Midterm

Mark the correct answer in each part of the following questions.

1. Consider the absent-minded secretary problem.

   (a) Suppose $n = 2k$, and that k of the addresses are in Nicaragua and the other k in Zimbabwe. The probability that the secretary will send all k letters intended to the Nicaraguan addressees to Zimbabwe and vice versa is:

   (i) $1/n!$.
   (ii) $1/2^{2n}$.
   (iii) $1/2^n$.
   (iv) $1/\binom{2n}{n}$.
   (v) None of the above.

   (b) Let us say (for the purposes of this question only) that a possible way of placing the letters in the envelopes is *cyclic* if, for some ordering of the addressees, the letter intended for addressee 1 has been sent to addressee 2, the letter intended for addressee 2 has been sent to addressee 3, ..., the letter intended for addressee $n-1$ has been sent to addressee n, and the letter intended for addressee n has been sent to addressee 1. (For example, for $n = 3$, if the letter intended for addressee 1 has been sent to addressee 3, the letter intended for 3 has been sent to addressee 2, and the letter intended for addressee 2 has been sent to addressee 1, then the placing of the letters is cyclic. However, if the letter intended for addressee 1 has been sent to addressee 3, the letter intended for 3 has been sent to addressee 1, and the letter intended for addressee 2 has been sent to addressee 2, then the placing of the letters is not not such.)

   Denote by p_n the probability that the placing of the letters is cyclic.

   (i) $p_n = 1/n!$ for all sufficiently large n.
   (ii) $p_n = 1/n$ for all sufficiently large n.
   (iii) $p_n = 1/(n-1)$ for all sufficiently large n.
   (iv) $p_n \longrightarrow_{n\to\infty} 1$.
   (v) None of the above.

(c) Let us say that a pair of letters i and j (with $1 \le i < j \le n$) has been *switched* if the letter intended for addressee i has been sent to addressee j and vice versa. Denote by p'_n the probability that no pair of letters has been switched.

(i) $p'_n \longrightarrow_{n\to\infty} 1/e^2$.
(ii) $p'_n \longrightarrow_{n\to\infty} 1/e$.
(iii) $p'_n \longrightarrow_{n\to\infty} 1/\sqrt{e}$.
(iv) $p'_n \longrightarrow_{n\to\infty} 1-1/e$.
(v) None of the above.

2. A die is tossed n times.

(a) The probability that the sum of upfaces is $n+3$ is

(i) $\frac{n+\binom{n}{2}+\binom{n}{3}}{6^n}$.
(ii) $\frac{n+\binom{n}{2}+2\binom{n}{3}}{6^n}$.
(iii) $\frac{n+2\binom{n}{2}+\binom{n}{3}}{6^n}$.
(iv) $\frac{n+2\binom{n}{2}+2\binom{n}{3}}{6^n}$.
(v) None of the above.

(b) Suppose it is known that the sum of upfaces has been $n+2$. The probability that the outcome of the first toss was 1 is

(i) $\frac{n-2}{n}$.
(ii) $\frac{n-1}{n+1}$.
(iii) $\frac{n-1}{n}$.
(iv) $\frac{n}{n+1}$.
(v) None of the above.

(c) Let X be the random variable counting the j's between 1 and $n-1$, for which the outcome of the j-th toss is distinct from that of the $(j+1)$-st toss.

(i) $X \sim U[0, n-1]$.
(ii) $X \sim U[1, n-1]$.
(iii) $X \sim B(n-2, 5/6)$.
(iv) $X \sim B(n-1, 5/6)$.
(v) None of the above.

(d) Now suppose that the die has the following properties: In a sequence of tosses, in the first toss, each of the 6 possible outcomes has a probability of 1/6. Later, if the preceding k tosses had the same outcome j (but the toss just before these k tosses had a different outcome), then the probability for the next toss to result in a j is $\frac{k}{k+1}$, while each of the others has the same probability $\frac{1}{5(k+1)}$. (For example, if the results up to this point have been $3, 2, 2, 2, 6, 5, 5, 5$, then next time the die will show a 5 with probability 0.75 and each of $1, 2, 3, 4, 6$ with probability 0.05.) Let the die be tossed an infinite number of times. The probability that it will show the same number in all tosses is

(i) 0.
(ii) $\frac{1}{e^6}$.
(iii) $\frac{1}{e^5}$.
(iv) $\frac{1}{e}$.
(v) None of the above.

3. Reuven and Shimon participate in the following game. At the first stage of the game, three coins are tossed again and again, until for the first time all three show the same side. If this stage has been finished after n steps, where n is even, Reuven receives a prize. The, whether or not n is even or not, a bunch of cards of size n is formed, in which one card is marked, while all others are not. Shimon draws a card randomly. If he draws the marked card, he wins a prize. (Note that it is not necessarily the case that exactly one of the two players wins a prize.)

   (a) The probability for Reuven to win a prize is

   (i) 3/7.
   (ii) 3/16.
   (iii) 9/16.
   (iv) 1/2.
   (v) None of the above.

   (b) The probability for Shimon to win a prize is

   (i) $\ln \frac{4}{3}$.
   (ii) $2 \ln \frac{4}{3}$.
   (iii) $\frac{2}{3} \cdot \ln 2$.
   (iv) $\ln 2$.
   (v) None of the above.

4. Let A, B, C be events with positive probabilities in a probability space.

(a)

- **(i)** If $P(A|B) > P(A)$ and $P(B|C) > P(B)$, then $P(A|C) > P(A)$.
- **(ii)** If $P(A|B) > P(A)$ and $P(A|C) > P(A)$, then $P(A|B \cup C) > P(A)$.
- **(iii)** If $P(A|B) > P(A)$, then $P(B|A) > P(B)$.
- **(iv)** If $B \supset C$ and $P(B) > P(C)$, then $P(A|B) < P(A|C)$.
- **(v)** None of the above.

Solutions

1. (a) There are $\binom{2k}{k}$ possibilities of choosing the k recipients for the letters addressed to Zimbabwe. Only one satisfies the condition, and therefore the probability is $\frac{1}{\binom{2k}{k}}$.

   Thus, (iv) is true.

   (b) Let $L_1, L_2, \ldots, L_n$ be the letters, and $E_1, E_2, \ldots, E_n$ the corresponding envelopes. Start with an arbitrary letter L_{i_1}. It must be placed in some envelope E_{i_2} with $i_2 \neq i_1$, which happens with probability $\frac{n-1}{n}$. Now consider the letter L_{i_2}. It must be placed in some E_{i_3}, with $i_3 \neq i_1$, which happens with probability of $\frac{n-2}{n-1}$. (Note that we do not have to require that $i_3 \neq i_2$, as E_{i_2} is already occupied.) Continuing this way, we see that the required probability is

$$\frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \ldots \cdot \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{n}.$$

   Thus, (ii) is true.

   (c) Let $A_{i,j}$ be the event that letters i and j have been switched. Thus the event in question is

$$\overline{\bigcup_{1 \leq i < j \leq n} A_{i,j}}.$$

   Consider the event $A_{i_1,j_1} \cap A_{i_2,j_2} \cap \ldots \cap A_{i_r,j_r}$. If, for some $1 \leq k < m \leq r$, we have $\{i_k, j_k\} \cap \{i_m, j_m\} \neq \emptyset$, then

$$P\left(A_{i_1,j_1} \cap A_{i_2,j_2} \cap \ldots \cap A_{i_r,j_r}\right) = 0.$$

   Let $A_{i_1,j_1}, A_{i_2,j_2}, \ldots, A_{i_r,j_r}$ be r events with pairwise disjoint indices. By the multiplication rule

$$P\left(A_{i_1,j_1} \cap A_{i_2,j_2} \cap \ldots \cap A_{i_r,j_r}\right) = \frac{1}{n(n-1)} \cdot \frac{1}{(n-2)(n-3)} \cdot \ldots \cdot \frac{1}{(n-(2r-2))(n-(2r-1))}.$$

   The number of events $A_{i_1,j_1} \cap A_{i_2,j_2} \cap \ldots \cap A_{i_r,j_r}$ with pairwise disjoint indices is

$$\frac{\binom{n}{2}\binom{n-2}{2} \cdot \ldots \cdot \binom{n-2(r-1)}{2}}{r!}.$$

Let $N = \left\lfloor \frac{n}{2} \right\rfloor$. Altogether

$$P\left(\bigcup_{1\le i<j\le n} A_{i,j}\right) = \frac{\binom{n}{2}}{1!}\cdot\frac{1}{n(n-1)} - \frac{\binom{n}{2}\binom{n-2}{2}}{2!}\cdot\frac{1}{n(n-1)(n-2)(n-3)}$$
$$+\frac{\binom{n}{2}\binom{n-2}{2}\binom{n-4}{2}}{3!}\cdot\frac{1}{n(n-1)\cdot\ldots\cdot(n-5)}$$
$$+\ldots+(-1)^{N-1}+\frac{\binom{n}{2}\cdot\ldots\cdot\binom{n-2(N-1)}{2}}{N!}\cdot\frac{1}{n!}$$

$$=\frac{1}{1!}\cdot\frac{1}{2}-\frac{1}{2!}\cdot\frac{1}{2^2}+\frac{1}{3!}\cdot\frac{1}{2^3}-\ldots+(-1)^{N-1}\frac{1}{2^N}\cdot\frac{1}{N!}.$$

Thus

$$p'_n = 1-\frac{1}{1!}\cdot\frac{1}{2}+\frac{1}{2!}\cdot\frac{1}{2^2}-\frac{1}{3!}\cdot\frac{1}{2^3}+\ldots+(-1)^N\frac{1}{2^N}\cdot\frac{1}{N!},$$

and therefore $\lim_{k\to\infty} p'_n = 1/\sqrt{e}$.

Thus, (iii) is true.

2. (a) In order to receive a sum of $n+3$, exactly one of the following events must occur:

   A_1: We receive $n-3$ outcomes of 1 and three outcomes of 2.

   A_2: We receive $n-2$ outcomes of 1, one outcome of 2 and one outcome of 3.

   A_3: We receive $n-1$ outcomes of 1 and one outcome of 4.

   The sample space consists of 6^n points. In the event A_1, there are $\binom{n}{3}$ possibilities for placing the three outcomes of 2 among the n possible locations, and thus $P(A_1) = \frac{\binom{n}{3}}{6^n}$. In the event A_2, there are $n(n-1)$ possible locations for placing the outcomes 2 and 3, and therefore $P(A_2) = \frac{2\binom{n}{2}}{6^n}$. Similarly, $P(A_3) = \frac{n}{6^n}$. The events are pairwise disjoint, and thus the probability of receiving the sum $n+3$ is

$$P(A_1\cup A_2\cup A_3) = P(A_1)+P(A_2)+P(A_3) = \frac{n+2\binom{n}{2}+\binom{n}{3}}{6^n}.$$

   Thus, (iii) is true.

   (b) Let A be the event that the sum of upfaces is $n+2$, and B be the event that the result of the first toss is 1. We need to find $P(B|A)$.

Similarly to the preceding part, the probability of A is

$$P(A) = \frac{n + \binom{n}{2}}{6^n} = \frac{\binom{n+1}{2}}{6^n}.$$

The event $A \cap B$ is the event that the outcome of first toss is 1 and the sum of the upfaces is $n+2$. Hence,

$$P(A \cap B) = \frac{1}{6} \cdot \frac{n - 1 + \binom{n-1}{2}}{6^{n-1}} = \frac{\binom{n}{2}}{6^n}.$$

Altogether,

$$\frac{P(A \cap B)}{P(A)} = \frac{\binom{n}{2}/6^n}{\binom{n+1}{2}/6^n} = \frac{n-1}{n+1}.$$

Thus, (ii) is true.

(c) First we note that the possible values of X are $0, 1, 2, \ldots, n-1$. The die tosses are independent, and the probability that the $(j+1)$-st toss will yield a different result from the j-th toss is $\frac{5}{6}$. Altogether, we have a sequence of $n-1$ independent trials, with a probability $\frac{5}{6}$ for success in each, and X is the number of successes. It follows that $X \sim B\left(n-1, \frac{5}{6}\right)$.

Thus, (iv) is true.

(d) The event in question is $\bigcap_{k=1}^{\infty} A_k$, where A_k is the event that, in the first k tosses, the die shows the same result. By the multiplication rule

$$\begin{aligned} P(A_k) &= P(A_1)\, P(A_2|A_1)\, P(A_3|A_2) \cdot \ldots \cdot P(A_k|A_{k-1}) \\ &= 1 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{k-1}{k} = \frac{1}{k}. \end{aligned}$$

As the sequence of events A_k is decreasing

$$P\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k \to \infty} P(A_k) = 0.$$

Thus, (i) is true.

3. (a) Let A_k be the event that all three coins show the same side for the first time at the k-th toss. Let R be the event that Reuven win a

prize. Then:

$$P(R) = \sum_{i=1}^{\infty} P(A_{2i}) = \sum_{i=1}^{\infty} \left(\frac{3}{4}\right)^{2i-1} \cdot \frac{1}{4} = \frac{3}{7}.$$

Thus, (i) is true.

(b) Let S be the event that Shimon wins a prize. By the law of total probability:

$$\begin{aligned} P(S) &= \sum_{i=1}^{\infty} P(S|A_i) P(A_i) = \sum_{i=1}^{\infty} \frac{1}{i} \cdot \left(\frac{3}{4}\right)^{i-1} \cdot \frac{1}{4} = \frac{1}{3} \sum_{i=1}^{\infty} \frac{1}{i} \cdot \left(\frac{3}{4}\right)^{i} \\ &= -\frac{1}{3} \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \cdot \left(-\frac{3}{4}\right)^{i} = -\frac{1}{3} \cdot \ln\left(1 - \frac{3}{4}\right) = -\frac{1}{3} \cdot \ln 2^{-2} \\ &= \frac{2}{3} \cdot \ln 2. \end{aligned}$$

Thus, (iii) is true.

4. (a) Consider (i). Take the experiment in which we toss a die twice. Let A be the event that both outcomes are 1, B the event that we receive the same outcome in both tosses and C the event that both outcomes are 2. Then:

$$\begin{aligned} \frac{1}{6} &= P(A|B) > P(A) = \frac{1}{36}, \\ 1 &= P(B|C) > P(B) = \frac{1}{6}, \\ 0 &= P(A|C) \ngtr P(A) = \frac{1}{36}. \end{aligned}$$

Thus, (i) is false.

Now consider (ii). Take the experiment in which we toss a die once. Let A be the event that we receive one of the outcomes $1, 3$ or 5, B the event that we receive one of the outcomes $1, 2, 3$ or 5 and C the event that we receive one of the outcomes $1, 3, 4, 5$ or 6. Then:

$$\begin{aligned} \frac{3}{4} &= P(A|B) > P(A) = \frac{1}{2}, \\ \frac{3}{5} &= P(A|C) > P(A) = \frac{1}{2}, \\ \frac{1}{2} &= P(A|B \cup C) \ngtr P(A) = \frac{1}{2}. \end{aligned}$$

Thus, (ii) is false.

Now consider (iv). Take the experiment in which we toss a die twice. Let A be the event that both outcomes are 2, B the event that the outcome in the first toss is 1, and C the event that both outcomes are 1. Hence, $B \supset C$,

$$\frac{1}{6} = P(B) > P(C) = \frac{1}{36},$$
$$0 = P(A|B) \not< P(A|C) = 0.$$

Thus, (iv) is false.

Let us show now that (iii) is true. Indeed,

$$P(A|B) > P(A) \Rightarrow \frac{P(A \cap B)}{P(B)} > P(A) \Rightarrow P(A \cap B) > P(A)\, P(B)\,.$$

Therefore:

$$P(B) < \frac{P(A \cap B)}{P(A)} = P(B|A)\,.$$

Thus, (iii) is true.