Review Questions

Mark the correct answer in each part of the following questions.

1. Let A be a set of size 6. Consider the graph $G = (V, E)$, where $V = 2^A$ (the set of all subsets of A) and $E = \{(B, C) : B, C \subseteq A, |B| \neq |C|\}$.

   (a) The independence number of G is

   (i) $\alpha(G) = 6$.
   (ii) $\alpha(G) = 7$.
   (iii) $\alpha(G) = 20$.
   (iv) $\alpha(G) = 36$.
   (v) none of the above.

   (b) The chromatic number of G is

   (i) $\chi(G) = 6$.
   (ii) $\chi(G) = 7$.
   (iii) $\chi(G) = 8$.
   (iv) $\chi(G) = 9$.
   (v) none of the above.

   (c) Let H be the subgraph of G, induced by some clique of size $\omega(G)$ and one additional vertex (altogether, $\omega(G) + 1$ vertices). The number $\tau(H)$ of spanning trees of H

   (i) cannot be determined without knowing the exact vertices of H.
   (ii) is 0 since H is disconnected.
   (iii) is necessarily $5 \cdot 7^3$.
   (iv) is necessarily $6 \cdot 8^4$.
   (v) is none of the above.

2. Define (for the purpose of this question only) a *semi-Latin square* to be an $n \times n$ square, where $n = 2m$ is even, satisfying the same requirements as does a Latin square, except for the following change regarding the columns: Instead of requiring that all entries in each column be distinct, we require that, in each column, the first m entries should be distinct, but the last m entries should be entries that already appeared among the first m. (For example, the square

3	1	4	2
2	4	1	3
2	1	4	3
3	4	1	2

is a semi-Latin square with $n = 4$.)

(a) Denote by a_n the lower bound obtained in class for the number of $n \times n$ Latin squares, namely:

$$a_n = \frac{n!^{2n}}{n^{n^2}}.$$

Employ the method, used to arrive at the bound a_n, to find a lower bound on the number of $n \times n$ (with $n = 2m$) semi-Latin squares. The bound we obtain is:

(i) $\frac{m^{m^2}}{m!^m} \cdot a_n$.

(ii) $\frac{m^{2m^2}}{m!^{2m}} \cdot a_n$.

(iii) $\frac{m^{2m^2}}{(2m)!^m} \cdot a_n$.

(iv) $\frac{(2m)^{m^2}}{(2m)!^m} \cdot a_n$.

(v) none of the above.

(b) Now recall the method we used to bound the number of Latin squares from above. Employing the same method to bound the number of semi-Latin squares, we arrive at the upper bound:

(i) $(n-0)!^{\frac{n}{n-0}}(n-1)!^{\frac{n}{n-1}}\ldots(n-(m-1))!^{\frac{n}{n-(m-1)}}(m-1)!^{2(m-1)}$.

(ii) $(n-0)!^{\frac{n}{n-0}}(n-1)!^{\frac{n}{n-1}}\ldots(n-(m-1))!^{\frac{n}{n-(m-1)}}(m-1)!^{2m}$.

(iii) $(n-0)!^{\frac{n}{n-0}}(n-1)!^{\frac{n}{n-1}}\ldots(n-(m-1))!^{\frac{n}{n-(m-1)}}m!^{2m}$.

(iv) $(n-0)!^{\frac{n}{n-0}}(n-1)!^{\frac{n}{n-1}}\dots(n-(m-1))!^{\frac{n}{n-(m-1)}}(m+1)!^{2(m+1)}$.

(v) none of the above.

3. Let G be a graph with 100 vertices and 1000 edges, and V_0 a clique of size 7 in G. The vertices of V_0 are colored in 7 distinct colors. The coefficient of k^{92} in $\chi(G, V_0, k)$ is:

   (i) -993.

   (ii) -985.

   (iii) -979.

   (iv) -972.

   (v) none of the above.

4. In the matrix-tree theorem and its proof we have defined matrices Q, C, that satisfied the equality $CC^T = Q$. Now consider the matrix C for a connected graph G on $n \geq 3$ vertices, which is not a tree. The sum of all columns of C is

   (i) necessarily $\mathbf{0}$.

   (ii) $\mathbf{0}$ if and only if G is (isomorphic to) C_n.

   (iii) $\mathbf{0}$ if G is (isomorphic to) K_n, but the converse is not true.

   (iv) is necessarily distinct from $\mathbf{0}$.

   (v) none of the above.

Solutions

1. (a) Since subsets of A are neighbors in G if and only if they are of distinct size, a set of vertices is independent in G if and only if all of them are sets of the same size. Now A includes sets of sizes 0 through 6, specifically $\binom{6}{k}$ sets of each size k is this range. The maximum is obtained for $k = 3$, for which $\binom{6}{3} = 20$. Thus, the set of all subsets of A of size 3 is an independent set of cardinality 20 (and is the only one of this cardinality).

   Thus, (iii) is true.

   (b) Cliques are collections of subsets of A, all of different sizes. By taking one subset of each size $0, 1, \ldots, 6$, we thus get a maximum clique. Hence $\omega(G) \geq 7$, and therefore $\chi(G) \geq 7$. On the other hand, assigning to each subset B of A the color $|B|$, we clearly obtain a proper coloring of G.

   Thus, (ii) is true.

   (c) In view of the discussion in the preceding part, H must consist of 7 subsets $B_0, B_1, \ldots, B_6$ of A, of sizes $0, 1, \ldots, 6$, respectively, and one additional $B \subseteq A$. Whatever B is, it neighbors all B_i's, except for the one which is of the same size as B. It follows that H is K_8, with one edge removed.

   The number of spanning trees of K_8 is $8^{8-2} = 8^6$. To find $\tau(G)$, we need to find the number of spanning trees of K_8, containing the edge which is missing in G. Now each spanning tree of K_8 consists of 7 edges, out of the total $\binom{8}{2} = 28$ edges of K_8. By symmetry, each edge of K_8 belongs to the same number of spanning trees of K_8, so it belongs to $\frac{7}{28} \cdot 8^6$ spanning trees. It follows that:

$$\tau(G) = 8^6 - \frac{7}{28} \cdot 8^6 = 6 \cdot 8^5.$$

   Thus, (v) is true.

2. (a) In the case of a Latin square, we have seen that, when we fill it in row after row, there are at least $(n-k)^n \cdot \frac{n!}{n^n}$ possibilities for filling row $k+1$ for each k. In our case, for the first m rows there is no difference, as the constraints for these rows are exactly the same as for Latin squares, so that we have at least

$$n^n \cdot \frac{n!}{n^n} \cdot (n-1)^n \cdot \frac{n!}{n^n} \cdot \ldots \cdot (m+1)^n \cdot \frac{n!}{n^n}$$

possibilities for filling in the top half of the square. However, when we continue, things become different from the case of Latin squares. Namely, when filling in any of these rows, we have m possibilities for each entry, and therefore at least $m^n \cdot \frac{n!}{n^n}$ possibilities for the entire row. In other words, the lower bound for the number of possibilities for filling in row $m+k$, for any k in the range $[1,m]$, is $m^n/(m-k+1)^n$ times the corresponding bound for Latin squares. Thus, the lower bound we obtain is

$$\frac{m^n}{m^n} \cdot \frac{m^n}{(m-1)^n} \cdot \frac{m^n}{(m-2)^n} \cdot \ldots \cdot \frac{m^n}{1^n} a_n = \frac{m^{mn}}{m!^n} a_n = \frac{m^{2m^2}}{m!^{2m}} a_n.$$

Thus, (ii) is true.

(b) Similarly to the preceding part, the number of possibilities for filling in the top half of the square is, just as in the case of Latin squares, bounded above by

$$(n-0)!^{\frac{n}{n-0}} \cdot (n-1)!^{\frac{n}{n-1}} \cdot \ldots \cdot (n-(m-1))!^{\frac{n}{n-(m-1)}}.$$

In the bottom part of the square, for each entry of each row we have m possibilities, and therefore for each row we have at most $m!^{n/m} = m!^2$ possibilities. Consequently, the number of possibilities for the bottom part is bounded above by $(m!^2)^m$, and altogether we obtain the upper bound

$$(n-0)!^{\frac{n}{n-0}} \cdot (n-1)!^{\frac{n}{n-1}} \cdot \ldots \cdot (n-(m-1))!^{\frac{n}{n-(m-1)}} \cdot m!^{2m}.$$

Thus, (iii) is true.

3. Assume first that G is connected. Recall the formula we have proved in class:

$$\chi(G, V_0, k) = \sum_{G' \preceq G} \mu(G', G) k^{|V(G')|-|V_0|}. \tag{1}$$

The contribution of $G' = G$ to the sum on the right-hand side is $k^{100-7} = k^{93}$. The contribution of each G', obtained from G by contracting a single edge, is $-k^{92}$. Now the edges we contract are only those with at least one endpoint not in V_0. Since V_0 is a clique, there are $\binom{7}{2} = 21$ edges with both endpoints in V_0. Hence there are $1000-21 = 979$ edges with at least one endpoint outside V_0. Hence the total contribution to the right-hand side of (1) is $-979k^{92}$. Graphs G' obtained by contracting two or more edges of G contribute monomials of degree at most 91 in k. Hence the coefficient of k^{92} in $\chi(G, V_0, k)$ is -979.

Now suppose G is not necessarily connected. Let $G_1, \ldots, G_r$ be the connected components of G, where, without loss of generality, G_1 is the component containing the clique V_0. Let $G_i = (V_i, E_i)$. By the considerations above, applied to G_1, the polynomial $\chi(G_1, V_0, k)$ is monic of degree $|V_1| - |V_0|$, and the coefficient of $k^{|V_1|-|V_0|-1}$ is $-(|E_1| - 21)$. The initial coloring of V_0 is irrelevant to $G_2, \ldots, G_r$. Hence each G_i with $i \geq 2$ can be colored in $\chi(G_i, k)$ ways, where the polynomial $\chi(G_i, k)$ is monic of degree $|V_i|$, and the coefficient of $k^{|V_i|-1}$ is $-|E_i|$. Now:

$$\chi(G, V_0, k) = \chi(G_1, V_0, k) \cdot \prod_{i=2}^{r} \chi(G_i, k).$$

The product is a polynomial of degree 93, and the coefficient of k^{92} is easily seen to be

$$-(|E_1| - 21) - |E_2| - \ldots - |E_r| = -|E| + 21 = -979.$$

Thus, (iii) is true.

4. Due to the definition of C, the entries in the first row are in any case either 0 or 1. Now all these entries are 0 only if V_1 has no neighbors, which contradicts the fact that G is connected. Hence already the first entry of the sum of the columns of C is strictly positive.

   Thus, (iv) is true.