t) &= e\int_1^t \int_1^{t/x} xe^{-xy}dydx \\
&= e\int_1^t x\cdot\left[-\frac{1}{x}e^{-xy}\right]_1^{t/x} dx = e\int_1^t x\left(\frac{1}{x}e^{-x} - \frac{1}{x}e^{-t}\right)dx = \\
&= e\int_1^t \left(e^{-x} - e^{-t}\right)dx = e\left[-e^{-x} - xe^{-t}\right]_1^t \\
&= e\left(-e^{-t} - te^{-t} + e^{-1} + e^{-t}\right) = 1 - te^{1-t}.
\end{aligned}$$

Thus , (i) is true.

4. **(a)** We have

$$1 = c\iint_{[0,1]^2}\left(x+y^2\right)dydx - c\iint_S\left(x+y^2\right)dydx,$$

where $S = \{(x,y) : 0 \le x, y \le 1, x^2+y^2 \le 1\}$. Now:

$$\begin{aligned}
&\int_0^1 \int_0^1 \left(x+y^2\right) dydx - \int_0^1 \int_0^{\pi/2} \left(r^2\cos\theta + r^3\sin^2\theta\right) d\theta dr \\
&= \int_0^1 \left[xy + \frac{y^3}{3}\right]_0^1 dx - \int_0^1 r^2 dr \int_0^{\pi/2} \cos\theta d\theta - \int_0^1 r^3 dr \int_0^{\pi/2} \sin^2\theta d\theta \\
&= \int_0^1 \left(x + \frac{1}{3}\right) dx - \left[\frac{r^3}{3}\right]_0^1 \cdot [\sin\theta]_0^{\pi/2} - \left[\frac{r^4}{4}\right]_0^1 \cdot \frac{1}{2} \cdot \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2} \\
&= \left[\frac{x^2}{2} + \frac{x}{3}\right]_0^1 - \frac{1}{3} - \frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{1}{2} + \frac{1}{3} - \frac{1}{3} - \frac{\pi}{16} = \frac{8-\pi}{16}.
\end{aligned}$$

Hence $c = \dfrac{16}{8-\pi}$.

Thus, (iii) is true.

(b) We have

$$\begin{aligned}
E(X) &= c\int_0^1 \int_0^1 \left(x^2 + xy^2\right) dydx - c\int_0^1 \int_0^{\pi/2} \left(r^3\cos^2\theta + r^4\cos\theta\sin^2\theta\right) d\theta dr \\
&= c\int_0^1 \left[x^2 y + \frac{xy^3}{3}\right]_0^1 dx - c\int_0^1 r^3 dr \int_0^{\pi/2} \cos^2\theta d\theta - c\int_0^1 r^4 dr \int_0^{\pi/2} \cos\theta\sin^2\theta d\theta \\
&= c\int_0^1 \left(x^2 + \frac{x}{3}\right) dx - c\cdot\left[\frac{r^4}{4}\right]_0^1 \cdot \frac{1}{2}\cdot\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} - c\cdot\left[\frac{r^5}{5}\right]_0^1 \cdot \left[\frac{\sin^3\theta}{3}\right]_0^{\pi/2} \\
&= c\cdot\left[\frac{x^3}{3} + \frac{x^2}{6}\right]_0^1 - c\cdot\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} - c\cdot\frac{1}{5}\cdot\frac{1}{3} = c\left(\frac{1}{3} + \frac{1}{6} - \frac{1}{15} - \frac{\pi}{16}\right) = c\left(\frac{13}{30} - \frac{\pi}{16}\right),
\end{aligned}$$

and

$$\begin{aligned}
E(Y) &= c\int_0^1 \int_0^1 \left(xy + y^3\right) dydx - c\int_0^1 \int_0^{\pi/2} \left(r^3\cos\theta\sin\theta + r^4\sin^3\theta\right) d\theta dr \\
&= c\int_0^1 \left[\frac{xy^2}{2} + \frac{y^4}{4}\right]_0^1 dx - c\int_0^1 r^3 dr \int_0^{\pi/2} \cos\theta\sin\theta d\theta - c\int_0^1 r^4 dr \int_0^{\pi/2} \sin^3\theta d\theta \\
&= c\int_0^1 \left(\frac{x}{2} + \frac{1}{4}\right) dx - c\cdot\left[\frac{r^4}{4}\right]_0^1 \cdot \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} - c\cdot\left[\frac{r^5}{5}\right]_0^1 \cdot \int_0^{\pi/2} \frac{3\sin\theta - \sin 3\theta}{4} d\theta \\
&= c\cdot\left[\frac{x^2}{4} + \frac{x}{4}\right]_0^1 - c\cdot\frac{1}{4}\cdot\frac{1}{2} - c\cdot\frac{1}{5}\cdot\frac{1}{4}\cdot\left[-3\cos\theta + \frac{\cos 3\theta}{3}\right]_0^{\pi/2} \\
&= c\left(\frac{1}{4} + \frac{1}{4} - \frac{1}{8} - \frac{1}{20}\left(3 - \frac{1}{3}\right)\right) = c\left(\frac{3}{8} - \frac{2}{15}\right) = c\cdot\frac{29}{120}.
\end{aligned}$$

Also

$$
\begin{aligned}
E(XY) =\ & c\int_0^1\int_0^1\left(x^2y+xy^3\right)dydx - c\int_0^1\int_0^{\pi/2}\left(r^4\cos^2\theta\sin\theta + r^5\cos\theta\sin^3\theta\right)d\theta dr \\
=\ & c\int_0^1\left[\frac{x^2y^2}{2}+\frac{xy^4}{4}\right]_0^1 dx - c\int_0^1 r^4dr\int_0^{\pi/2}\cos^2\theta\sin\theta d\theta \\
& - c\int_0^1 r^5dr\int_0^{\pi/2}\cos\theta\sin^3\theta d\theta \\
=\ & c\int_0^1\left(\frac{x^2}{2}+\frac{x}{4}\right)dx - c\cdot\left[\frac{r^5}{5}\right]_0^1\cdot\left[-\frac{\cos^3\theta}{3}\right]_0^{\pi/2} - c\cdot\left[\frac{r^6}{6}\right]_0^1\cdot\left[\frac{\sin^4\theta}{4}\right]_0^{\pi/2} \\
=\ & c\cdot\left[\frac{x^3}{6}+\frac{x^2}{8}\right]_0^1 - c\cdot\frac{1}{5}\cdot\frac{1}{3} - c\cdot\frac{1}{6}\cdot\frac{1}{4} \\
=\ & c\left(\frac{1}{6}+\frac{1}{8}-\frac{1}{15}-\frac{1}{24}\right) = c\cdot\frac{11}{60}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\operatorname{Cov}(X,Y) &= c\cdot\frac{11}{60} - c^2\left(\frac{13}{30}-\frac{\pi}{16}\right)\cdot\frac{29}{120} \\
&= \frac{16}{8-\pi}\cdot\frac{11}{60} - \left(\frac{16}{8-\pi}\right)^2\cdot\left(\frac{13}{30}-\frac{\pi}{16}\right)\cdot\frac{29}{120} \\
&= \frac{44}{15\left(8-\pi\right)} - \frac{16\cdot 16}{\left(8-\pi\right)^2}\cdot\frac{13\cdot 16-30\pi}{30\cdot 16}\cdot\frac{29}{120} \\
&= \frac{44\cdot 15\left(8-\pi\right)}{225\left(8-\pi\right)^2} - \frac{2}{\left(8-\pi\right)^2}\cdot\frac{13\cdot 8-15\pi}{15}\cdot\frac{29}{15} \\
&= \frac{44\cdot 15\cdot 8 - 44\cdot 15\cdot\pi - 2\cdot 13\cdot 8\cdot 29 + 2\cdot 15\cdot 29\cdot\pi}{225\left(8-\pi\right)^2} \\
&= \frac{-752+210\pi}{225\left(8-\pi\right)^2}.
\end{aligned}
$$

Thus, (i) is true.

(c) We have

$$
P\left(X>Y\right) = c\iint_A\left(x+y^2\right)dydx,
$$

where $A = \{(x,y) : 0 \le x, y \le 1,\ x^2 + y^2 \le 1,\ x > y\}$. Thus:

$$\begin{aligned}
P(X > Y) &= c \iint_{[0,1]^2 \cap A} \left(x + y^2\right) dydx - c \iint_{S \cap A} \left(x + y^2\right) dydx \\
&= c \int_0^1 \int_0^x \left(x + y^2\right) dydx - c \int_0^1 \int_0^{\pi/4} \left(r^2 \cos\theta + r^3 \sin^2\theta\right) d\theta dr \\
&= c \int_0^1 \left[xy + \frac{y^3}{3}\right]_0^x dx - c \int_0^1 r^2 dr \int_0^{\pi/4} \cos\theta d\theta - c \int_0^1 r^3 dr \int_0^{\pi/4} \sin^2\theta d\theta \\
&= c \int_0^1 \left(x^2 + \frac{x^3}{3}\right) dx - c \cdot \left[\frac{r^3}{3}\right]_0^1 \cdot [\sin\theta]_0^{\pi/4} - c \cdot \left[\frac{r^4}{4}\right]_0^1 \cdot \frac{1}{2} \cdot \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/4} \\
&= c \cdot \left[\frac{x^3}{3} + \frac{x^4}{12}\right]_0^1 - c \cdot \frac{1}{3} \cdot \frac{\sqrt{2}}{2} - c \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \left(\frac{\pi}{4} - \frac{1}{2}\right) \\
&= \frac{16}{8 - \pi} \left(\frac{1}{3} + \frac{1}{12} - \frac{\sqrt{2}}{6} - \frac{\pi}{32} + \frac{1}{16}\right) = \frac{46 - 16\sqrt{2} - 3\pi}{6(8 - \pi)}.
\end{aligned}$$

Thus, (iii) is true.