Numerical Integration – Review Questions

Mark the correct answer in each part of the following questions.

1. (a) The expression $(b-a) \cdot f(2a/3 + b/3)$ has been suggested as an approximation to $\int_a^b f(x)dx$. The best possible upper bound on the absolute error in terms of $M = \sup_{x\in[a,b]} |f'(x)|$ and the length of the interval is:

   (i) $\frac{5}{18}M(b-a)^2$.
   (ii) $\frac{1}{3}M(b-a)^2$.
   (iii) $\frac{4}{9}M(b-a)^2$.
   (iv) $\frac{1}{2}M(b-a)^2$.
   (v) None of the above.

   (b) We are given some numerical integration method. It is known that, when using this method to approximate $\int_a^b f(x)dx$ for a function f which is 9 times continuously differentiable on $[a,b]$, the error is

$$E = \frac{(b-a)^{10}}{10^4} f^{(9)}(\eta)$$

   for some intermediate point $\eta \in (a,b)$. Suppose we approximate $\int_0^{\pi/2} \sin 2x dx$ by dividing the interval of integration into three equal sub-intervals and employing the given method for the integral over each of the sub-intervals. Then:

   (i) $-\frac{\pi^{10}}{6^{10}\cdot 10^4} \leq E \leq \frac{\pi^{10}}{6^{10}\cdot 10^4}$.
   (ii) $-\frac{\pi^{10}}{6^{9}\cdot 10^4} \leq E \leq \frac{\pi^{10}}{6^{9}\cdot 10^4}$.
   (iii) $-\frac{\pi^{10}}{2\cdot 3^{10}\cdot 10^4} \leq E \leq \frac{\pi^{10}}{2\cdot 3^{10}\cdot 10^4}$.
   (iv) $-\frac{\pi^{10}}{2\cdot 3^{9}\cdot 10^4} \leq E \leq \frac{\pi^{10}}{2\cdot 3^{9}\cdot 10^4}$.
   (v) None of the above.

Remark: Obviously, each of the first four claims implies all following ones. Mark only the first one (if any) that is correct.

(c) We approximate $\int_1^2 \frac{dx}{\sqrt{x}}$ by dividing the interval into n sub-intervals of equal length and approximating the integral on each by the trapezoid rule. Then the total error is approximately

(i) $-\frac{2-\sqrt{2}}{8n}$.

(ii) $-\frac{2-\sqrt{2}}{4n}$.

(iii) $-\frac{2-\sqrt{2}}{2n}$.

(iv) $-\frac{2-\sqrt{2}}{n}$.

(v) None of the above.

2. (a) Denote by $\mathcal{C}$ the vector space of all real continuous functions on $[-1, 1]$ and by $\mathcal{P}$ its subspace of all polynomial functions. Put

$$G = \left\{ f \in \mathcal{C} : \int_{-1}^{1} f(x)dx = f(-1/\sqrt{3}) + f(1/\sqrt{3}) \right\},$$

and $G' = G \cap \mathcal{P}$.

(i) G consists of all polynomials of degree not exceeding 3.

(ii) G' consists of all polynomials of degree not exceeding 3. G strictly contains G', but it is still a finite-dimensional vector space.

(iii) G' strictly contains the subspace of all polynomials of degree not exceeding 3, but it is still a finite-dimensional vector space. G is an infinite-dimensional vector space.

(iv) G' is an infinite-dimensional vector space and $G \supsetneq G'$.

(v) None of the above.

(b) We are looking for an approximation formula of the form

$$\int_0^1 f(x)dx \approx \frac{f(x_1) + f(x_2) + f(x_3)}{3},$$

with appropriate $x_1, x_2, x_3 \in [0, 1]$, that has zero error for polynomials of degree up to 3.

(i) It is impossible to find such a formula. It would be possible if we required it to be with zero error only for polynomials of degree up to 2.

(ii) Choosing $x_1 = \frac{1}{6}, x_2 = \frac{1}{2}, x_3 = \frac{5}{6}$, we obtain a formula as required.

(iii) Choosing $x_1 = \frac{1}{\sqrt{3}}, x_2 = \frac{1}{2}, x_3 = 1 - \frac{1}{\sqrt{3}}$, we obtain a formula as required.

(iv) Choosing $x_1 = \frac{1}{2} - \frac{\sqrt{2}}{4}, x_2 = \frac{1}{2}, x_3 = \frac{1}{2} + \frac{\sqrt{2}}{4}$, we obtain a formula as required.

(v) None of the above.

Solutions

1. (a) Recall that the error when using the rectangle rule is bounded by $\frac{1}{2}M(b-a)^2$. The approximation suggested in the question means that we basically use the rectangle rule twice, once for $\int_a^{(2a+b)/3} f(x)dx$ and once for $\int_{(2a+b)/3}^b f(x)dx$. More precisely, for the first of these integrals we take the value of the function at the right endpoint of the interval instead of at the left endpoint, but it clearly makes no difference as far as the absolute error goes. Hence the absolute error for the first integral is bounded by

$$\frac{1}{2} \sup_{x\in[a,(2a+b)/3]} |f'(x)| \left(\frac{b-a}{3}\right)^2$$

and for the second – by

$$\frac{1}{2} \sup_{x\in[(2a+b)/3,b]} |f'(x)| \left(\frac{2(b-a)}{3}\right)^2.$$

Thus, the total absolute error is bounded by

$$\frac{1}{2} \sup_{x\in[a,(2a+b)/3]} |f'(x)| \left(\frac{b-a}{3}\right)^2 + \frac{1}{2} \sup_{x\in[(2a+b)/3,b]} |f'(x)| \left(\frac{2(b-a)}{3}\right)^2,$$

which is at most $\frac{5}{18}M(b-a)^2$. To see that the bound cannot be improved, note that the worst case for the rectangle rule is when

f' is identically M (or identically $-M$) throughout the interval. In our case, if we take f this way, the errors in approximating the first integral and in approximating the second will be of opposite signs, and thus partly cancel each other. Rather, the worst case is when f' is identically M throughout $[a, (2a+b)/3]$ and identically $-M$ throughout $[(2a+b)/3, b]$ (or vice versa). Obviously, in this case f is not differentiable at the point $(2a+b)/3$ itself, but smoothing the function a bit near this point we can attain an error arbitrarily close to the bound above.

Thus, (i) is true.

(b) Denoting by E_1, E_2, E_3 the errors when approximating the integral over the left, middle and right thirds of the interval, respectively, and noting that $(\sin 2x)^{(9)} = 2^9 \cos 2x$, we obtain

$$\begin{aligned} E &= E_1 + E_2 + E_3 \\ &= \tfrac{(\pi/6)^{10}}{10^4} \cdot 2^9 (\cos 2\eta_1 + \cos 2\eta_2 + \cos 2\eta_3) \end{aligned}$$

for some points η_1, η_2, η_3 in those intervals. Now

$$\cos 2\eta_1 \in [1/2, 1], \cos 2\eta_2 \in [-1/2, 1/2], \cos 2\eta_3 \in [-1, -1/2].$$

Consequently,

$$\frac{(\pi/6)^{10}}{10^4} \cdot 2^9 (1/2 + (-1/2) + (-1)) \le E \le \frac{(\pi/6)^{10}}{10^4} \cdot 2^9 (1 + 1/2 + (-1/2)),$$

that is:

$$-\frac{\pi^{10}}{2 \cdot 3^{10} \cdot 10^4} \le E \le \frac{\pi^{10}}{2 \cdot 3^{10} \cdot 10^4}.$$

Thus, (iii) is true.

(c) Let $f(x) = 1/\sqrt{x}$. The error in each sub-interval $[1 + (i-1)/n, 1 + i/n], 1 \le i \le n$, is

$$-\frac{1}{12n^3} f''(\eta_i), \qquad (\eta_i \in (1 + (i-1)/n, 1 + i/n).$$

Hence the total error is

$$\sum_{i=1}^{n} -\frac{1}{12n^3} f''(\eta_i) = -\frac{1}{12n^2} \sum_{i=1}^{n} \frac{1}{n} f''(\eta_i).$$

The sum on the right-hand side is a Riemann sum of the function f'' on the interval $[1, 2]$. Thus,

$$\begin{aligned} E &\approx -\tfrac{1}{12n^2} \int_1^2 f''(x)dx \\ &= -\tfrac{1}{12n^2} \cdot (f'(2) - f'(1)) \\ &= \tfrac{2^{-3/2} - 1^{-3/2}}{24n^2} = -\tfrac{4-\sqrt{2}}{96n^2}. \end{aligned}$$

(Note that the mere fact that all claims (i)-(iv) are false follows from the fact that, when approximating an integral by dividing the interval into n equal sub-intervals and using the trapezoid rule for each, the error is bounded by C/n^2 for some constant C. The above calculations serve to exemplify how we estimate the error rather than just bounding the absolute error from above.)

Thus, (v) is true.

2. (a) Since both expressions $\int_{-1}^1 f(x)dx$ and $f(-1/\sqrt{3}) + f(1/\sqrt{3})$ are linear functionals on the space of all real (continuous) functions on $[-1, 1]$, the set of functions f for which they assume the same value, namely G, is indeed a subspace of $\mathcal{C}$. By the theory we learned, G contains all polynomials of degree not exceeding 3. Also, it is obvious that G contains all odd functions on $[-1, 1]$. Hence, G' contains all polynomials spanned by monomials of odd degrees (as well as numerous other polynomials) and is therefore infinite-dimensional. G contains non-polynomial odd functions (such as sin), and in particular it strictly contains G'.

   Thus, (iv) is true.

   (b) For the formula in question to be exact for all polynomials up to degree 3, it needs to hold for the polynomials $1, x, x^2, x^3$. Namely, the following equalities need to hold:

$$\begin{aligned} \tfrac{1+1+1}{3} &= 1, \\ \tfrac{x_1+x_2+x_3}{3} &= \tfrac{1}{2}, \\ \tfrac{x_1^2+x_2^2+x_3^2}{3} &= \tfrac{1}{3}, \\ \tfrac{x_1^3+x_2^2+x_3^3}{3} &= \tfrac{1}{4}. \end{aligned}$$

In principle, one would not expect a system of 4 equations in 3 variables to have a solution. However, in our case the first equation is in fact an identity. A routine calculation shows that the choice $x_1 = \frac{1}{2} - \frac{\sqrt{2}}{4}, x_2 = \frac{1}{2}, x_3 = \frac{1}{2} + \frac{\sqrt{2}}{4}$ indeed yields a solution of the system.

Thus, (iv) is true.