Final #1

Mark the correct answer in each part of the following questions.

1. Reuven and Shimon play the following game. Each of them in turn tosses a die, Reuven being the first. The game continues until, for the first time, either Reuven gets one of the two results $1, 6$ or Shimon gets one of the four results $2, 3, 4, 5$. Suppose the game ends after n tosses. If n is odd, Reuven gets from Shimon 2^n shekels, while if n is even then Shimon gets from Reuven 2^n shekels. Let X be the total number of tosses in the game, R – Reuven's win, and N_i the number of tosses in which the die showed i, $1 \leq i \leq 6$. (For example, if Reuven got 5, then Shimon got 1, Reuven got 3, Shimon got 6, Reuven got 2, and Shimon got 5, the random variables assume the following values: $X = 6, R = -64, N_1 = N_2 = N_3 = N_6 = 1, N_4 = 0, N_5 = 2$.)

   (a)

   (i) $P(R > 0) = \dfrac{3}{7} \cdot P(R < 0)$.

   (ii) $P(R > 0) = \dfrac{1}{2} \cdot P(R < 0)$.

   (iii) $P(R > 0) = \dfrac{3}{4} \cdot P(R < 0)$.

   (iv) $P(R > 0) = P(R < 0)$.

   (v) None of the above.

   (b) $P(N_5 = 20 | X = 100) =$

(i) $\dfrac{\binom{50}{20} \cdot \ 3^{30}}{4^{50}}$.

(ii) $\dfrac{\binom{50}{20} \cdot \ 3^{31}}{4^{50}}$.

(iii) $\dfrac{\binom{51}{20} \cdot \ 3^{30}}{4^{51}}$.

(iv) $\dfrac{\binom{51}{20} \cdot \ 3^{31}}{4^{51}}$.

(v) None of the above.

(c) $E(X) =$

(i) $\dfrac{13}{7}$.

(ii) $\dfrac{15}{7}$.

(iii) $\dfrac{17}{7}$.

(iv) $\dfrac{19}{7}$.

(v) None of the above.

(d)

(i) $E(R) = -20$ and $V(R) = 54$.

(ii) $E(R) = -10$ and $V(R)$ is infinite.

(iii) The series defining $E(R)$ converges conditionally, so that R does not have expectation and variance.

(iv) The series defining $E(R)$ diverges to $-\infty$, so that R does not have expectation and variance.

(v) None of the above.

2. The life length X of Sunlight+ light bulbs (in years) is distributed Exp(1). Student A bought a single Sunlight+ light bulb, while students B and C bought n light bulbs each. However, B installed all light bulbs in parallel, while C uses only one bulb at a time, and replaces it when it burns out.

   (a) The probability that A's light bulb will still work when all of B's light bulbs will burn out is

      (i) $\frac{1}{2}$.

      (ii) $\frac{1}{n+1}$.

      (iii) $\frac{1}{2n}$.

      (iv) $\frac{1}{2^n}$.

      (v) None of the above.

   (b) The probability that A's light bulb will still work when all of C's light bulbs will burn out is

      (i) $\frac{1}{n+1}$.

      (ii) $\frac{1}{2n}$.

      (iii) $\frac{1}{2^n}$.

      (iv) $\frac{1}{3^n-1}$.

      (v) None of the above.

   (c) The power of Sunlight+ light bulbs changes in the course of their life in such a way that the total energy, consumed by a bulb whose life length is X, is $Y = e^{\alpha X}$, where $\alpha \in (0, 1/2)$ is a constant.

(i) $\rho(X,Y) = \dfrac{\sqrt{1-\alpha^2}}{\sqrt{1+2\alpha}}$.

(ii) $\rho(X,Y) = 1-\alpha$.

(iii) $\rho(X,Y) = \sqrt{1-\alpha}$.

(iv) $\rho(X,Y) = \dfrac{\sqrt{1-2\alpha}}{1-\alpha}$.

(v) None of the above.

3. Suppose that (X,Y) is a two-dimensional continuous random variable with joint density function, defined by

$$f_{X,Y}(x,y) = \begin{cases} cx^2 e^{-(x^2+y^2)}, & x, y \geq 0,\ 1 \leq x^2+y^2 \leq 4, \\ \\ 0, & \text{otherwise}, \end{cases}$$

for an appropriate constant c.

(a) $c =$

(i) $\dfrac{e^4}{\pi(2e^3-5)}$.

(ii) $\dfrac{2e^4}{\pi(2e^3-5)}$.

(iii) $\dfrac{4e^4}{\pi(2e^3-5)}$.

(iv) $\dfrac{8e^4}{\pi(2e^3-5)}$.

(v) None of the above.

(b) $P(X > Y) =$

(i) $\frac{1}{2}$.

(ii) $\frac{e-1}{\pi}$.

(iii) $\frac{1}{2} + \frac{1}{\pi}$.

(iv) $\frac{1}{2} + \frac{e}{2\pi}$.

(v) None of the above.

(c) $E(1/X) =$

(i) $c \cdot \left(\frac{1}{2e} - \frac{1}{e^4} + \frac{\sqrt{\pi}\Phi(2\sqrt{2})}{2} - \frac{\sqrt{\pi}\Phi(\sqrt{2})}{2} \right)$.

(Here, Φ denotes the standard normal distribution function.)

(ii) $c \cdot \left(\frac{1}{e} - \frac{1}{e^4} + \frac{\sqrt{\pi}\Phi(2\sqrt{2})}{2} - \frac{\sqrt{\pi}\Phi(\sqrt{2})}{2} \right)$.

(iii) $c \cdot \left(\frac{1}{2e} - \frac{1}{e^4} + \frac{\sqrt{\pi}\Phi(2)}{2} - \frac{\sqrt{\pi}\Phi(1)}{2} \right)$.

(iv) $c \cdot \left(\frac{1}{e} - \frac{1}{e^4} + \frac{\sqrt{\pi}\Phi(2)}{2} - \frac{\sqrt{\pi}\Phi(1)}{2} \right)$.

(v) None of the above.

(d) Let $D = \sqrt{X^2 + Y^2}$. The density function f_D is given by:

(i) $f_D(d) = \begin{cases} \frac{c\pi d^3 e^{-d^2}}{8}, & 1 \le d \le 2, \\ 0, & \text{otherwise.} \end{cases}$

(ii) $f_D(d) = \begin{cases} \frac{c\pi d^3 e^{-d^2}}{6}, & 1 \le d \le 2, \\ 0, & \text{otherwise.} \end{cases}$

(iii) $f_D(d) = \begin{cases} \dfrac{c\pi d^3 e^{-d^2}}{4}, & 1 \leq d \leq 2, \\ \\ 0, & \text{otherwise.} \end{cases}$

(iv) $f_D(d) = \begin{cases} \dfrac{c\pi d^3 e^{-d^2}}{3}, & 1 \leq d \leq 2, \\ \\ 0, & \text{otherwise.} \end{cases}$

(v) None of the above.

4. Let $(U_n)_{n=1}^{\infty}$ be sequence of independent continuous $U(-1,1)$-distributed random variables.

(a) Let N_1 be the number of integers n between 1 and 10^6 for which $U_n > 1 - 10^{-6}$. Then $P(N_1 = 1) \approx$

(i) $\dfrac{1}{e^2}$.

(ii) $\dfrac{1}{2\sqrt{e}}$.

(iii) $\dfrac{1}{e}$.

(iv) $\dfrac{1}{\sqrt{e}}$.

(v) None of the above.

(b) Let N_2 be the number of integers n between 1 and 10^6 for which $|U_n| \leq 0.9$. Then $P(|N_2 - 900000| \leq 600) \approx$

(i) 0.34.

(ii) 0.48.

(iii) 0.68.

(iv) 0.95.

(v) None of the above.

(c) The following variable has an exponential distribution:

(i) $-\ln|U_1|$.

(ii) $e^{1/|U_1|}$.

(iii) $\operatorname{tg}(\pi|U_1|)$.

(iv) $\dfrac{1}{|U_1|} - 1$.

(v) None of the above.

(d) Define three additional sequences $(X_n)_{n=1}^{\infty}, (Y_n)_{n=1}^{\infty}, (Z_n)_{n=1}^{\infty}$ by

$$X_n = (2 + \sin n)U_n, \qquad Y_n = \sqrt{n}U_n, \qquad Z_n = nU_n,$$

for $n = 1, 2, \ldots$. Recall the theorem proved in class, according to which a sequence of independent identically distributed random variables, with finite expectation and variance, satisfies the weak law of large numbers. Here we would like to find out to which of the three sequences defined above the same technique would apply to prove that it satisfies the weak law of large numbers. (Note: We are not asking which of the sequences satisfies in fact the law. We only ask for which of them the technique of the proof we employed in the quoted theorem applies.)

(i) The technique works for all three sequences $(X_n)_{n=1}^{\infty}, (Y_n)_{n=1}^{\infty}$, and $(Z_n)_{n=1}^{\infty}$.

(ii) The technique works for $(X_n)_{n=1}^{\infty}$ and $(Y_n)_{n=1}^{\infty}$, but not for $(Z_n)_{n=1}^{\infty}$.

(iii) The technique works for $(X_n)_{n=1}^{\infty}$, but not for $(Y_n)_{n=1}^{\infty}$ and $(Z_n)_{n=1}^{\infty}$.

(iv) The technique works for none of the sequences $(X_n)_{n=1}^{\infty}, (Y_n)_{n=1}^{\infty}$, and $(Z_n)_{n=1}^{\infty}$.

(v) None of the above.

Solutions

1. (a) By the law of total probability

$$
\begin{aligned}
P(R > 0) &= \sum_{n=1}^{\infty} P(R > 0 \cap X = n) \\
&= \sum_{n=0}^{\infty} P(R > 0 \cap X = 2n+1) \\
&= \sum_{n=0}^{\infty} \left(\frac{2}{3} \cdot \frac{1}{3}\right)^{n} \cdot \frac{1}{3} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{9}\right)^{n} = \frac{3}{7}.
\end{aligned}
$$

Now, $P(R < 0) = 1 - P(R > 0) = \frac{4}{7}$, and thus $P(R > 0) = \frac{3}{4} \cdot P(R < 0)$.

Thus, (iii) is true.

(b) Given that the game has ended after 100 tosses, there were 50 times Reuven received one of the results $2, 3, 4, 5$, there were 49 times Shimon received one of the results $1, 6$, and at the 100-th toss Shimon received one of the results $2, 3, 4, 5$. By the independence of the tosses, the probability of receiving 5 at each of the tosses $1, 3, \ldots, 97, 99, 100$ is $\frac{1}{4}$. Thus, given that the game ended at the 100-th toss, the number of times the result 5 has occurred is $B\left(51, \frac{1}{4}\right)$-distributed. Hence:

$$
P(N_{20} | X = 100) = \binom{51}{20} \cdot \left(\frac{1}{4}\right)^{20} \cdot \left(\frac{3}{4}\right)^{31} = \frac{\binom{51}{20} \cdot 3^{31}}{4^{51}}.
$$

Thus, (iv) is true.

(c) We have:

$$
\begin{aligned}
E(X) &= \sum_{k=1}^{\infty} k \cdot P(X=k) \\
&= \sum_{k=0}^{\infty} (2k+1) \cdot P(X=2k+1) + \sum_{k=1}^{\infty} 2k \cdot P(X=2k) \\
&= \sum_{k=0}^{\infty} (2k+1) \cdot \left(\frac{2}{3} \cdot \frac{1}{3}\right)^{k} \cdot \frac{1}{3} + \sum_{k=1}^{\infty} 2k \cdot \left(\frac{2}{3} \cdot \frac{1}{3}\right)^{k-1} \frac{2}{3} \cdot \frac{2}{3} \\
&= \frac{1}{3} \sum_{k=0}^{\infty} \left(\frac{2}{9}\right)^{k} + \frac{2}{3} \sum_{k=0}^{\infty} k \cdot \left(\frac{2}{9}\right)^{k} + \frac{8}{9} \sum_{k=1}^{\infty} k \cdot \left(\frac{2}{3} \cdot \frac{1}{3}\right)^{k-1} \\
&= \frac{1}{3} \sum_{k=0}^{\infty} \left(\frac{2}{9}\right)^{k} + \frac{2}{3} \sum_{k=0}^{\infty} k \cdot \left(\frac{2}{9}\right)^{k} + \frac{9}{2} \cdot \frac{8}{9} \sum_{k=1}^{\infty} k \cdot \left(\frac{2}{9}\right)^{k-1} \cdot \frac{2}{9} \\
&= \frac{1}{3} \sum_{k=0}^{\infty} \left(\frac{2}{9}\right)^{k} + \left(\frac{2}{3} + 4\right) \sum_{k=0}^{\infty} k \cdot \left(\frac{2}{9}\right)^{k} \\
&= \frac{1}{3} \cdot \frac{1}{1-2/9} + \frac{14}{3} \cdot \frac{2/9}{(1-2/9)^2} = \frac{15}{7}.
\end{aligned}
$$

Thus, (ii) is true.

(d) Suppose the game has ended after X tosses. The winnings of Reuven depend only on the pairity of X. If X is odd, Reuven wins 2^X shekels from Shimon, whereas if X is even, Reuven pays

Shimon 2^X shekels. Thus $R = -\left(-2\right)^X$. We have:

$$\begin{aligned}
E\left(R\right) &= \sum_{k=1}^{\infty} -\left(-2\right)^k \cdot P\left(X = k\right) \\
&= \sum_{k=0}^{\infty} 2^{2k+1} \cdot P\left(X = 2k+1\right) - \sum_{k=1}^{\infty} 2^{2k} \cdot P\left(X = 2k\right) \\
&= \sum_{k=0}^{\infty} 2^{2k+1} \cdot \left(\frac{2}{3} \cdot \frac{1}{3}\right)^k \frac{1}{3} + \sum_{k=1}^{\infty} 2^{2k} \cdot \left(\frac{2}{3} \cdot \frac{1}{3}\right)^{k-1} \frac{2}{3} \cdot \frac{2}{3} \\
&= 2 \cdot \frac{1}{3} \sum_{k=0}^{\infty} \left(\frac{8}{9}\right)^k - 2^2 \cdot \frac{4}{9} \sum_{k=1}^{\infty} 4^{k-1} \cdot \left(\frac{2}{9}\right)^{k-1} \\
&= \frac{2}{3} \sum_{k=0}^{\infty} \left(\frac{8}{9}\right)^k - \frac{16}{9} \sum_{k=0}^{\infty} \left(\frac{8}{9}\right)^k \\
&= \frac{1}{1 - 8/9} \cdot \left(\frac{2}{3} - \frac{16}{9}\right) = 9 \cdot \frac{-10}{9} = -10.
\end{aligned}$$

Similarly:

$$\begin{aligned}
E\left(R^2\right) &= \sum_{k=1}^{\infty} 2^{2k} \cdot P\left(X = k\right) \\
&= \sum_{k=0}^{\infty} 4^{2k+1} \cdot P\left(X = 2k+1\right) + \sum_{k=1}^{\infty} 4^{2k} \cdot P\left(X = 2k\right) \\
&> \sum_{k=0}^{\infty} 4^{2k+1} \cdot P\left(X = 2k+1\right) = \sum_{k=0}^{\infty} 4^{2k+1} \cdot \left(\frac{2}{3} \cdot \frac{1}{3}\right)^k \frac{1}{3} \\
&= 4 \cdot \frac{1}{3} \sum_{k=0}^{\infty} \left(\frac{32}{9}\right)^k = \infty.
\end{aligned}$$

Therefore, $E(R) = -10$ and $V(R) = \infty$.

Thus, (ii) is true.

2. (a) The life length of the light bulbs is a continuous variable, and thus the probability of two light bulbs burning out at the same exact moment is 0. By symmetry, the probability that any of the $n+1$ light bulbs will outlive all others is $\frac{1}{n+1}$. In particular, the probability that A's light bulb will outlive all of B's is $\frac{1}{n+1}$.

Thus, (ii) is true.

(b) Let X_A be the life length of A's light bulb, and $Y_1, \dots, Y_n$ the life lengths of C's. Denote by p the probability in question. Then:

$$\begin{aligned} p = &\ P\left(X_A > Y_1 + Y_2 + \ldots + Y_n\right) \\ = &\ P\left(X_A > Y_1, X_A > Y_1, \ldots, X_A > Y_1 + \ldots + Y_n\right) \\ = &\ P\left(X_A > Y_1\right) \cdot P\left(X_A > Y_1 + Y_2 | X_A > Y_1\right) \\ &\ \cdot \ldots \cdot P\left(X_A > Y_1 + \ldots + Y_n | X_A > Y_1 + \ldots + Y_{n-1}\right). \end{aligned}$$

By the memorylessness property of the exponential distribution:

$$p = P\left(X_A > Y_1\right) \cdot \ldots \cdot P\left(X_A > Y_n\right).$$

By symmetry, each of the factors on the right-hand side is $\frac{1}{2}$, and therefore

$$p = \frac{1}{2^n}.$$

Thus, (iii) is true.

(c) We have

$$\begin{aligned} E\left(e^{\alpha X}\right) &= \int_0^\infty e^{\alpha x} \cdot e^{-x} dx = \int_0^\infty e^{(\alpha-1)x} dx \\ &= \left[\frac{e^{(\alpha-1)x}}{\alpha - 1}\right]_0^\infty = \frac{1}{1-\alpha}, \end{aligned}$$

and by the same token

$$E\left(\left(e^{\alpha X}\right)^2\right) = E\left(e^{2\alpha X}\right) = \frac{1}{1-2\alpha}.$$

Thus

$$V\left(e^{\alpha X}\right) = \frac{1}{1-2\alpha} - \left(\frac{1}{1-\alpha}\right)^2 = \frac{\alpha^2}{(1-2\alpha)(1-\alpha)^2}.$$

Now,

$$\begin{aligned}
E\left(Xe^{\alpha X}\right) &= \int_0^\infty xe^{\alpha x} \cdot e^{-x} dx = \int_0^\infty x \cdot e^{(\alpha-1)x} dx \\
&= \left[\frac{x}{\alpha-1} \cdot e^{(\alpha-1)x}\right]_0^\infty - \int_0^\infty \frac{1}{\alpha-1} \cdot e^{(\alpha-1)x} dx \\
&= \left(\frac{1}{1-\alpha}\right)^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
\operatorname{Cov}\left(X, e^{\alpha X}\right) &= \left(\frac{1}{1-\alpha}\right)^2 - E(X) \cdot \frac{1}{1-\alpha} \\
&= \left(\frac{1}{1-\alpha}\right)^2 - \frac{1}{1-\alpha} \\
&= \frac{1-(1-\alpha)}{(1-\alpha)^2} = \frac{\alpha}{(1-\alpha)^2}.
\end{aligned}$$

Altogether

$$\begin{aligned}
\rho\left(X, e^{\alpha X}\right) &= \frac{\operatorname{Cov}\left(X, e^{\alpha X}\right)}{\sqrt{V(X) \cdot V(e^{\alpha X})}} \\
&= \frac{\alpha/(1-\alpha)^2}{\sqrt{1 \cdot \alpha^2/(1-2\alpha)(1-\alpha)^2}} = \frac{\sqrt{1-2\alpha}}{1-\alpha}.
\end{aligned}$$

Thus, (iv) is true.

3. (a) We have

$$1 = c \iint_S x^2 e^{-(x^2+y^2)} dxdy,$$

where $S = \{(x,y) : x, y \geq 0, 1 \leq x^2 + y^2 \leq 4\}$. Now:

$$\begin{aligned}
1 &= c \int_1^2 \int_0^{\pi/2} r^2 \cos^2\theta \cdot e^{-r^2} \cdot r dr d\theta \\
&= c \int_1^2 r^2 \cdot r e^{-r^2} dr \int_0^{\pi/2} \cos^2\theta d\theta \\
&= c \cdot \left(\left[-\frac{1}{2} r^2 \cdot e^{-r^2} \right]_1^2 + \int_1^2 r \cdot e^{-r^2} dr \right) \cdot \left(\frac{1}{2} \cdot \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2} \right) \\
&= c \cdot \left(\left[-\frac{1}{2} r^2 \cdot e^{-r^2} \right]_1^2 + \left[-\frac{1}{2} \cdot e^{-r^2} \right]_1^2 \right) \cdot \left(\frac{1}{2} \cdot \frac{\pi}{2} \right) \\
&= c \cdot \frac{\pi}{8} \left(-4e^{-4} + e^{-1} - e^{-4} + e^{-1} \right) = c \cdot \frac{\pi}{8} \left(-5e^{-4} + 2e^{-1} \right).
\end{aligned}$$

Hence, $c = \dfrac{8e^4}{\pi\left(2e^3 - 5\right)}$.

Thus, (iv) is true.

(b) We have

$$P(X > Y) = c \iint_A x^2 e^{-(x^2+y^2)} dxdy,$$

where $A = \{(x, y) : x, y \geq 0, 1 \leq x^2 + y^2 \leq 4,\ x > y\}$. Now:

$$
\begin{aligned}
P\left(X > Y\right) &= c \int_1^2 \int_0^{\pi/4} r^2 \cos^2\theta \cdot e^{-r^2} \cdot r dr d\theta \\
&= c \int_1^2 r^2 \cdot r e^{-r^2} dr \int_0^{\pi/4} \cos^2\theta d\theta \\
&= \frac{c}{2} \cdot \left(-5e^{-4} + 2e^{-1}\right) \cdot \left(\frac{1}{2} \cdot \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/4}\right) \\
&= \frac{c}{4} \cdot \left(-5e^{-4} + 2e^{-1}\right) \cdot \left(\frac{\pi}{4} + \frac{1}{2}\right) \\
&= \frac{8e^4}{\pi\left(2e^3 - 5\right)} \cdot \frac{1}{4} \cdot \left(-5e^{-4} + 2e^{-1}\right) \cdot \left(\frac{\pi}{4} + \frac{1}{2}\right) \\
&= \frac{2}{\pi} \cdot \left(\frac{\pi}{4} + \frac{1}{2}\right) = \frac{1}{2} + \frac{1}{\pi}.
\end{aligned}
$$

Thus, (iii) is true.

(c) We have

$$
\begin{aligned}
E(1/X) &= c \iint_S \frac{1}{x} \cdot x^2 e^{-(x^2+y^2)} dx dy = c \iint_S x e^{-(x^2+y^2)} dx dy \\
&= c \int_1^2 r \cdot r e^{-r^2} dr \int_0^{\pi/2} \cos\theta d\theta \\
&= c \cdot \left(\left[-\frac{1}{2} r \cdot e^{-r^2}\right]_1^2 + \int_1^2 \frac{1}{2} e^{-r^2} dr\right) \cdot [\sin\theta]_0^{\pi/2} \\
&= c \cdot \left(\frac{1}{2}e^{-1} - e^{-4} + \frac{1}{2} \cdot \sqrt{\pi} \int_1^2 \frac{1}{\sqrt{2\pi \cdot 1/2}} e^{-r^2/(2\cdot\frac{1}{2})} dr\right) \cdot 1 \\
&= c \cdot \left(\frac{1}{2}e^{-1} - e^{-4} + \frac{\sqrt{\pi}}{2} \cdot P(1 \leq W \leq 2)\right),
\end{aligned}
$$

where $W \sim N(0, \dfrac{1}{2})$. Now $W/\frac{1}{\sqrt{2}} \sim N(0, 1)$, and thus

$$
P(1 \leq W \leq 2) = P(\sqrt{2} \leq \sqrt{2}W \leq 2\sqrt{2}) = \Phi(2\sqrt{2}) - \Phi(\sqrt{2}).
$$

Altogether

$$E(1/X) = c \cdot \left(\frac{1}{2} e^{-1} - e^{-4} + \frac{\sqrt{\pi}}{2} \left(\Phi(2\sqrt{2}) - \Phi(\sqrt{2}) \right) \right).$$

Thus, (i) is true.

(d) The variable D assumes values in $(1,2)$, and therefore for $t \in (1,2)$ we have

$$F_D(t) = P\left(\sqrt{X^2 + Y^2} \le t \right) = c \iint_{B_t} x^2 e^{-(x^2+y^2)} dxdy,$$

where $B_t = \{(x,y) : x, y \ge 0, 1 \le x^2 + y^2 \le t^2\}$. Now:

$$\begin{aligned} F_D(t) &= c \int_1^t r^2 \cdot r e^{-r^2} dr \int_0^{\pi/2} \cos^2 \theta d\theta \\ &= c \cdot \left(\left[-\frac{1}{2} r^2 \cdot e^{-r^2} \right]_1^t + \left[-\frac{1}{2} \cdot e^{-r^2} \right]_1^t \right) \cdot \frac{\pi}{4} \\ &= -\frac{1}{2} \cdot \frac{c\pi}{4} \cdot \left(t^2 e^{-t^2} - e^{-1} + e^{-t^2} - e^{-1} \right). \end{aligned}$$

Now,

$$\begin{aligned} f_D(t) = F_D'(t) &= -\frac{c\pi}{8} \cdot \left(2t \cdot e^{-t^2} + t^2 e^{-t^2} \cdot (-2t) + e^{-t^2} \cdot (-2t) \right) \\ &= -\frac{c\pi}{8} \cdot e^{-t^2} \cdot (-2t) \cdot t^2 = \frac{c\pi e^{-t^2} t^3}{4}. \end{aligned}$$

Thus, (iii) is true.

4. (a) Let $X_n = 1$ if $U_n > 1 - 10^{-6}$, and $X_n = 0$ otherwise. We may express N_1 in terms of the X_n's:

$$N_1 = X_1 + \ldots + X_{10^6}.$$

We have

$$P(X_n = 1) = P\left(U_n > 1 - 10^{-6}\right) = \frac{1 - (1 - 10^{-6})}{1 - (-1)} = \frac{10^{-6}}{2}.$$

Thus, $X_n \sim B\left(1, \frac{10^{-6}}{2}\right)$ and $N_1 \sim B\left(10^6, \frac{10^{-6}}{2}\right)$. As 10^6 is very large, and $\frac{10^{-6}}{2}$ very small, we use the Poissonian approximation to the binomial distribution to obtain $P\left(N_1 = 1\right) \approx P(W = 2)$, where $W \sim P\left(\lambda\right)$, with $\lambda = 10^6 \cdot \frac{10^{-6}}{2} = \frac{1}{2}$. Namely:

$$P\left(N_1 = 1\right) \approx \frac{\left(1/2\right)^1}{1!} \cdot e^{-1/2} = \frac{1}{2\sqrt{e}}.$$

Thus, (ii) is true.

(b) Let $Y_n = 1$ if $|U_n| < 0.9$, and $Y_n = 0$ otherwise. We may express N_2 in terms of the Y_n's:

$$N_2 = Y_1 + \ldots + Y_{10^6}.$$

We have:

$$P\left(Y_n = 1\right) = P\left(|U_n| \le 0.9\right) = \frac{0.9 - (-0.9)}{1 - (-1)} = 0.9.$$

Thus, $Y_n \sim B(1, 0.9)$, so that $\mu = E(Y_n) = 0.9$, and $\sigma^2 = V(Y_n) = 0.09$. Denote by p the probability in question. The variables Y_n are independent and thus, by the central limit theorem:

$$\begin{aligned}
p &= P\left(|N_2 - 900000| \le 600\right) \\
&= P\left(-\frac{600}{1000 \cdot 0.3} \le \frac{N_2 - 10^6\mu}{\sqrt{10^6} \cdot \sigma} \le \frac{600}{1000 \cdot 0.3}\right) \\
&= P\left(-2 \le \frac{N_2 - 10^6\mu}{\sqrt{10^6} \cdot \sigma} \le 2\right) \\
&\approx \Phi(2) - \Phi(-2) = 2 \cdot \Phi(2) - 1 = 0.9545.
\end{aligned}$$

Thus, (iv) is true.

(c) Consider (i). Let $A = -\ln|U_1|$. The variable $|U_1|$ assumes values in $(0,1)$, and therefore:

$$\begin{aligned} F_A(t) &= P(-\ln|U_1| \le t) = P\left(|U_1| \ge e^{-t}\right) \\ &= 1 - P\left(|U_1| < e^{-t}\right) \\ &= 1 - P\left(-e^{-t} < U_1 < e^{-t}\right) \\ &= 1 - \frac{e^{-t} - (-e^{-t})}{1-(-1)} = 1 - e^{-t}, \qquad t \ge 0. \end{aligned}$$

Hence

$$F_A(t) = \begin{cases} 0, & t < 0, \\ 1 - e^{-t}, & t \ge 0, \end{cases}$$

and

$$f_A(t) = \begin{cases} 0, & t < 0, \\ e^{-t}, & t \ge 0, \end{cases}$$

so that (i) is true.

Consider (ii). Let $B = e^{1/|U_1|}$. The variable B assumes values (e, ∞), so that (ii) is false.

Consider (iii). Let $C = \tan(\pi|U_1|)$. The variable C assumes values $(-\infty, \infty)$, so that (iii) is false.

Consider (iv). Let $D = \dfrac{1}{|U_1|} - 1$. The variable D assumes values in $(0, \infty)$. For $t \in (0, \infty)$ we have

$$\begin{aligned} F_D(t) &= P\left(\frac{1}{|U_1|} - 1 \le t\right) = P\left(\frac{1}{|U_1|} \le t + 1\right) \\ &= P\left(|U_1| \ge \frac{1}{t+1}\right) = 1 - F_{|U_1|}\left(\frac{1}{t+1}\right) \\ &= 1 - \frac{1}{t+1} = \frac{t}{t+1}. \end{aligned}$$

Therefore

$$F_D(t) = \begin{cases} 0, & t < 0, \\ \dfrac{t}{t+1}, & t \ge 0, \end{cases}$$

and

$$f_D(t) = \begin{cases} 0, & t < 0, \\ \dfrac{1}{(t+1)^2}, & t \geq 0, \end{cases}$$

so that (iv) is false.

Thus, (i) is the only true claim.

(d) Recall that the proof in class used Chebyshev's inequality to prove that a sequence of independent identically distributed random variables satisfies the weak law of large numbers. Note that all three sequences $(X_n)_{n=1}^{\infty}$, $(Y_n)_{n=1}^{\infty}$ and $(Z_n)_{n=1}^{\infty}$ are special cases of the sequence $(W_n)_{n=1}^{\infty}$ defined by $W_n = a_n U_n$, $n \geq 1$, for an appropriate sequence $(a_n)_{n=1}^{\infty}$ of positive numbers. We have

$$E(W_n) = E(a_n U_n) = a_n \cdot E(U_n) = a_n \cdot \frac{-1+1}{2} = 0,$$

and

$$V(W_n) = V(a_n U_n) = a_n^2 V(U_n) = a_n^2 \cdot \frac{(1-(-1))^2}{12} = a_n^2 \cdot \frac{1}{3}.$$

Denote $\overline{W}_n = \dfrac{1}{n} \sum_{k=1}^{n} W_k$. We have

$$E(\overline{W}_n) = \frac{1}{n} \sum_{k=1}^{n} E(W_k) = 0,$$

and by the independence of W_k

$$V(\overline{W}_n) = \frac{1}{n^2} \sum_{k=1}^{n} V(W_k) = \frac{1}{n^2} \sum_{k=1}^{n} a_k^2 \cdot \frac{1}{3} = \frac{1}{3n^2} \sum_{k=1}^{n} a_k^2.$$

By Chebyshev's inequality

$$P\left(|\overline{W}_n| \geq \varepsilon\right) \leq \frac{V(\overline{W}_n)}{\varepsilon^2} = \frac{1}{3n^2\varepsilon^2} \sum_{k=1}^{n} a_k^2, \qquad \varepsilon > 0.$$

Consider $(X_n)_{n=1}^{\infty}$. We have $a_n = 2 + \sin n$ for $n = 1, 2, \ldots$. Thus, for every $\varepsilon > 0$

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{3n^2\varepsilon^2} \sum_{k=1}^{n} a_k^2 &= \lim_{n\to\infty} \frac{1}{3n^2\varepsilon^2} \sum_{k=1}^{n} (2 + \sin k)^2 \\ &\leq \lim_{n\to\infty} \frac{9n}{3n^2\varepsilon^2} = 0.\end{aligned}$$

Thus, we may employ the same technique of proof for the sequence $(X_n)_{n=1}^{\infty}$.

Consider $(Y_n)_{n=1}^{\infty}$. We have $a_n = \sqrt{n}$ for $n = 1, 2, \ldots$. Thus, for every $\varepsilon > 0$

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{3n^2\varepsilon^2} \sum_{k=1}^{n} a_k^2 &= \lim_{n\to\infty} \frac{1}{3n^2\varepsilon^2} \sum_{k=1}^{n} \left(\sqrt{k}\right)^2 \\ &= \lim_{n\to\infty} \frac{1}{3n^2\varepsilon^2} \sum_{k=1}^{n} k \\ &= \lim_{n\to\infty} \frac{1}{3n^2\varepsilon^2} \cdot \frac{n(n+1)}{2} = \frac{1}{6\varepsilon^2} \neq 0.\end{aligned}$$

Thus, we cannot employ the same technique of proof for the sequence $(Y_n)_{n=1}^{\infty}$.

Consider $(Z_n)_{n=1}^{\infty}$. We have $a_n = n$ for $n = 1, 2, \ldots$. Thus, for every $\varepsilon > 0$

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{3n^2\varepsilon^2} \sum_{k=1}^{n} a_k^2 &= \lim_{n\to\infty} \frac{1}{3n^2\varepsilon^2} \sum_{k=1}^{n} k^2 \\ &= \lim_{n\to\infty} \frac{1}{3n^2\varepsilon^2} \cdot \frac{n(n+1)(2n+1)}{6} = \infty.\end{aligned}$$

Thus, we cannot employ the same technique of proof for the sequence $(Z_n)_{n=1}^{\infty}$.

Thus, (iii) is true.