Probability Theory for EE Students

Exercises

1 Probability Spaces

1. A fair coin is tossed n times. Find the probability that

(a) there is exactly one head.

(b) there are exactly two heads.

(c) the sequence of results is palindromic.

2. Assuming all sex distributions to be equally probable, what proportion of families with exactly six children should be expected to have 3 boys and 3 girls?

3. Find the probability that in a randomly shuffled full deck of cards

(a) the four aces are located consecutively in the deck.

(b) the locations of the aces in the deck form an arithmetic sequence of difference 7.

(c) the locations of the aces in the deck form an arithmetic sequence.

4. Suppose that n people get on an elevator that stops at k floors. Assuming that each person has the same probability of getting out at each of the floors, find the probability that they all get off at distinct floors.

5. Each of n given sticks is broken into one long and one short part. The $2n$ parts are paired, and a new stick is formed out of each pair. Find the probability that

(a) the pairs will be the same as in the original sticks.

(b) all long parts are paired with short parts.

6. In a town of $(n+1)$ inhabitants, a person tells a rumor to a second person, who in turn repeats it to a third person, etc. At each step the recipient of the rumor is chosen at random from the n people available.

(a) Find the probability that the rumor will be told r times without:

(i) returning to the originator.

(ii) being repeated to any person.

(b) The same, when at each step the rumor is told to a group of N randomly chosen people, for some arbitrary fixed N. (The situation in part (a) is the special case $N = 1$.)

7. Twenty five books are placed on a shelf, among them six of the volumes of Harry Potter. Find the probability that the six volumes are arranged on the shelf according to the correct order (not necessarily adjacent to each other).

8. A number a is chosen randomly from the set $\{1, 2, \ldots, n\}$. Find the probability p_n that $a^2 \bmod 10 = 1$. Find $\lim_{n \longrightarrow \infty} p_n$.

9. A number a is chosen randomly from $\{0, 1, 2, \ldots, 10^n - 1\}$. Find the probability p that a is a k-digit number, i.e., $a = c_k 10^{k-1} + c_{k-1} 10^{k-2} + \ldots + c_1$, where $0 \le c_i \le 9$ for $i = 1, 2, \ldots, k-1$ and $0 < c_k \le 9$.

10. Two numbers X and Y are chosen randomly with replacement from the set $\{1, 2, \ldots, n\}$, where $n \ge 4$. Let

$$p_2 = P\left(X^2 - Y^2 \bmod 2 = 0\right)$$

and

$$p_3 = P\left(X^2 - Y^2 \bmod 3 = 0\right).$$

Which of the probabilities p_2 and p_3 is larger?

11. Let $M = 2^{\{1,2,\ldots,n\}}$. Two elements A_1 and A_2 of M are chosen randomly (with replacement). Find the probability that $A_1 \bigcap A_2$ is empty.

12. Two numbers X_1 and X_2 are drawn randomly from the set $\{1, 2, \ldots, n\}$ without replacement. Find $P(X_2 > X_1)$.

13. Three numbers X_1, X_2 and X_3 are drawn randomly from the set $\{1, 2, \ldots, n\}$ without replacement. Find the probability that $X_1 < X_2 < X_3$.

14. A coin is tossed until the first time a head turns up. Find the probability that the coin is tossed

(a) an even number of times.

(b) exactly k times.

(c) at least k times.

15. A coin is tossed until the same result appears twice in a row. Find the probability that the coin is tossed

(a) less than 6 times.

(b) an even number of times.

16. Consider an experiment with possible outcomes $w_1, w_2, \ldots, w_N$, where it is known that each outcome w_{j+1} is twice as likely as outcome w_j, $j = 1, 2, \ldots, N-1$. Find $P(A_k)$, where $A_k = \{w_1, w_2, \ldots, w_k\}$.

17. A die and two coins are tossed.

(a) Describe the sample space of the experiment.

(b) Describe the following events:

(i) A_1 – the die shows "4".

(ii) A_2 – the coin shows the same face in both tosses.

(iii) $A_1 \cup A_2$.

(iv) $A_1 \cap A_2$.

(v) $A_1 \cap \overline{A_2}$.

18. An urn, containing 10 blue and 20 red balls, is given. A coin is tossed 3 times. If it shows more heads than tails, we draw from the urn 2 balls without replacement. If it shows more tails, we draw from the urn 3 balls.

(a) Describe the sample space of the experiment.

(b) Describe the following events:

(i) We draw at least one red ball.

(ii) The number of blue balls we draw does not exceed that of the red balls.

(iii) No two consecutively drawn balls are of the same color.

19. A, B and C are events in a probability space $(\Omega, \mathcal{B}, P)$. Express the following events in terms of the three given events:

(a) At least one of the three events occurs.

(b) At most one of the three events occurs.

(c) Either one or all three events occur.

(d) A is not the only event to occur.

20. $(A_n)_{n=1}^{\infty}$ is a sequence of events in a probability space $(\Omega, \mathcal{B}, P)$. Express the following events in terms of the given events:

(a) Exactly three of the A_n's occur.

(b) Only finitely many of the A_n's occur.

(c) At least one out of every 5 consecutive A_n's occurs.

(d) All but finitely many of the A_n's occur.

(e) At least one out of every 5 consecutive A_n's occurs, and at least one out of every 3 consecutive ones does not.

21. Let $\Omega = [0,1]$ and suppose the probability of each sub-interval of Ω is equal to its length. Find

(a) $P\left(\bigcup_{i=1}^{\infty}[\frac{1}{2i+1}, \frac{1}{2i}]\right)$.

(b) the probability of the set of points whose infinite decimal expansion does not contain the digit 7.

(c) the probability of the set of points whose infinite hexadecimal expansion does not contain the digit D.

(d) the probability of the set of points whose infinite decimal expansion contains infinitely many occurrences of the digit 7.

22.

(a) Prove that, for any events $A_1, A_2, \ldots, A_n$, we have $P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$.

(b) Prove the analogous statement for infinite sequences of events.

23. For each of the following collections $\mathcal{B}$ of subsets of Ω, determine whether it is a σ-field:

(a) $\Omega = \mathbf{N}$, $\mathcal{B}$ is the collection of sets containing the number 1.

(b) $\Omega = \mathbf{N}$, $\mathcal{B}$ is the collection of all finite sets of even size and complements of such sets.

(c) $\Omega = \mathbf{N}$, $\mathcal{B}$ is the collection of all infinite sets and $\emptyset$.

(d) $\Omega = [0,1]$, $\mathcal{B}$ is the collection of all finite sets and their complements.

(e) $\Omega = [0,1]$, $\mathcal{B}$ is the collection of all sets whose elements may be listed in a sequence and their complements.

2 Basic Combinatorics

24. Car license plates consist of 5 digits and 5 letters. How many different license plates are possible

(a) if all digits precede the letters?

(b) if no two letters may be adjacent?

(c) if there is no restriction?

25. A full deck consists of 52 cards. How many sets of 8 cards are made of pairs of cards (namely, contain either two aces or none, either two kings or none, etc.)?

26. A computer with k processors receives n jobs. How many possibilities are there to assign the jobs?

27. A *word* over an alphabet Σ is a finite sequence of elements of Σ. Let $|\Sigma| = r$. How many words of length n are there over Σ

(a) if there are no restrictions?

(b) if adjacent letters must be distinct?

(c) if each letter σ_i has to appear some prescribed number n_i of times (where $\sum_{i=1}^{r} n_i = n$)?

(d) if the word has to be a palindrome (i.e., the last letter is the same as the first, the second last the same as the second, etc)?

28. Recall that Stirling's formula provides the following approximation for $n!$:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n .$$

Use the idea of the integral test (for testing the convergence of infinite series) to prove the following weak form of Stirling's formula:

$$e\left(\frac{n}{e}\right)^n \leq n! \leq e\left(\frac{n+1}{e}\right)^{n+1}.$$

29. Consider the middle binomial coefficients $\binom{2n}{n}$.

(a) What asymptotics does Stirling's formula yield for them?

(b) Employing elementary means only, derive upper and lower bounds which are at most polynomially worse than the "precise" asymptotics found in the first part.

30. Prove combinatorially the following identities:

(a) $\binom{n}{k} = \binom{n}{n-k}$.

(b) $\sum_{j=0}^{n} \binom{n}{j}^2 = \binom{2n}{n}$.

(c) $\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}$.

(d) $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$.

(e) $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

(f) $\sum_{k=1}^{n} k \cdot \binom{n}{k} = n \cdot 2^{n-1}$.

(g) $\sum_{k=2}^{n} k(k-1) \cdot \binom{n}{k} = n(n-1) \cdot 2^{n-2}$.

31. $2n$ balls are chosen at random from a total of $2n$ red balls and $2n$ blue balls. Find a combinatorial expression for the probability that the chosen balls are equally divided in color. Use Stirling's formula to estimate this probability.

32. How many permutations are there of the word "committee"?

33. Find the number of even-sized subsets of $\{1, 2, \ldots, n\}$.

34. Express the number of multiplications (of numbers) one needs to perform to multiply two matrices in terms of the sizes of the matrices.

35. Find the number of $m \times n$ matrices of 0's and 1's with mutually distinct rows.

36. In New Manhattan there are 1000 east-west streets and 100 north-south avenues. In how many ways can one get from the southwest corner of New Manhattan to the northeast corner if it is allowed to move only north or east during the whole trip?

37. How many vectors of length $2n$, consisting of 1's, 2's and 3's, are there, such that the number of 3's is the same as the number of 1's and 2's combined?

3 Elementary Probability Calculations

38. An urn contains B blue and R red balls. A random sample of size n is drawn from the urn. Find the probability that the sample contains exactly b blue balls, if the sample is drawn:

(a) with replacement.

(b) without replacement.

39. An urn contains M_i balls of color i for $i = 1, 2 \ldots, N$. A random sample of size n is drawn from the urn without replacement. Find the probability that

(a) the sample contains exactly n_i balls of color i for $i = 1, 2 \ldots, N$.

(b) each of the colors is represented.

40. Itzik and Shmulik take a course with 3 possible grades: 0, 56 and 100. The probability to get 56 is 0.3 for Itzik and 0.4 for Shmulik. The probability that neither gets 0 but at least one gets 56 is 0.1. Find the probability that at least one gets 56 but neither gets 100.

41. X_i, $i = 1, 2, \ldots, n$, are drawn randomly from the set $\{1, 2, \ldots, N\}$ without replacement. Denote by Y_i their order statistics:

$$\min_{1 \leq i \leq n} X_i = Y_1 < Y_2 < Y_3 < \ldots < Y_n = \max_{1 \leq i \leq n} X_i.$$

(a) Find $P(Y_m \leq M < Y_{m+1})$.

(b) Find the limit of the probability in part (a) if $N, M \longrightarrow \infty$ with $M/N \longrightarrow \theta \in [0, 1]$ (for constant m and n).

42. Thirteen cards are drawn randomly from a full deck of 52 cards. Find the probability of getting:

(a) a full hand.

(b) no aces.

(c) no aces and exactly one king and one queen.

(d) one ace, one king, one queen, etc.

43. Find the probability that, out of a set of n people, at least two have their birthdays in the same month. (Assume the months are equally likely.)

44. A sequence of length n over $\{0, 1, 2\}$ is chosen randomly. Find the probability that the sequence:

(a) starts with a 0.

(b) starts and ends with a 0, and contains exactly m additional 0's.

(c) contains digit i exactly n_i times, $i = 0, 1, 2$.

45. A drug is assumed to be effective with unknown probability p. To estimate p, the drug is tried on n different patients. It is found to be effective on m of them. The *method of maximum likelihood* for estimating p suggests that we choose the value of p yielding the highest probability to the result actually obtained at the experiment. Show that the maximum likelihood estimate for p is $\frac{m}{n}$.

46. Prove the following claims:

(a) For sufficiently small $x \geq 0$ we have

$$e^{-x-x^2} \leq 1 - x \leq e^{-x}.$$

(b) If $(\alpha_n)_{n=1}^{\infty}$ is a sequence of numbers in the open interval $(0, 1)$, then $\prod_{n=1}^{\infty}(1 - \alpha_n) = 0$ if and only if $\sum_{n=1}^{\infty} \alpha_n = \infty$.

47. Numbers $X_1, X_2, \ldots, X_k$ are chosen randomly from the set $\{1, 2, \ldots, n\}$ with replacement.

(a) Find the probability that all chosen numbers are distinct.

(b) Find the limit of the probability calculated in part (a) if $n \longrightarrow \infty$ and $\frac{k}{\sqrt{n}} \longrightarrow \theta$ for some $0 < \theta < \infty$. (Hint: Employ Exercise 46.a.)

48. Numbers $X_1, X_2, \ldots, X_k$ are chosen randomly without replacement from the set $\{1, 2, \ldots, 2n\}$.

(a) Find the probability that $|X_i - X_j| \neq n$ for $i \neq j$.

(b) Find the limit of the expression you found if $n \longrightarrow \infty$ and $\frac{k}{\sqrt{n}} \longrightarrow \theta$ for some $0 < \theta < \infty$. (Hint: Employ Exercise 46.a.)

49. Given any sequence $(A_n)_{n=1}^{\infty}$ of events, denote by $\limsup_{n\to\infty} A_n$ the set of points in the sample space belonging to infinitely many of the sets A_n, and by $\liminf_{n\to\infty} A_n$ the set of points belonging to all sets A_n from some place on.

(a) Let

$$A_n = \begin{cases} [0, 1+1/n], & n \text{ even}, \\ [1/2, 2+1/n], & n \text{ odd}. \end{cases}$$

Find the sets $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$.

(b) Prove that, for any sequence $(A_n)_{n=1}^{\infty}$ of events, $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$ are also events.

50. Let A be a finite set of size n. A subset R of A is chosen randomly. Find the probability that:

(a) $|R|$ (the size of R) is even.

(b) $|R|$ is divisible by 3; is 1 modulo 3. (Hint: Let $\omega = e^{2\pi i/3}$ be a root of unity of order 3. What is the value of $1 + \omega^k + \omega^{2k}$ as a function of k?)

(c) $|R|$ is divisible by 4; is 1,2,3 modulo 4.

51. A finite sequence of length n consisting of distinct numbers is permuted randomly (all $n!$ orderings being equi-probable). The numbers are read one by one, and put into a binary search tree T. Find the probability that:

(a) T is of height $n-1$.

(b) the left subtree and the right subtree of T contain $\frac{n-1}{2}$ nodes each (assuming n is odd).

(c) T is perfectly balanced (namely, contains all 2^k possible nodes at level k for each $k \le h$, where h is the height of the tree; we assume here that $n = 2^{h+1} - 1$).

4 Conditional Probability

52. Let the probability p_n that a family has exactly n children be αp^n for $n \ge 1$. Suppose that all sex distributions of n children have the same probability. Find the probability that a family has

(a) exactly k boys and l girls.

(b) exactly k boys.

53. Given are $N+1$ urns, each containing N blue and white balls. The kth urn contains k blue and $N-k$ white balls, $k = 0, 1, 2, \dots \mathbf{N}$. An urn is chosen at random and a random ball is drawn from it $n+1$ times with replacement. Let $A = \{$the first n balls turn out to be blue$\}$ and $B = \{$the $(n+1)$'st ball is blue$\}$.

(a) Find $P(B \mid A)$.

(b) Show that for large N the probability in part (a) is approximately $\frac{n+1}{n+2}$. (Hint: Use $\int_0^1 x^n dx$.)

54. Two players are playing a game until one of them is ruined. At each round, one of them wins 1 dollar from the other. At the beginning of the game, the first player has x dollars and the second y dollars. The probability of each of them to win any game is $1/2$. Find the probabilities P_1 and P_2 for the first and the second player, respectively, to win the whole match.

55. An urn contains initially w white and b black balls. A ball is drawn at random. It is replaced and, moreover, another ball of the same color is put into the urn. The procedure is repeated indefinitely. Let $P_k(n)$ be the probability of drawing k white (and $n-k$ black) balls in the first n trials.

(a) Given that the second ball was black, what is the probability that the first was black as well?

(b) Prove the recurrence relation

$$P_k(n+1) = P_k(n)\frac{b+n-k}{w+b+n} + P_{k-1}(n)\frac{w+k-1}{w+b+n},$$

where we set $P_{-1}(k) = 0$.

(c) Using the preceding part (or directly) prove that:

$$P_k(n) = \binom{n}{k} \cdot \frac{w+b-1}{w+b+n-1} \cdot \frac{\binom{w+b-2}{w-1}}{\binom{w+b+n-2}{w+k-1}}.$$

56. We choose a random graph $G = (V, E)$ on n vertices as follows. (See https://en.wikipedia.org/wiki/Graph_(discrete_mathematics) and https://en.wikipedia.org/wiki/Random_graph.) V is any set of size n (say, $V = \{1, 2, \dots, n\}$). For any two distinct $v_1, v_2 \in G$, the edge (v_1, v_2) belongs to E with probability p and does not belong to it with probability $q = 1 - p$, where $0 < p < 1$ is arbitrary fixed. Show that the probability of G being connected converges to 1 as $n \longrightarrow \infty$.

57.

(a) The final exam at the snipers course of the US Army consists of an infinite sequence of shots. At the n-th trial, the student shoots at an $2n \times 2n$ square target. To pass the exam, one needs to hit the central 1×1 subsquare for some n. If one fails all trials, he fails the course. Assume that the probability of a certain student to hit any portion of the square is proportional to the area of that portion. What is his probability of failure?

(Note: You may write your answer in a simple way by using Wallis's formula

$$\frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot \ldots \cdot 2n \cdot 2n}{1 \cdot 3 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1) \cdot (2n+1)} \underset{n\to\infty}{\longrightarrow} \frac{\pi}{2}.$$

See https://en.wikipedia.org/wiki/Wallis_product.)

(b) In the final exam at the snipers course of the Marines, the targets are 1-dimensional. At the n-th trial, the student shoots at a target of length $(n + 1)$. To pass the exam, one needs to never hit the target near the right or the left endpoints; if he hits the rightmost $1/2$ meter or the leftmost $1/2$ meter, he fails. Assume that the probability of a certain student to hit any portion of the target is proportional to the length of that portion. Find his probability of passing the exam.

58. Three people toss a die repeatedly until all three obtain the same result. Then they continue tossing their dice as many times as before.

(a) What is the probability that, in the course of the second stage of the experiment, they received three different results exactly k times?

(b) Assuming that at the second stage they received three different results k times, what is the probability that the first stage took n steps?

59. A parent particle splits at the end of its life into 0, 1 or 2 particles, with probabilities $1/4$, $1/2$ and $1/4$, respectively. Starting with a single particle and denoting by X_i the number of particles in the i-th generation, find

(a) $P(X_2 > 0)$.

(b) $P(X_1 = 2|X_2 = 1)$.

(c) $P(X_1 = 1|X_3 > 0)$.

60. An urn contains 5 balls, r of them red and the others green, the value of r being unknown. (The possible values of r between 0 and 5 are *a priori* equally reasonable.) Mr. X is to draw a ball at random from the balls in the urn, and you are required to guess the color of the ball he will draw. You are allowed to hold one of the following three experiments prior to making your guess:

(1.) Draw a single ball from the urn.

(2.) Draw three balls with replacement from the urn.

(3.) Draw three balls without replacement from the urn.

(The balls are returned to the urn after the experiment.) What is the probability of making the right guess if you

a. use option 1 and guess that Mr. X will draw a ball of the same color as you did?

b. use option 2 and guess that Mr. X will draw a ball of the same color as the majority among the three balls you have drawn?

c. use option 3 and guess that Mr. X will draw a ball of the same color as the majority among the three balls you have drawn?

61. A and B throw alternately a pair of dice. A wins if he scores a total of 6 before B scores a total of 7; otherwise B wins. If A starts the game, what is his probability of winning?

62. N players $A_1, A_2, \dots, A_N$ toss a biased coin whose probability of a head is p. A_1 starts the game, A_2 plays next, etc. The first player to get a head wins. Find the probability of each of the players to be the winner.

63. An urn contains initially W white and B black balls. A random sample of size n is drawn. Find the probability that the jth ball in the sample is black, given that the sample contains b black balls. Consider sampling

(a) with replacement.

(b) without replacement.

64. The game of craps is played as follows. The player tosses repeatedly a pair of dice. If he scores a total of 7 or 11 at his first toss he wins, whereas if his total is 2, 3 or 12 he loses. In any other case the game continues until for the first time he scores a total equal to either his first total or to 7. In the first of these cases he wins, and in the second – loses.

(a) Find the player's total probability of winning.

(b) Write a computer program estimating the same probability.

65. Construct examples of three events A, B and C

(a) which are pairwise independent but not independent.

(b) satisfying $P(A \cap B \cap C) = P(A)P(B)P(C)$, without any two of them being independent.

66. For events A, B and C, prove or disprove:

(a) If A and B are independent, then $P(A \cap B|C) = P(A|C)P(B|C)$.

(b) If $P(A) > P(B)$, then $P(A|C) > P(B|C)$.

(c) If $P(A) = 0$, then $P(A \cap B) = 0$.

(d) If $P(A|B) \geq P(A)$, then $P(B|A) \geq P(B)$.

(e) If $P(B|\overline{A}) = P(B|A)$, then A and B are independent.

(f) If $P(A) = a$ and $P(B) = b$, then $P(A|B) \geq \frac{a+b-1}{b}$.

(g) If $P(A) = P(B) > 0$, then $P(A|B) = P(B|A)$.

(h) If $P(A|B) = P(B|A)$, then $P(A) = P(B)$.

(i) If $P(A|B) = P(B|A)$, with $P(A \cup B) = 1$ and $P(A \cap B) > 0$, then $P(A) > \frac{1}{2}$.

(j) If $P(A) = a$ and $P(B) = b$, then $P(\overline{A} \cap \overline{B}) \geq 1 - a - b$.

67. A coin with probability p for a head is tossed n times. Suppose $A = \{$a head is obtained in the first toss$\}$ and $B_k = \{$exactly k heads are obtained$\}$. For which n and k are A and B_k independent?

68. It is known that each of four given people A, B, C and D tells the truth in any given instance with probability $\frac{1}{3}$. Suppose that A makes a statement, and then D says that C says that B says that A is telling the truth. What is the probability that A is actually telling the truth?

69. A coin is tossed twice. Consider the following events:
A : Head on the first toss.
B : Head on the second toss.
C : Same outcomes in the two tosses.

(a) Are $A,\ B,\ C$ pairwise independent?

(b) Are A, B, C independent?

(c) Show that C is independent of A and B but not of $A \cap B$.

70. Same as Problem 50 if the set R is selected by including each element of A in it with probability p and excluding it with probability $1 - p$ (distinct elements being independent).

5 Discrete Distributions

71. Denote $b(n, p, j) = \binom{n}{j} p^j (1-p)^{n-j}$ for $0 \leq j \leq n$ and $0 < p < 1$. Determine, for any fixed n and p, the value, or values, of j which maximize $b(n, p, j)$.

72. An urn contains 10 white balls, marked by the numbers $1, 2, \ldots, 10$, and 3 blue balls, marked by $11, 12, 13$. Five balls are drawn randomly

(i) with replacement;

(ii) without replacement.

For each of the cases (i) and (ii) find the distribution of the

(a) number of white balls in the sample.

(b) minimum of the numbers marked on the balls.

(c) maximum of the numbers marked on the balls.

(d) number of stage at which the first white ball is drawn (6 if all balls in the sample are blue).

(e) number of balls marked by even numbers.

(f) number of balls marked by 2.

73. Same as the preceding problem, with the blue balls marked by $1, 2, 3$.

74. The traffic flow at a certain street is such that the probability of a car passing during any given second is p. Moreover, different seconds are independent. Suppose that the pedestrian can cross the street only if no car is to pass during the next three seconds. Find the probability that pedestrian has to wait for exactly $k = 0, 1, 2, 3, 4$ seconds. (Regard seconds as indivisible time units.)

75. Two people toss a coin n times each. Let P_n be the probability that they score the same number of heads. Show that for large n we have $P_n \approx \frac{1}{\sqrt{\pi n}}$.

76. The following functions are probability functions of certain random variables. In each case, find the parameter c and calculate the distribution function F:

(a)

$$p(x) = \begin{cases} c(N-x), & x = 0, 1, \ldots, N-1, \qquad (N \geq 1), \\ 0, & \text{otherwise.} \end{cases}$$

(b)

$$p(x) = \begin{cases} \frac{c}{x(x+1)}, & x = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

(c)

$$p(x) = \begin{cases} \frac{c}{x(x+1)(x+2)}, & x = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

(d)

$$p(x) = \begin{cases} \frac{c}{x(x+1)(x+3)}, & x = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

(e)

$$p(x) = \begin{cases} \frac{c}{x}, & x = 2^k 3^l, \ k, l = 0, 1, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

(f)

$$p(x) = \begin{cases} \frac{c}{x^2}, & x = 2^k 3^l, \ k, l = 0, 1, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

77. Suppose all rational numbers are ordered in some way, r_1 being the first, r_2 – the second, etc. Let X be a random variable assuming each rational value r_k with probability $P(X = r_k) = 1/2^k$. At which points is the distribution function F_X continuous and at which not?

78. M people are chosen at random out of n couples. Find the distribution of the number of couples within them.

79. Recall that the probability function of the binomial distribution converges, as $n \to \infty$ and $p \to 0$ with $np = \lambda$ constant, to the probability function of the Poissonian distribution with parameter λ. Show that the convergence is uniform over the set of non-negative integers.

80. Suppose a family of hypergeometric distributions $H(n, a, b)$ is given, with $a, b \to \infty$ such that $a/b \to C$ for some $0 < C < \infty$ and n constant. Show that the distributions converge to a certain binomial distribution and specify its parameters.

81. Consider the distribution of the number of failures, obtained in a sequence of independent experiments with success probability p, until r successes are obtained. Let $r \to \infty$, $p \to 1$ and $r(1-p) = \lambda$. Show that the distributions converge to the $P(\lambda)$ distribution.

82. Find the limiting distribution of the number of letters sent to their right destination in the absent-minded secretary problem.

83. Show that the number of trials, in a sequence of independent trials with a success probability of p in each, until the r-th success is $\overline{B}(p, r)$-distributed.

6 Expectation

84. Find $E(X^2)$ and $E(X^3)$ if X is a random variable distributed according to the following distribution:

(a) $X \sim U[a, b]$.

(b) $X \sim B(n, p)$.

(c) $X \sim H(n, a, b)$.

(d) $X \sim G(p)$.

(e) $X \sim \overline{B}(p, r)$.

(f) $X \sim P(\lambda)$.

85. Find $E(X)$ if:

(a) $X = 2^Y$, where $Y \sim U[a, b]$.

(b) $X = 2^Y$, where $Y \sim B(n, p)$.

(c) $X = \sin Y$, where $Y \sim B(n, p)$. (Hint: Recall that $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$.)

(d) $X = 2^Y$, where $Y \sim G(p)$.

(e) $X = 2^Y$, where $Y \sim P(\lambda)$.

(f) $X = \sin Y$, where $Y \sim P(\lambda)$. (Hint: See (c).)

86. Consider a system consisting of n independent components, each of which either functions or fails, with functioning probabilities $p_i, \quad i = 1, 2, \dots, n$. A system is *serial* if it works only if all its components function and *parallel* if it works if at least one of its components functions. Let

$$Y = \begin{cases} 1, & \text{the system works,} \\ -1, & \text{the system fails.} \end{cases}$$

Find $E(Y)$ for a serial system and for a parallel system.

87. A random subset S of $\{1, 2, \dots, m\}$ is chosen by selecting n consecutive times a random element of $\{1, 2, \dots, m\}$ and letting S consist of all elements chosen at least once in the process. Let $X = |S|$. Find $E(X)$ if:

(a) at each step, all the numbers have equal probabilities of being selected.

(b) the probabilities of $1, 2, \dots, m$ to be selected are $p_1, p_2, \dots, p_m$, respectively.

88. An urn contains N balls enumerated from 1 to N. Let X be the largest number drawn in n drawings, when random sampling with replacement is used.

(a) Find $E(X)$ as a function of N and n.

(b) Approximate $E(X)$ for large N and constant n.

(c) Approximate $E(X)$ for large n and constant N.

89. A royal family has children until it has a boy or until it has three children, whichever comes first. Find the expected number of boys and the expected number of girls in the family.

90. A multiple-choice exam is given. A problem has four possible answers, of which exactly one is correct. A student is allowed to mark any subset of the possible answers. The student receives 3 points if the correct answer is marked, and is penalized by 1 point for each wrong answer. Suppose the student marks his answers randomly, where each possible answer is marked with some probability.

(Distinct answers are not necessarily marked with the same probability.) Show that the expected score of the student is 0.

91. A coin is tossed until the first time a head turns up. If this occurs at the nth step, then the player wins $2^n/n$ dollars if n is odd, and loses $2^n/n$ dollars if n is even. Let X be the amount the player wins. Find $E(X)$ if it exists.

92. A number of people are to pass a blood test each. The test may be held in one of two ways:

(1) Each person is tested separately.

(2) The group is divided into groups of size k each. (Assume the whole group to be large, so that we may neglect the inaccuracy due to the fact that the total size is not exactly divisible by k.) The blood samples of the people in each of these groups are pooled and analyzed together. If the test is negative, this one test shows that the result is negative for each of the k people, and the test suffices. If it is positive, each of the k people must be tested separately, in which case $k+1$ tests are required in all for the k people.

Assume that the probability p that the test yields a positive result for a person is the same for all people and that distinct people are independent.

(a) What is the expected value of the number X of tests carried out per person under plan (2)?

(b) For small p, show that the value of k which will minimize $E(X)$ is approximately $\frac{1}{\sqrt{p}}$.

93. Consider random variables as in Problem 76.

(a) Which of these variables have finite expectation? Find their expectation.

(b) Same for the squares of these variables.

94. Let G be a random graph as in Problem 56. Let v_1, v_2 be arbitrary fixed vertices. Show that the expected value of the distance between v_1 and v_2 converges to $2-p$ as $n \longrightarrow \infty$ (where we take the distance as n if there exists no path between the two vertices).

95. We select n people randomly.

(a) Find the expected number of people whose birthday is distinct from that of all others.

(b) For which value(s) of n is this expected number maximal?

(c) Estimate the expected number for the value of n found in (b) without using a calculator.

96. In Problem 78, find the expected number of couples within the people selected.

97. Two algorithms, *randomPerm1* and *randomPerm2*, have been proposed for generating random permutations of $1, 2, \ldots, n$. The first of these consists of choosing n random integers between 1 and n until the chosen n-tuple forms a permutation. The second algorithm consists of n steps; at each step we repeatedly choose a random integer between 1 and n until the chosen integer is distinct from all those chosen earlier. Find the expected number of selections of random integers for each of these algorithms.

98. Find the expected number of letters sent to their right destination in the absent-minded secretary problem.

7 Continuous Distributions

99. Let $U \sim U(0, 1)$. Find the density function and the distribution function of the following random variables:

(a) $X = 3U + 2$.

(b) $X = U^3$.

(c) $X = -\ln U$.

(d) $X = \frac{1}{U+1}$.

(e) $X = \ln U + 1$.

(f) $X = |U - \frac{1}{2}|$.

(g) $X = (U - \frac{1}{2})^2$.

100. Let $f_1(x)$ and $f_2(x)$ be density functions and $\theta_1,\ \theta_2$ constants.

(a) Find a sufficient condition on θ_1 and θ_2 for the function $\theta_1 f_1(x) + \theta_2 f_2(x)$ to be a density function.

(b) Is the condition found in (a) always necessary?

(c) Prove that, for every θ_1 and θ_2 not satisfying the condition, there exist densities $f_1(x)$ and $f_2(x)$ such that the function $\theta_1 f_1(x) + \theta_2 f_2(x)$ is not a density function.

101. For what values of the parameters a, b is the function f, defined by

$$f(x) = \begin{cases} ax + b, & -1 \le x \le 1, \\ 0, & \text{otherwise}, \end{cases}$$

a density function?

102. The following functions are density functions of certain random variables. In each case find all possible values of the parameter c and calculate the distribution function F:

(a)

$$f(x) = \begin{cases} cx^3, & 3 \le x \le 5, \\ 0, & \text{otherwise}. \end{cases}$$

(b)

$$f(x) = \begin{cases} c(e^{-2x} + e^{-3x}), & x \ge 0, \\ 0, & \text{otherwise}. \end{cases}$$

(c)

$$f(x) = c(e^{-2|x|} + e^{-3|x|}), \qquad -\infty < x < \infty.$$

(d)

$$f(x) = \begin{cases} 0, & x < -1, \\ -2cx, & -1 \le x \le 0, \\ 3ce^{-x}, & x > 0. \end{cases}$$

(e)

$$f(x) = \begin{cases} \frac{c-1}{(1+x)^c}, & x > 0, \\ 0, & \text{otherwise}. \end{cases}$$

103. Suppose that the distribution function $F_X(x)$ of a certain random variable X is strictly increasing on the whole real line. Let $U \sim U(0, 1)$.

(a) Show that the random variable $W = F_X(X)$ is distributed $U(0, 1)$.

(b) Show that the random variable $Y = F_X^{-1}(U)$ is distributed according to the distribution function $F_X(x)$.

104. Let X be the amount of bread (measured in hundreds of kilograms) a certain bakery sells a day. It is known that X has the density function:

$$f(x) = \begin{cases} cx, & 0 \le x < 3, \\ c(6-x), & 3 \le x < 6, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of c.

(b) Find the probability of the events $A = \{X > 3\}$ and $B = \{1.5 \le X \le 4.5\}$. Are A and B independent?

105. A mirror is mounted on a vertical axis, and is free to revolve about that axis. The axis of the mirror is at a distance of 1 meter from a straight wall of infinite length. A pulse of light is sent to the mirror, and the reflected ray hits the wall. Suppose that the angle between the reflected ray and the line perpendicular to the wall and running through the axis of the mirror is distributed $U\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Let X be the distance between the point on the wall hit by the reflected ray and the point on the wall closest to the axis of the mirror. Find the density function and the distribution function of X.

106. Find the expectation of random variables distributed on $[-1,1]$ with the following density functions:

(a) $f(x) = |x|$.

(b) $f(x) = 1 - |x|$.

(c) $f(x) = \frac{3}{2}x^2$.

(d) $f(x) = \frac{4}{3}(1 - x^2)$.

(e) $f(x) = \frac{\pi}{4}\cos(\pi x/2)$.

(f) $f(x) = \frac{x+1}{2}$.

(g) $f(x) = \frac{3(x+1)^2}{8}$.

107. The lifetime of the super light bulb, measured in hours, is a random variable T with density function

$$f(t) = 0.01te^{-0.1t}\ .$$

Find the expected lifetime of the bulb.

108. A point (U_1, U_2) is selected uniformly in the square $[0,1] \times [0,1]$. Find:

(a) $E(U_1^2 + U_2^2)$.

(b) $E((U_1 + U_2)^2)$.

(c) $E(-\ln U_1 - \ln U_2)$.

(d) $E(\max(U_1, U_2))$.

(e) $E(\min(U_1, U_2))$.

109. Find $E(X^n)$ for any positive integer n if:

(a) $X \sim U(a,b)$.

(b) $X \sim \mathrm{Exp}(\lambda)$.

110. Find $E(X)$ (in case it exists) if X is distributed according to any of the distributions defined in question 102.

8 Variance and Covariance

111. Find $V(X)$ (if it exists) if X is distributed according to any of the distributions defined in question 85.

112. Find the variance of a $\overline{B}(p,r)$ random variable.

113. An urn contains n balls of l distinct weights – n_1 of weight x_1 each, n_2 balls of weight x_2, and so forth. Let $m = \frac{\sum_{j=1}^{l} n_j x_j}{n}$ be the mean ball weight. We select r balls randomly without replacement. (Assume that balls of different weights still have the same probability of being selected.) Let W be the total weight of the selected balls. Find $E(W)$ and $V(W)$.

114. Find $E(X)$ and $V(X)$ if X is distributed according to the following probability functions:

(a)

$$p(x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!(1-e^{-\lambda})}, & x = 1, 2, \ldots, \ (\lambda > 0), \\ 0, & \text{otherwise.} \end{cases}$$

(b)

$$p(x) = \begin{cases} \dfrac{(1-p)^x}{-x \ln p}, & x = 1, 2, \ldots, \ (0 < p < 1), \\ 0, & \text{otherwise.} \end{cases}$$

115. Let X be a random variable with $E(X) = \mu$ and $V(X) < \infty$. Show that the function $v(s) = E\left((X - s)^2\right)$ assumes its minimum at the point $s = \mu$.

116. Let X and Y be two random variables having finite variances. Assume that all four variables X, Y, $X + Y$ and $X - Y$ have the same distribution. Show that $P(X = Y = 0) = 1$.

117. Find $V(X)$ (if it exists) if X is distributed according to any of the distributions defined in questions 102 and 106.

118. Find $V(e^U + e^{1-U})$, where $U \sim U(0, 1)$.

119. A die is tossed n times. Find the covariance of

(a) the number of ones and the number of sixes.

(b) the number of ones and the sum of all results.

(c) the number of even results and the sum of all results.

120. An urn contains a white balls and b black balls. Suppose m random balls are taken out of the urn without replacement, and then n more are taken out, where $m + n \le \min(a, b)$. Let X and Y denote the number of white balls in the first batch and in the second batch, respectively.

(a) Calculate $V(X)$ and $V(Y)$.

(b) Calculate $\mathrm{Cov}(X, Y)$.

(c) Is $\mathrm{Cov}(X, Y)$ positive or negative? Why should you have expected it without performing any calculations?

121. Find the variance of the number of letters sent to the right destination in the absent-minded secretary problem.

122. Let $X = e^U + c\left(U - \frac{1}{2}\right)$, where $U \sim U(0, 1)$ and c is a constant.

(a) Find $E(X)$.

(b) Find $V(X)$.

(c) For which value of c is $V(X)$ minimal?

(d) Same for $Y = e^{1-U} + c\left(U - \frac{1}{2}\right)$.

123. An coin is tossed n times. A change is said to occur when two consecutive results are different. Let S_n denote the total number of changes in the sequence. For example, if a coin is tossed for 5 times and the result is HTTHH, then $S_5 = 2$. Find $E(S_n)$ and $V(S_n)$ if the coin

(a) is fair.

(b) has probability p for a head.

124. Same as the preceding question for

(a) a fair die.

(b) a die with probabilities $p_1, p_2, \ldots, p_6$ for the outcomes $1, 2, \ldots, 6$, respectively.

9 Multi-Dimensional Distributions

125. The probability function of (X, Y) is given by the following table:

$y \backslash x$	1	2	3
1	1/12	1/6	1/12
2	1/6	1/4	1/12
3	1/12	1/12	0

Find:

(a) the probability for X to be odd.

(b) the probability for XY to be odd.

(c) $P(X + Y \geq 4)$.

(d) $E(X + Y)$.

(e) $V(X + Y)$.

126. Two cards are drawn without replacement from a full deck. Let X be the number of aces and Y the number of kings. Find:

(a) the probability function of (X, Y).

(b) $P(X \geq Y)$.

(c) $\mathrm{Cov}(X, Y)$.

127. A company manufactures items, each of which may have 0, 1, or 2 defects with probabilities 0.7, 0.2 and 0.1, respectively. If the item has 2 defects, the inspectors notice it and the item is replaced by one without defects before delivery. Let X be the original number of defects in an item produced and Y the number of defects in the corresponding delivered item. Find:

(a) the probability function of (X, Y).

(b) $E(X - Y)$.

(c) $V(X - Y)$.

128. Consider a sample of size 2 drawn without replacement from an urn containing three balls, numbered $1, 2$ and 3. Let X be the smaller of the drawn numbers and Y the larger. Find:

(a) the probability function of (X, Y).

(b) $E(X + Y)$.

(c) $V(X + Y)$.

129. A die is tossed n times. Let X_1 and X_2 be the number of 1's and of 2's, respectively. Find:

(a) the probability function of (X_1, X_2).

(b) $\mathrm{Cov}(X_1, X_2)$.

130. A point (U_1, U_2) is selected uniformly in $[0, 1] \times [0, 2]$. Find the distribution function and the density function of the following random variables:

(a) U_1.

(b) U_2.

(c) $X = \sqrt{U_2/2}$.

(d) $V = \max(U_1, U_2)$.

(e) $Y = \min(U_1, U_2)$.

(f) $W = U_1 \cdot U_2$.

131. Suppose that (X, Y) is a two-dimensional continuous random variable with joint density function defined by:

$$f_{X,Y}(x, y) = \begin{cases} cy^2, & 0 \le x \le 2,\ 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find:

(a) the value of the constant c.

(b) $P(X+Y>2)$.

(c) $P(Y<\frac{1}{2})$.

(d) $P(X\leq 1)$.

(e) $P(X=3Y)$.

(f) $E(X)$.

(g) $\mathrm{Cov}(X,Y)$.

132. Suppose that (X,Y) is a two-dimensional continuous random variable with joint density function defined by:

$$f_{X,Y}(x,y)=\begin{cases} c(x^2+y), & 0\leq y\leq 1-x^2,\\ 0, & \text{otherwise.}\end{cases}$$

Find:

(a) the value of the constant c.

(b) $P(0\leq X\leq \frac{1}{2})$.

(c) $P(Y\leq X+1)$.

133. Suppose that (X,Y) is a two-dimensional continuous random variable with joint density function defined by:

$$f_{X,Y}(x,y)=\begin{cases} 2, & 0\leq x\leq y\leq 1,\\ 0, & \text{otherwise.}\end{cases}$$

Find:

(a) $\mathrm{Cov}(X,Y)$.

(b) $\mathrm{Cov}(X+Y,Y-X)$.

(c) the marginal density function of X.

134. Suppose that (X,Y) is a two-dimensional continuous random variable with joint density function defined by:

$$f_{X,Y}(x,y)=\begin{cases} c(2x^2+y^2), & x^2+y^2\leq 1,\\ 0, & \text{otherwise.}\end{cases}$$

Find:

(a) the value of the constant c.

(b) $E(X^2)$.

(c) $\rho(X, Y)$.

135. Suppose that (X, Y) is a two-dimensional continuous random variable with joint density function defined by:

$$f_{X,Y}(x, y) = \begin{cases} c, & x^2 + y^2 \leq R^2, \\ 0, & \text{otherwise.} \end{cases}$$

Find:

(a) the value of the constant c.

(b) the density functions of X and Y.

(c) the distribution function of the distance D of a random point from the origin.

(d) $E(D)$.

136. Suppose that (X, Y) is a two-dimensional continuous random variable with joint density function defined by:

$$f_{X,Y}(x, y) = \begin{cases} cxy, & 0 \leq x \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find:

(a) the value of the constant c.

(b) $\mathrm{Cov}(X, Y)$.

(c) $\mathrm{Cov}(X + Y, X - Y)$.

(d) $V(X + Y)$.

137. Suppose that (X, Y) is a two-dimensional continuous random variable with joint density function defined by:

$$f_{X,Y}(x, y) = \begin{cases} \frac{c}{\sqrt{x^2+y^2}}, & x, y \geq 0,\ 1 \leq x^2 + y^2 \leq 4, \\ 0, & \text{otherwise.} \end{cases}$$

Find:

(a) the value of the constant c.

(b) $E(X)$.

(c) $V(X)$.

(d) $\rho(X, Y)$.

138. Let $F_{X,Y}$ be the joint distribution function of (X, Y) and F_X and F_Y be the marginal distribution functions of X and Y, respectively. Prove:

$$F_X(x)+F_Y(y)-1 \leq F_{X,Y}(x,y) \leq \sqrt{F_X(x) \cdot F_Y(y)}, \qquad -\infty < x, y < \infty.$$

10 Independence

139. Suppose that $X \sim U(0,1)$ and $Y \sim U(-1,0)$ and are independent random variables. Find the density function of $Z = X - Y$.

140. Let g and h be non-decreasing functions. Prove that for any random variable X we have

$$\mathrm{Cov}(g(X), h(X)) \geq 0.$$

(Hint: Let X and Y be independent identically distributed random variables. Consider the non-negative random variable $(g(X) - g(Y))(h(X) - h(Y))$.)

141. Let $U \sim U(0,1)$ and let h be a monotonic function. Prove that

$$\mathrm{Cov}(h(U), h(1-U)) \leq 0.$$

(Hint: Use the result of question 140.)

142. Let $X_i,\ i = 1, 2, \ldots, m$, be independent random variables distributed according to one of the following possibilities:

(a) $X_i \sim P(\lambda_i)$.

(b) $X_i \sim B(n_i, p)$.

(c) $X_i \sim G(p)$.

Let $Y = \sum_{i=1}^{m} X_i$. Prove that, correspondingly:

(a) $Y \sim P\left(\sum_{i=1}^{m} \lambda_i\right)$.

(b) $Y \sim B\left(\sum_{i=1}^{m} n_i, p\right)$.

(c) $Y \sim \overline{B}(m, p)$.

143. Let $X_i,\ i = 1, 2$, be independent random variables. Find the distribution of $X_1 + X_2$ if:

(a) $X_i \sim U[a_i, b_i],\ i = 1, 2$.

(b) $X_i \sim U(a_i, b_i),\ i = 1, 2$.

144. Show that for every k there exist k dependent random variables, any $k-1$ of which are independent.

145. A die is rolled twice. Let X_i denote the outcome of the ith roll, and put $S = X_1 + X_2$ and $D = |X_1 - X_2|$.

(a) Show that $E(SD) = E(S)E(D)$.

(b) Are S and D independent?

146. Show that, if X and Y are random variables assuming only two values each, and $E(XY) = E(X)E(Y)$, then X and Y are independent.

147. Let X and Y be discrete random variables, each assuming only finitely many possible values. Let $P_{ij} = P_{X,Y}(x_i, y_j)$. Show that X and Y are independent if and only if the matrix of probabilities $P = (P_{ij})$ is of rank 1.

148. A tetrahedron, with the numbers 1, 2, 3 and 4 marked on its faces, is tossed twice. Let X be the smaller of the two outcomes and Y the larger. Find $\rho(X, Y)$.

149. Let $X \sim U[-k, k]$ and $Y = X^2$.

(a) Find $\rho(X, Y)$.

(b) Explain intuitively the result of part (a) .

150. Let X and Y be independent identically distributed random variables, and put $S = X + Y$ and $D = X - Y$. Find:

(a) $\rho(X, S)$.

(b) $\rho(Y, D)$.

(c) $\rho(S, D)$.

151. Suppose that the probability function of (X_1, X_2, X_3) is trinomial with parameters n and (p_1, p_2, p_3). Find

(a) $E(X_1)$.

(b) $V(X_1)$.

(c) $\rho(X_1, X_2)$.

152. Let $X \sim P(\lambda)$. After a value for X is selected randomly, a fair coin is tossed X times. Let Y be the number of heads in these tosses. Find:

(a) the probability function of (X, Y).

(b) $E(Y)$ and $V(Y)$.

(c) $\rho(X, Y)$.

153. A fair die is tossed twice. Let X be the number of sixes in the two tosses and Y the number of even outcomes.

(a) Find the probability function of (X, Y).

(b) Are X and Y independent?

(c) Find $\rho(X, Y)$.

154. Let $X, Y_1, \ldots, Y_k, Z_1, \ldots, Z_k$ be independent random variables, each with mean 0 and finite variance σ^2. The random variable N is independent of all these variables, and its probability function is given by

$$p_N(j) = \begin{cases} p_j, & j = 1, 2, \ldots, k, \\ 0, & \text{otherwise}, \end{cases}$$

(where $p_j \geq 0$, $\sum_{j=1}^{k} p_j = 1$). Define random variables U, V by:

$$U = X + \sum_{j=1}^{N} Y_j, \qquad V = X + \sum_{j=1}^{N} Z_j \, .$$

(a) Show that N, UV are not independent, yet they are uncorrelated.

(b) Find $\rho(U, V)$.

155. Let $X \sim U(0, 1)$. Find:

(a) $\rho(X^2, X^3)$.

(b) $\rho\left(X, e^X\right)$.

(c) $\rho(\sin 2\pi X, \cos 2\pi X)$.

(d) $\rho(-\ln X, \ln^2 X)$.

156. Let X and Y be independent random variables with $E(X) = E(Y) = 0$ and $V(X) = V(Y) = 1$. Let $-1 \leq c \leq 1$ and $Z = cX + \sqrt{1 - c^2} \cdot Y$. Find:

(a) $\rho(X, Z)$.

(b) $\rho(Y, Z)$.

157. Let (X, Y) be uniformly distributed in the triangle with vertices $(0,0)$, $(0,2)$ and $(1,0)$. Find:

(a) the density functions of X and Y.

(b) $\rho(X, Y)$.

158. Let (X, Y) be uniformly distributed in the square with vertices $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$.

(a) Find $\rho(X, Y)$.

(b) Are X and Y independent? uncorrelated?

(c) Calculate $P(XY > 0)$.

(d) Calculate $P(X^2 + Y^2 < \frac{1}{4})$.

11 Normal Distribution

159. Calculate $\int_0^\infty e^{-x^2} dx$ by the following steps:

(a) Prove that:

$$(1 + x)e^{-x} \leq 1, \qquad -\infty < x < \infty.$$

(b) Using part (a), show that

$$1 - x^2 \leq e^{-x^2}, \qquad 0 < x < 1,$$

and

$$e^{-x^2} \leq \frac{1}{1 + x^2}, \qquad x > 0.$$

(c) Show that for every positive integer n

$$\int_0^1 (1 - x^2)^n dx = \frac{(2n)!!}{(2n+1)!!} \ ,$$

and

$$\int_0^\infty \frac{1}{(1 + x^2)^n} dx = \frac{(2n-3)!!}{(2n-2)!!} \cdot \frac{\pi}{2} \ .$$

(Here, $m!!$ is defined as $1 \cdot 3 \cdot 5 \cdot \ldots \cdot m$ for odd m and as $2 \cdot 4 \cdot 6 \cdot \ldots \cdot m$ for even m.)

(d) Using parts (b) and (c), prove that for every positive integer n:

$$\sqrt{n}\frac{(2n)!!}{(2n+1)!!} \le \int_0^\infty e^{-x^2}dx \le \sqrt{n}\frac{(2n-3)!!}{(2n-2)!!}\frac{\pi}{2} .$$

(e) Using part (d) and Wallis's formula $\frac{2\cdot 2\cdot 4\cdot 4\cdot \ldots \cdot 2n\cdot 2n}{1\cdot 3\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)\cdot (2n+1)} \xrightarrow[n\to\infty]{} \frac{\pi}{2}$, show that

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2} .$$

160. Let X be a random variable with density function

$$f_X(x) = c(2e^{-x^2} + 3e^{-2x^2}), \qquad -\infty < x < \infty.$$

Find:

(a) the value of c.

(b) $E(X)$ and $V(X)$.

161. Show that the even-order moments of the standard normal distribution are given by:

$$E\left(Z^{2k}\right) = (2k-1)!!, \qquad k \ge 1.$$

(See Problem 159.(b) for the notation.)

162. Find $E(|X|)$ and $E(|X|^3)$ if $X \sim N(0,1)$.

163. Find $E(X^3)$ if $X \sim N(\mu, \sigma^2)$.

12 Limit Theorems

164. Let g be a positive increasing function. Prove that for any random variable X:

$$P(|X| \ge t) \le \frac{E(g(|X|))}{g(t)} .$$

165. Let $(X_n)_{n=1}^\infty$ be a sequence of identically distributed random variables with mean μ and variance σ^2. Let $(Y_n)_{n=1}^\infty$ be a sequence of random variables, each taking the values ± 1 with probabilities $\frac{1}{2}$. Assume that all variables are independent. Form the sequence:

$$S_n = Y_1X_1 + Y_2X_2 + \ldots + Y_nX_n, \qquad n = 1, 2, \ldots .$$

Show that $S_n/\sqrt{n}$ is asymptotically normal, and find its asymptotic mean and variance.

166. A random sample of size 54 is drawn from a population distributed according to the probability function

$$P(x) = \begin{cases} 1/3, & x = 3, 5, 8, \\ 0, & \text{otherwise.} \end{cases}$$

Estimate the probability for the sample mean to be between 5 and 5.2.

167. A random sample of size 200 is drawn from an Exp(2) population. Find a number d for which the probability for the sample mean to deviate from the population mean by more than d is approximately 0.05.

13 The Moment Generating Function

168. Find the moment generating function of X if:

(a) $X \sim B(n, p)$.

(b) $X \sim P(\lambda)$.

(c) $X \sim G(p)$.

(d) $X \sim U(a, b)$.

(e) $X \sim \mathrm{Exp}(\theta)$.

169. Let $X \sim P(\lambda_1), Y \sim P(\lambda_2)$ be independent. Find the moment generating function of $X + Y$

170. Let $X \sim U(0, 1)$. Using moment generating functions, calculate $E(X^n)$ and $V(X^n)$ for each n.

171. Let X and Y be independent random variables, $X, Y \sim U(0, a)$. Find the moment generating function of the following variables:

(a) $X + Y$.

(b) $X - Y$.

(c) $|X - Y|$.

(d) XY.

172. Let $Z_i \sim N(\mu_i, \sigma_i)$, $i = 1, 2, \ldots, n$, be independent. Let $Z = \frac{1}{n} \sum_{i=1}^{n} Z_i$. Find the moment generating function of Z.