Final #2

Mark the correct answer in each part of the following questions.

1. Reuven and Shimon play the following infinite-stage game. At each stage k, they toss $2k$ coins. If the sequence is palindromic (namely, for each $1 \leq j \leq k$, the result of the j-th toss is the same as that of the $(2k-j+1)$-st toss), Reuven gets from Shimon k shekels. If the sequence is anti-palindromic (namely, for each $1 \leq j \leq k$, the result of the j-th toss is the opposite of the result of the $(2k-j+1)$-st toss), Shimon gets from Reuven $k-1$ shekels. Finally, if all $2k$ tosses yield the same result, Shimon gets from Reuven $k(k-1) \cdot 3^k$ shekels. (Note that, in the latter case, Reuven also gets from Shimon k shekels.) Let R be the total revenues of Reuven and S the total revenues of Shimon. (Namely, R ignores the losses of Reuven and S ignores those of Shimon. For example, suppose that for $k = 1$ the resulting sequence was HT, for $k = 2$ it was HTTH, for $k = 3$ it was HHHHHH, and for no $k \geq 4$ did any of the players win anything. Then $R = 0 + 2 + 3 = 5$ and $S = (1-1) + 0 + 3 \cdot 2 \cdot 3^3 = 162$.)

   (a) Markov's inequality, applied to R, yields:

      (i) $P(R \geq 10) \leq 0.1$.

      (ii) $P(R \geq 10) \leq 0.2$.

      (iii) $P(R \geq 10) \leq 0.3$.

      (iv) $P(R \geq 10) \leq 0.4$.

      (v) None of the above.

   (b) Let $\mu_S = E(S)$. Then $\mu_S =$

      (i) 73.

      (ii) 97.

      (iii) 113.

(iv) 145.

(v) None of the above.

In the next 2 parts, assume that Reuven and Shimon repeat the whole infinite game infinitely many times.

(c) Let $S_1, S_2, \ldots, S_n$ be the revenues of Shimon in the first n games, where n is large. We would like to estimate $P\left(\bar{S}_n \leq \mu_S + \frac{1}{2\sqrt{n}}\right)$ using the central limit theorem.

(i) $P\left(\bar{S}_n \leq \mu_S + \frac{1}{2\sqrt{n}}\right) \approx \Phi(1/2)$.
(Here, Φ denotes the standard normal distribution function.)

(ii) $P\left(\bar{S}_n \leq \mu_S + \frac{1}{2\sqrt{n}}\right) \approx \Phi(1)$.

(iii) $P\left(\bar{S}_n \leq \mu_S + \frac{1}{2\sqrt{n}}\right) \approx \Phi(2)$.

(iv) It is impossible to use the central limit theorem to estimate the probability in question as the conditions of the theorem are not satisfied.

(v) None of the above.

(d) Let N be the number of the first game in which both Reuven and Shimon had positive revenues. The distribution of N is:

(i) Poissonian.

(ii) approximately, but not exactly, Poissonian.

(iii) geometric.

(iv) negative binomial, but not geometric.

(v) None of the above.

2. X is a continuous random variable, distributed uniformly with parameters 0 and 1. Let $Y = X^{-1/10}$ and $W = \lfloor 1/X \rfloor$. (Here, $\lfloor \cdot \rfloor$ is the integer part function; for example, if $X = 3/20$, then $W = 6$.)

(a) The density function f_Y is given by:

(i)
$$f_Y(y) = \begin{cases} 10y^{-11}, & y \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(ii)
$$f_Y(y) = \begin{cases} 10y^{-1/11}, & y \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(iii)
$$f_Y(y) = \begin{cases} 9y^{-10}, & y \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(iv)
$$f_Y(y) = \begin{cases} 9y^{-1/10}, & y \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(v) None of the above.

(b) The probability that W is odd is

(i) non-existent, as the series defining it converges conditionally and not absolutely.

(ii) $\ln 2$.

(iii) $2/e$.

(iv) $2/\pi$.

(v) None of the above.

(c) $V(Y) =$

(i) $\frac{5}{324}$.

(ii) $\frac{15}{324}$.

(iii) $\frac{25}{324}$.

(iv) $\frac{35}{324}$.

(v) None of the above.

(d) Denote

$$a_n = \rho\left(e^{-Y^2}, \ln\left(Y^n\right)\right), \qquad n = 1, 2, \ldots.$$

The sequence $(a_n)_{n=1}^{\infty}$ is

(i) increasing and converges to a negative number.

(ii) decreasing and converges to a number greater than -1.

(iii) is constant.

(iv) of some sign for all even n and of the opposite sign for all odd n.

(v) None of the above.

3. Suppose that (X, Y) is a two-dimensional continuous random variable with joint density function, defined by

$$f_{X,Y}(x, y) = \begin{cases} x + cy, & x, y \geq 0,\ x^2 + y^2 \leq 1, \\ \\ 0, & \text{otherwise}, \end{cases}$$

for an appropriate constant c.

(a) $c =$

(i) 1.

(ii) 2.

(iii) 3.

(iv) 4.

(v) None of the above.

(b) $P\left(X > \frac{4Y}{3} | X > \frac{Y}{2}\right) =$

(i) $\frac{1}{6}$.

(ii) $\dfrac{1}{3}$.

(iii) $\dfrac{1}{2}$.

(iv) $\dfrac{2}{3}$.

(v) None of the above.

(c) $E\left(e^{-(X^2+Y^2)}\right) =$

(i) $-\dfrac{3}{2e} - \dfrac{\sqrt{\pi}}{2} + \sqrt{\pi}\Phi(\sqrt{2})$.
(Here, Φ denotes the standard normal distribution function.)

(ii) $-\dfrac{1}{e} - \dfrac{3\sqrt{\pi}}{4} + \dfrac{3\sqrt{\pi}\Phi(\sqrt{2})}{4}$.

(iii) $-\dfrac{3}{2e} - \dfrac{3\sqrt{\pi}}{4} + \sqrt{\pi}\Phi(\sqrt{2})$.

(iv) $-\dfrac{1}{e} - \dfrac{\sqrt{\pi}}{2} + \dfrac{3\sqrt{\pi}\Phi(\sqrt{2})}{4}$.

(v) None of the above.

(d) Let $\Theta = \operatorname{arctg} \frac{Y}{X}$. The distribution function F_Θ is given by:

(i) $F_\Theta(\theta) = \begin{cases} 0, & \theta < 0, \\ \dfrac{\sin\theta - 2\cos\theta + 2}{3}, & 0 \le \theta \le \frac{\pi}{2}, \\ 1, & \theta > \pi/2. \end{cases}$

(ii) $F_\Theta(\theta) = \begin{cases} 0, & \theta < 0, \\ \dfrac{\sin\theta - 3\cos\theta + 3}{4}, & 0 \le \theta \le \frac{\pi}{2}, \\ 1, & \theta > \pi/2. \end{cases}$

(iii) $F_\Theta(\theta) = \begin{cases} 0, & \theta < 0, \\ \dfrac{\sin\theta - 2\sqrt{2}\cos\theta + 2\sqrt{2}}{1+2\sqrt{2}}, & 0 \le \theta \le \frac{\pi}{2}, \\ 1, & \theta > \pi/2. \end{cases}$

(iv) $F_\Theta(\theta) = \begin{cases} 0, & \theta < 0, \\ \dfrac{\sin\theta - 3\sqrt{2}\cos\theta + 3\sqrt{2}}{1+3\sqrt{2}}, & 0 \le \theta \le \frac{\pi}{2}, \\ 1, & \theta > \pi/2. \end{cases}$

(v) None of the above.

4. Let $(X_k)_{k=1}^{\infty}$ be sequence of independent Exp(2)-distributed random variables.

   (a) Let n be a large positive integer. Let us say (for the purposes of this question only) that an index i in the range between 1 and n is good if $X_i \le 1/n$; an index i in the range between $n+1$ and $2n$ is good if $1/2n \le X_i \le 1/n$. Let N be the number of good indices in the whole range between 1 and $2n$. Then $P(N = 1) \approx$

   (i) $\dfrac{2}{e^2}$.

(ii) $\dfrac{3}{2e^{3/2}}$.

(iii) $\dfrac{1}{e}$.

(iv) $\dfrac{1}{2\sqrt{e}}$.

(v) None of the above.

(b) Recall the theorem proved in class, according to which a sequence of independent identically distributed random variables, with finite expectation and variance, satisfies the weak law of large numbers. Here we would like to find out to which of the four sequences $(Y_{k1})_{k=1}^{\infty}, (Y_{k2})_{k=1}^{\infty}, (Y_{k3})_{k=1}^{\infty}, (Y_{k4})_{k=1}^{\infty}$, defined below, the theorem applies to prove that it satisfies the weak law of large numbers. (Note: We are not asking which of the sequences satisfies in fact the law. We only ask to which of them the theorem applies.)

(i) $Y_{k1} = X_k + X_{k+1},\ 1 \le k < \infty$.
(ii) $Y_{k2} = X_k \cdot X_{k+1},\ 1 \le k < \infty$.
(iii) $Y_{k3} = e^{X_k/2},\ 1 \le k < \infty$.
(iv) $Y_{k4} = e^{X_k},\ 1 \le k < \infty$.
(v) None of the above.

(c) For large n, we have $P\left(\sum_{i=1}^{n} X_i \le \frac{n}{2} - \sqrt{n}\right) \approx$

(i) $1 - \Phi(2)$.

(ii) $1 - \Phi(1/2)$.

(iii) $\Phi(1/2)$.

(iv) $\Phi(2)$.

(v) None of the above.

Solutions

1. (a) Let R_k be Reuven's revenue at the k-th stage of the game. Then:

$$R_k = \begin{cases} k, & \text{the word accepted is a palindrome,} \\ 0, & \text{otherwise.} \end{cases}$$

Notice that $R = \sum_{k=1}^{\infty} R_k$. In order to use Markov's inequality, first find $E\left(R\right)$:

$$\begin{aligned} E\left(R\right) &= E\left(\sum_{k=1}^{\infty} R_k\right) \\ &= \sum_{k=1}^{\infty} E\left(R_k\right) \\ &= \sum_{k=1}^{\infty} k \cdot P\left(R_k = k\right) \\ &= \sum_{k=1}^{\infty} k \cdot \frac{1}{2^k}. \end{aligned}$$

Now the right-hand side is the expectation of a $G(\frac{1}{2})$-distributed random variable. Hence, $E\left(R\right) = 2$. Applying Markov's inequality, we bound the probability in question:

$$\begin{aligned} P\left(R \geq 10\right) &\leq \frac{E\left(R\right)}{10} \\ &= \frac{2}{10}. \end{aligned}$$

Thus, (ii) is true.

(b) Similarly to (a), let S_k be Shimon's revenue at the k-th stage of the game. Then:

$$S_k = \begin{cases} k-1, & \text{the obtained sequence is an anti-palindrome,} \\ k(k-1)3^k, & \text{all } 2k \text{ tosses yield the same result,} \\ 0, & \text{otherwise.} \end{cases}$$

Applying similar calculations:

$$\begin{aligned} E\left(S\right) &= E\left(\sum_{k=1}^{\infty} S_k\right) \\ &= \sum_{k=1}^{\infty} E\left(S_k\right) \\ &= \sum_{k=1}^{\infty} (k-1)\cdot P\left(S_k = k-1\right) + k(k-1)3^k \cdot P\left(S_k = k(k-1)3^k\right) \\ &= \sum_{k=1}^{\infty} (k-1)\cdot \frac{1}{2^k} + k(k-1)3^k \cdot \frac{2}{2^{2k}} \\ &= \sum_{k=1}^{\infty} k\cdot\frac{1}{2^k} - \sum_{k=1}^{\infty}\frac{1}{2^k} + 2\cdot\left(\sum_{k=1}^{\infty} k^2 3^k \cdot \frac{1}{4^k} - \sum_{k=1}^{\infty} k3^k\cdot\frac{1}{4^k}\right) \\ &= 2 - 1 + 2\cdot\frac{3}{4}\cdot\frac{4}{1}\left(\sum_{k=1}^{\infty} k^2\cdot\left(\frac{3}{4}\right)^{k-1}\cdot\frac{1}{4} - \sum_{k=1}^{\infty} k\left(\frac{3}{4}\right)^{k-1}\cdot\frac{1}{4}\right). \end{aligned}$$

Notice that

$$\sum_{k=1}^{\infty} k^2\cdot\left(\frac{3}{4}\right)^{k-1} = E\left(X^2\right)$$

and

$$\sum_{k=1}^{\infty} k\left(\frac{3}{4}\right)^{k-1}\cdot\frac{1}{4} = E\left(X\right),$$

where X is a $G(\frac{1}{4})$-distributed random variable . Hence,

$$\sum_{k=1}^{\infty} k\left(\frac{3}{4}\right)^{k-1}\cdot\frac{1}{4} = 4$$

and

$$\sum_{k=1}^{\infty} k^2 \cdot \left(\frac{3}{4}\right)^{k-1} = V(X) + (E(X))^2$$
$$= 12 + 16 = 28.$$

Finally:

$$E(S) = 1 + 6 \cdot 24 = 145$$

Thus, (i) is true.

(c) In order to use the central limit theorem, we should first find out whether the conditions of the theorem are satisfied. Let us calculate $V(S_k) = V(S)$. Note that $S^2 \geq S_k^2$ for all $k \in \mathbb{N}$. Therefore:

$$\begin{aligned} E\left(S^2\right) &\geq E\left(S_k^2\right) \\ &= (k-1)^2 \cdot \frac{1}{2^k} + k^2(k-1)^2 3^{2k} \cdot \frac{2}{2^{2k}} \\ &= (k-1)^2 \cdot \frac{1}{2^k} + 2 \cdot k^2(k-1)^2 \cdot \left(\frac{9}{4}\right)^k \xrightarrow[k\to\infty]{} \infty \end{aligned}$$

Hence $V(S)$ is infinite and the conditions of the theorem are not satisfied.

Thus, (iv) is true.

(d) First, let us denote $p = P(S, R > 0)$. Notice that all games played are identical and independent. Let us say that at the n-th game we have a success if $S_n, R_n > 0$. Therefore N is the first success when repeating the same experiment independently, and the probability for success is p. It is clear that $0 < p < 1$, so by the definition of geometric distribution, $X \sim G(p)$.

Thus, (iii) is true.

2. (a) In order to find f_Y, let us first find F_Y:

$$\begin{aligned} F_Y(t) &= P(Y \le t) \\ &= P\left(X^{-\frac{1}{10}} \le t\right) \\ &= P\left(X \ge t^{-10}\right) \\ &= P\left(X > t^{-10}\right) \\ &= 1 - F_X\left(t^{-10}\right) \\ &= \begin{cases} 1 - t^{-10}, & t^{-10} \in [0,1], \\ 1, & \text{otherwise}, \end{cases} \\ &= \begin{cases} 1 - t^{-10}, & t \ge 1. \\ 1, & \text{otherwise}. \end{cases} \end{aligned}$$

Hence:

$$f_Y(y) = F_Y'(y) = \begin{cases} 10y^{-11}, & y \ge 1, \\ 0, & \text{otherwise} . \end{cases}$$

Thus, (i) is true.

(b) We have

$$\begin{aligned}
P\left(W \text{ is odd}\right) &= P\left(W \in \{1,3,5,7\dots\}\right) \\
&= \sum_{k=1}^{\infty} P\left(W = 2k-1\right) \\
&= \sum_{k=1}^{\infty} P\left(\left\lfloor \frac{1}{X} \right\rfloor = 2k-1\right) \\
&= \sum_{k=1}^{\infty} P\left(2k-1 \leq \frac{1}{X} < 2k\right) \\
&= \sum_{k=1}^{\infty} P\left(\frac{1}{2k} < X \leq \frac{1}{2k-1}\right) \\
&= \sum_{k=1}^{\infty} \left(F_X\left(\frac{1}{2k-1}\right) - F_X\left(\frac{1}{2k}\right)\right) \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{2k-1} - \frac{1}{2k}\right) \\
&= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \ln 2.
\end{aligned}$$

Thus, (i) is true.

(c) We have

$$\begin{aligned}
V\left(Y\right) &= E\left(Y^2\right) - \left(E\left(Y\right)\right)^2 \\
&= E\left(X^{-\frac{2}{10}}\right) - E\left(X^{-\frac{1}{10}}\right)^2 \\
&= \int_0^1 t^{-\frac{1}{5}}\,dt - \left(\int_0^1 t^{-\frac{1}{10}}\,dt\right)^2 \\
&= \frac{5}{4}\left(1-0\right) - \left(\frac{10}{9}\left(1-0\right)\right)^2 \\
&= \frac{5}{4} - \frac{100}{81} = \frac{5}{324}.
\end{aligned}$$

Thus, (i) is true.

(d) We have

$$a_n = \rho\left(e^{-Y^2}, \ln\left(Y^n\right)\right), \qquad n = 1, 2, \ldots.$$

First, let us verify that a_n is well defined. Indeed:

$$\begin{aligned} E\left(\left(e^{-Y^2}\right)^2\right) &= E\left(e^{-2Y^2}\right) \\ &= \int_1^\infty e^{-2y^2} \cdot 10y^{-11} dy < \infty, \end{aligned}$$

$$\begin{aligned} E\left(\left(\ln\left(Y^n\right)\right)^2\right) &= E\left(\left(n \cdot \ln Y\right)^2\right) \\ &= n^2 \cdot E\left(\left(\ln Y\right)^2\right) \\ &= n^2 \cdot \int_1^\infty \left(\ln y\right)^2 \cdot 10y^{-11} dy < \infty, \end{aligned}$$

and

$$E\left(\ln\left(Y^n\right) e^{-Y^2}\right) = n \cdot \int_1^\infty e^{-y^2} \ln y \cdot 10y^{-11} dy < \infty.$$

Similarly, $E\left(e^{-Y^2}\right)$ and $E\left(\ln\left(Y^n\right)\right)$ are well defined. Therefore, $V\left(e^{-Y^2}\right)$, $V\left(\ln\left(Y^n\right)\right)$ and $\operatorname{Cov}\left(\ln\left(Y^n\right), e^{-Y^2}\right)$ are well defined, which implies that a_n is well defined, for every $n \in \mathbb{N}$.
The correlation coefficient ρ is invariant to multiplication by scalars, and therefore:

$$\begin{aligned} a_n &= \rho\left(e^{-Y^2}, \ln\left(Y^n\right)\right) \\ &= \rho\left(e^{-Y^2}, n \cdot \ln Y\right) \\ &= \rho\left(e^{-Y^2}, \ln Y\right). \end{aligned}$$

That is, the sequence $(a_n)_{n=1}^{\infty}$ is constant

Thus, (iii) is true.

3. (a) We have

$$1 = \iint_S (x + cy) \cdot dxdy,$$

where $S = \{(x, y) : x, y \geq 0,\ x^2 + y^2 \leq 1\}$. Hence,

$$\begin{aligned}
1 &= \int_0^1 \int_0^{\pi/2} (r\cos\theta + cr\sin\theta) \cdot r dr d\theta \\
&= \int_0^1 r^2 dr \int_0^{\pi/2} (\cos\theta + c\sin\theta)\, d\theta \\
&= \frac{1}{3} \cdot \int_0^{\pi/2} (\cos\theta + c\sin\theta)\, d\theta \\
&= \frac{1}{3} \left[\sin\theta - c\cos\theta\right]_0^{\frac{\pi}{2}} \\
&= \frac{1 + c}{3}.
\end{aligned}$$

That is, $c = 2$.

Thus, (ii) is true.

(b) We have

$$\begin{aligned}
P\left(X > \frac{4Y}{3} \,|\, X > \frac{Y}{2}\right) &= \frac{P\left(X > \frac{4Y}{3} \bigcap X > \frac{Y}{2}\right)}{P\left(X > \frac{Y}{2}\right)} \\
&= \frac{P\left(X > \frac{4Y}{3}\right)}{P\left(X > \frac{Y}{2}\right)}.
\end{aligned}$$

Denoting $\theta_0 = \text{ arctg } \frac{3}{4}$:

$$\begin{aligned}
P\left(X > \frac{4Y}{3}\right) &= \int_0^1 \int_0^{\theta_0} (r\cos\theta + 2r\sin\theta)\cdot r dr d\theta \\
&= \frac{1}{3}\cdot\int_0^{\theta_0} (\cos\theta + 2\sin\theta)\, d\theta \\
&= \frac{1}{3}\left[\sin\theta - 2\cos\theta\right]_0^{\theta_0}.
\end{aligned}$$

Notice that calculating θ_0 is not necessary; $\sin\theta$ and $\cos\theta$ can be evaluated at θ_0 using basic trigonometry : $\sin\theta_0 = \frac{3}{5}$ and $\cos\theta_0 = \frac{4}{5}$.

$$\begin{aligned}
P\left(X > \frac{4Y}{3}\right) &= \frac{1}{3}\left[\sin\theta - c\cos\theta\right]_0^{\theta_0} \\
&= \frac{1}{3}\cdot\left(\frac{3}{5} - 0 + 2\cdot\left(1 - \frac{4}{5}\right)\right) \\
&= \frac{1}{3}.
\end{aligned}$$

Similarly, denoting $\theta_1 = \text{ arctg } 2$, we have $\sin\theta_1 = \frac{2}{\sqrt{5}}$ and $\cos\theta_1 = \frac{1}{\sqrt{5}}$.

$$\begin{aligned}
P\left(X > \frac{Y}{2}\right) &= \frac{1}{3}\left[\sin\theta - c\cos\theta\right]_0^{\theta_1} \\
&= \frac{1}{3}\cdot\left(\frac{2}{\sqrt{5}} - 0 + 2\cdot\left(1 - \frac{1}{\sqrt{5}}\right)\right) \\
&= \frac{2}{3}.
\end{aligned}$$

Altogether

$$\begin{aligned}
P\left(X > \frac{4Y}{3} \mid X > \frac{Y}{2}\right) &= \frac{P\left(X > \frac{4Y}{3}\right)}{P\left(X > \frac{Y}{2}\right)} \\
&= \frac{1}{2}.
\end{aligned}$$

Thus, (iii) is true.

(c) We have

$$
\begin{aligned}
E\left(e^{-\left(X^2+Y^2\right)}\right) &= \int_0^1 \int_0^{\pi/2} e^{-r^2}\left(r\cos\theta + 2r\sin\theta\right)\cdot r dr d\theta \\
&= \int_0^1 r^2 e^{-r^2} dr \int_0^{\pi/2} \left(\cos\theta + 2\sin\theta\right) d\theta \\
&= 3\cdot \int_0^1 r^2 e^{-r^2} dr \\
&= -\frac{3}{2}\cdot \int_0^1 r\cdot(-2r\cdot e^{-r^2}) dr.
\end{aligned}
$$

Using integration by parts and change of variable:

$$
\begin{aligned}
E\left(e^{-\left(X^2+Y^2\right)}\right) &= -\frac{3}{2}\cdot \int_0^1 r\cdot(-2r\cdot e^{-r^2}) dr \\
&= -\frac{3}{2}\cdot \left(\left[r\cdot e^{-r^2}\right]_0^1 - \int_0^1 e^{-r^2} dr\right) \\
&= -\frac{3}{2}\cdot \left(\frac{1}{e} - \sqrt{\pi}\int_0^1 \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}} ds\right) \\
&= -\frac{3}{2}\cdot \left(\frac{1}{e} - \sqrt{\pi}\left(\Phi\left(\sqrt{2}\right) - \Phi\left(0\right)\right)\right) \\
&= -\frac{3}{2e} - \frac{3}{4}\sqrt{\pi} + \frac{3}{2}\sqrt{\pi}\Phi\left(\sqrt{2}\right).
\end{aligned}
$$

Thus, (v) is true.

(d) For $\theta \in \left[0, \frac{\pi}{2}\right]$ we have

$$\begin{aligned}
F_\Theta(\theta) &= P\left(\operatorname{arctg}\frac{Y}{X} \leq \theta\right) \\
&= P\left(\frac{Y}{X} \leq \operatorname{tg}\theta\right) \\
&= P\left(X \geq Y\operatorname{tg}\theta\right) \\
&= \int_0^1 \int_0^\theta (r\cos\theta + 2r\sin\theta) \cdot r dr d\theta \\
&= \frac{1}{3} \cdot \int_0^\theta (\cos\theta + 2\sin\theta)\, d\theta \\
&= \frac{1}{3}\left[\sin\theta - 2\cos\theta\right]_0^\theta \\
&= \frac{\sin\theta - 2\cos\theta + 2}{3}.
\end{aligned}$$

It is clear that for $\theta < 0$ we have $F_\Theta(\theta) = 0$, and for $\theta > \frac{\pi}{2}$ we have $F_\Theta(\theta) = 1$.

Thus, (i) is true.

4. (a) Let us define new random variables $(Z_i)_{i=1}^{2n}$ as follows: for $1 \leq i \leq n$ let

$$\begin{aligned}
Z_i &= \begin{cases} 1, & i \text{ is "good"}, \\ 0, & \text{otherwise}, \end{cases} \\
&= \begin{cases} 1, & i \leq \frac{1}{n}, \\ 0, & \text{otherwise}. \end{cases}
\end{aligned}$$

For $n+1 \leq i \leq 2n$ let

$$\begin{aligned}
Z_i &= \begin{cases} 1, & i \text{ is "good"}, \\ 0, & \text{otherwise}, \end{cases} \\
&= \begin{cases} 1, & \frac{1}{n} \leq X_i \leq \frac{1}{2n}, \\ 0, & \text{otherwise}. \end{cases}
\end{aligned}$$

We have

$$P\left(Z_i = 1\right) = \begin{cases} P\left(X_i \le \frac{1}{n}\right), & 1 \le i \le n, \\ \\ P\left(\frac{1}{2n} \le X_i \le \frac{1}{n}\right), & n+1 \le i \le 2n, \end{cases}$$

$$= \begin{cases} F_{X_i}(\frac{1}{n}), & 1 \le i \le n, \\ \\ F_{X_i}(\frac{1}{n}) - F_{X_i}(\frac{1}{2n}), & n+1 \le i \le 2n, \end{cases}$$

$$= \begin{cases} 1 - e^{-\frac{1}{n}}, & 1 \le i \le n, \\ \\ e^{-\frac{1}{2n}} - e^{-\frac{1}{n}}, & n+1 \le i \le 2n. \end{cases}$$

Denote: $S_1 = \sum_{i=1}^{n} Z_i$, $S_2 = \sum_{i=n+1}^{2n} Z_i$.
Since $(Z_i)_{i=1}^{2n}$ are independent identically distributed random variables such that $Z_i \sim B(1, 1 - e^{-\frac{1}{n}})$ for $1 \le i \le n$ and $Z_i \sim B(1, e^{-\frac{1}{2n}} - e^{-\frac{1}{n}})$ for $n+1 \le i \le 2n$, it is obvious that $S_1 \sim B(n, 1 - e^{-\frac{1}{n}})$ and $S_2 \sim B(n, e^{-\frac{1}{2n}} - e^{-\frac{1}{n}})$. Using L'Hpital's rule, one can easily check that:

$$\lim_{n\to\infty} n \cdot \left(1 - e^{-\frac{1}{n}}\right) = 1$$

and

$$\lim_{n\to\infty} n \cdot \left(e^{-\frac{1}{2n}} - e^{-\frac{1}{n}}\right) = \frac{1}{2}.$$

By the theorem proved in class, the distributions of S_1 and S_2 converge to Poisson distributions with parameters 1 and $\frac{1}{2}$ respectively.

Since n is large, S_1 is distributed approximately $P(1)$ and S_2 is distributed approximately $P(\frac{1}{2})$.
Notice that $N = \sum_{i=1}^{2n} Z_i = S_1 + S_2$, and distributed approximately $P(\frac{1}{2}+1)$ as a sum of two independent approximately Poisson distributed random variables with parameters 1 and $\frac{1}{2}$.
Hence $P\left(N_1 = 1\right) \approx \frac{3}{2}e^{-\frac{3}{2}}$.

Thus, (ii) is true.

(b) It it intuitively clear that $(Y_{k1})_{k=1}^{\infty}$ and $(Y_{k2})_{k=1}^{\infty}$ do not satisfy the independence condition. Let us prove it formally by showing that for any two consecutive indices, the covariance is not zero:

$$\begin{aligned}
E\left(Y_k \cdot Y_{k-1}\right) &= E\left(\left(X_k + X_{k-1}\right)\left(X_k + X_{k+1}\right)\right) \\
&= E\left(X_k \cdot X_{k-1}\right) + E\left(X_k \cdot X_{k+1}\right) \\
&+ E\left(X_{k-1} \cdot X_{k+1}\right) + E\left(X_k^2\right) \\
&= E\left(X_k\right) E\left(X_{k-1}\right) + E\left(X_k\right) E\left(X_{k+1}\right) \\
&+ E\left(X_{k-1}\right) E\left(X_{k+1}\right) + E\left(X_k^2\right) \\
&= 3E\left(X_k\right)^2 + E\left(X_k^2\right).
\end{aligned}$$

On the other hand:

$$\begin{aligned}
E\left(Y_k\right) E\left(Y_{k-1}\right) &= E\left(X_k + X_{k-1}\right) E\left(X_k + X_{k+1}\right) \\
&= \left(E\left(X_k\right) + E\left(X_{k-1}\right)\right)\left(E\left(X_k\right) + E\left(X_{k+1}\right)\right) \\
&= 4E\left(X_k\right)^2.
\end{aligned}$$

That is $E\left(Y_k \cdot Y_{k-1}\right) \neq E\left(Y_k\right) E\left(Y_{k-1}\right)$, so that $\mathrm{Cov}(Y_k, Y_{k-1}) \neq 0$, and in particular Y_k and Y_{k-1} are not independent. Similarly, one can easily show that Y_{k2} and $Y_{k-1,2}$ are not independent.

Now, let us show that $(Y_{k4})_{k=1}^{\infty}$ does not satisfy the condition of finite variance:

$$\begin{aligned}
E\left(Y_{k4}^2\right) &= E\left(e^{2X_k}\right) \\
&= \int_0^{\infty} e^{2t} \cdot e^{-2t} \cdot dt \\
&= \int_0^{\infty} 1 \cdot dt = \infty.
\end{aligned}$$

Therefore $V(Y_{k4}) = \infty$.

All (X_k) are independent, so it is clear that all (Y_{k3}) are independent as each Y_{k3} is a function X_k. Now,

$$\begin{aligned} E(Y_{k3}) &= E\left(e^{\frac{X_k}{2}}\right) \\ &= \int_0^\infty e^{\frac{1}{2}t} \cdot e^{-2t} \cdot dt = \int_0^\infty e^{\frac{-3}{2}t} \cdot dt < \infty, \end{aligned}$$

so that:

$$E\left(Y_{k3}^2\right) = E\left(e^{X_k}\right) = \int_0^\infty e^t \cdot e^{-2t} \cdot dt = \int_0^\infty e^{-t} \cdot dt < \infty.$$

Therefore, $E(Y_{k3}), V(Y_{k3}) < \infty$.
Since all X_k's are identically distributed, so are the Y_{k3}'s.

Thus, (iii) is true.

(c) Notice that the sequence $(X_k)_{k=1}^\infty$ satisfies the conditions of the central limit theorem.
Let $E(X_i) = \mu$, $\sqrt{V(X_k)} = \sigma$. Since $X_i \sim \text{Exp}(2)$ we have $\mu = \frac{1}{2}$ and $\sigma = \frac{1}{2}$. Hence:

$$\begin{aligned} P\left(\sum_{i=1}^n X_i \leq \frac{n}{2} - \sqrt{n}\right) &= P\left(\sum_{i=1}^n X_i - \frac{n}{2} \leq -\sqrt{n}\right) \\ &= P\left(\frac{\sum_{i=1}^n X_i - \frac{n}{2}}{n} \leq -\frac{\sqrt{n}}{n}\right) \\ &= P\left(\bar{X}_n - \frac{1}{2} \leq -\frac{1}{\sqrt{n}}\right) \\ &= P\left(\frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \leq -\frac{\frac{1}{\sqrt{n}}}{\frac{\sigma}{\sqrt{n}}}\right) \\ &= P\left(\frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \leq -\frac{1}{\sigma}\right) \\ &= P\left(\frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \leq -2\right). \end{aligned}$$

Since n is large, we may apply the central limit theorem:

$$\begin{aligned} P\left(\sum_{i=1}^{n} X_i \leq \frac{n}{2} - \sqrt{n}\right) &\approx \Phi\left(-2\right) \\ &= 1 - \Phi\left(2\right). \end{aligned}$$

Thus, (i) is true.