Midterm

Mark the correct answer in each part of the following questions.

1. Consider the problem of handshakes in a party attended by n couples, discussed in class. Suppose now that the hostess received from the other participants some answers (not necessarily distinct answers, as in the version presented in class). She believes them all, except for her husband. Let k_1 be the number of hands her husband has shaken, k_2 the number of hands she has shaken, and $k_3, k_4, \ldots, k_{2n}$ the numbers reported by the other participants.

   (i) The values of $k_2, k_3, k_4, \ldots, k_{2n}$ determine uniquely that of k_1, so the hostess will in any case know whether her husband has lied or not.

   (ii) Some values of $k_2, k_3, k_4, \ldots, k_{2n}$ determine uniquely that of k_1, but there are values $k_2, k_3, k_4, \ldots, k_{2n}$ for which every value of k_1 between 0 and $2n-2$ are possible.

   (iii) Some values of $k_2, k_3, k_4, \ldots, k_{2n}$ determine uniquely that of k_1. Other values $k_2, k_3, k_4, \ldots, k_{2n}$ do not determine uniquely that of k_1, but they always give some information, namely there are always some values between 0 and $2n-2$ that k_1 may not assume, given $k_2, k_3, k_4, \ldots, k_{2n}$.

   (iv) None of the above.

2. A family $\mathcal{G}$ of graphs is (for the purpose of this question only) a *Havel-Hakimi family* if every graph $G \in \mathcal{G}$, with at least 2 vertices, has the following property: If $d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ are the degrees of the vertices of G, then there exists a graph $G' \in \mathcal{G}$ such that

$d_2 - 1, \ldots, d_{k+1} - 1, d_{k+2}, \ldots, d_n$ (where $k = d_1$) is the sequence of degrees of its vertices.

Consider the following claims:
I. The family of complete graphs is a Havel-Hakimi family.
II. The family of bi-partite graphs is a Havel-Hakimi family.
III. The family of circles is a Havel-Hakimi family.
IV. The family of trees is a Havel-Hakimi family.

(i) Exactly one of the claims above is true.

(ii) Exactly two of the claims above are true.

(iii) Exactly three of the claims above are true.

(iv) All four claims above are true.

3. Let $G = (V, E)$, where $V = \{1, 2, \ldots, 40\}$ and $(u, v) \in E$ if one of the numbers u, v divides the other.

   (a) The clique number of G is

   (i) $\omega(G) = 3$.
   (ii) $\omega(G) = 4$.
   (iii) $\omega(G) = 5$.
   (iv) $\omega(G) = 6$.
   (v) None of the above.

   (b) The independence number of G is

   (i) $\alpha(G) = 20$.
   (ii) $\alpha(G) = 21$.
   (iii) $\alpha(G) = 22$.
   (iv) $\alpha(G) = 23$.
   (v) None of the above.

   (c) The chromatic number of G is

   (i) $\chi(G) = 4$.

(ii) $\chi(G) = 5$.

(iii) $\chi(G) = 6$.

(iv) $\chi(G) = 7$.

(v) None of the above.

4. Let X_3 be the graph whose proper colorings we connected in class to Sudoku squares. The coefficient of k^{80} in the polynomial $\chi(X_3, k)$ is:

   (i) -810.

   (ii) -972.

   (iii) -1620.

   (iv) -1944.

   (v) None of the above.

5. Given two graphs G_1, G_2, consider the following possible three claims:

   I. There exist infinitely many values of k, for which the number of proper colorings of G_1 using k colors is equal to the number of proper colorings of G_2 using k colors.

   II. There exist infinitely many values of k, for which the number of proper colorings of G_1 using k colors is smaller than the number of proper colorings of G_2 using k colors.

   III. There exist infinitely many values of k, for which the number of proper colorings of G_1 using k colors is larger than the number of proper colorings of G_2 using k colors.

   (i) Claim I holds if and only if G_1 and G_2 are isomorphic.

   (ii) For every two graphs G_1, G_2, exactly one of the three claims I-III holds.

(iii) For any two graphs G_1, G_2, it is possible that both claims I and II hold, and it is also possible that both claims I and III hold, but it is impossible that both claims II and III hold.

(iv) For any two graphs G_1, G_2, it is possible that all three claims I, II and III hold.

(v) None of the above.

6. We employ the greedy coloring algorithm, presented in class, to color C_n. Let k denote the number of colors the algorithm has actually used.

   (i) For every n and ordering of the vertices, we have $k = \chi(C_n)$.

   (ii) For infinitely many values of n, for every ordering of the vertices we will have $k = \chi(C_n)$. There are also infinitely many values of n, for which we will get $k = \chi(C_n)$ for some orderings and $k > \chi(C_n)$ for others.

   (iii) For infinitely many values of n, for every ordering of the vertices we will have $k = \chi(C_n)$. There are also infinitely many values of n, for which we will get $k > \chi(C_n)$ for every ordering.

   (iv) For every sufficiently large n, there exist orderings of the vertices for which $k = \chi(C_n)$, and there exist orderings for which $k > \chi(C_n)$.

   (v) None of the above.

Solutions

1. The example we have discussed in class shows that, for some values of $k_2, k_3, k_4, \ldots, k_{2n}$, the value of k_1 is uniquely determined. In fact, there other values for which it is much easier to figure out what k_1 is, for example when all other k_i's are 0 or when all are $2n - 2$.

On the other hand, for some values of $k_2, k_3, k_4, \dots, k_{2n}$, the value of k_1 is not uniquely determined. For example, suppose $k_3 = k_4 = 1$, while $k_2 = k_6 = k_6 = \dots = k_{2n} = 0$. One interpretation of this data is that the hostess's husband has shaken hands with participants 3 and 4, in which case $k_1 = 2$. It is also possible that 3 and 4 shook hands (assuming they are not husband and wife), and $k_1 = 0$.

By the handshakes lemma, the number of indices i for which k_i is odd must be even. Hence the hostess will in any case know the value of k_1 modulo 2.

Thus, (iii) is true.

2. The sequence of vertex degrees of K_n is $(n-1, n-1, \dots, n-1)$. The transformation in question maps it to the sequence $(n-2, n-2, \dots, n-2)$, which is the sequence of vertex degrees of K_{n-1}.

   The transformation always maps the sequence of vertex degrees of some graph to that of a graph obtained from it by removing some vertex. Since the removal of a vertex from a bi-partite graph yields again a bi-partite graph, the second family is also a Havel-Hakimi family.

   The sequence of vertex degrees of C_n is $(2, 2, \dots, 2)$. Applying the transformation to this sequence, we obtain the sequence $(1, 1, 2, 2, \dots, 2)$, which is not the sequence of vertex degrees of a circle.

   The sequence $(n-1, 1, \dots, 1)$ is the sequence of vertex degrees of a star on n vertices. Applying the transformation to this sequence, we obtain the sequence $(0, 0, \dots, 0)$, which is the sequence of vertex degrees of $\bar{K}_{n-1}$.

   Thus, (ii) is true.

3. (a) The set $\{1, 2, 4, 8, 16, 32\}$ is clearly a clique, and hence $\omega(G) \geq 6$.

On the other hand, let $C = \{u_1, u_2, \ldots, u_k\}$ be a clique, where, say, $u_1 < u_2 < \ldots < u_k$. Since u_j divides u_{j+1} for each j, we have $u_{j+1} \geq 2u_j$, which implies that $u_j \geq 2^{j-1}$ for each j. Since $2^6 = 64 > 40$, there exists no clique of size 7 (and, in fact, the clique specified above is the only one of size 6). Hence $\omega(G) \leq 6$.

Altogether, $\omega(G) = 6$.

Thus, (iv) is true.

(b) The set $\{21, 22, 23, \ldots, 40\}$ is clearly independent, and hence $\alpha(G) \geq 20$.

Now for each odd $u \in V$, denote $C_u = \{u, 2u, 2^2u, \ldots, 2^ku\}$, where k is the largest integer for which $2^ku \leq 40$. We have $V = \cup_{r=1}^{20} C_{2r-1}$, where the union is disjoint. As each C_j is a clique, an independent set may include at most one element of each C_{2r-1}. Hence $\alpha(G) \leq 20$.

Altogether, $\alpha(G) = 20$.

Thus, (i) is true.

(c) Due to part (a) and the inequality $\chi(G) \geq \omega(G)$, we certainly have $\chi(G) \geq 6$.

On the other hand, notice that u may (properly) divide v only if the length of the prime power factorization (counting each prime according to its multiplicity in the factorization) of u is shorter than that of v. (We agree that the length of the prime power factorization of 1 is 0.) Hence, denoting by F_j the set of all integers in V whose prime power factorization is of length j, there are no edges within any of the sets F_j. By the considerations of part (a), we have $V = \cup_{j=0}^{5} F_j$. Coloring all elements of each F_j by color j, we obtain a proper coloring of G by 6 colors. Hence $\chi(G) \geq 6$.

Altogether, $\chi(G) = 6$.

Thus, (iii) is true.

4. We have seen that the coefficient of k^{n-1} in the chromatic polynomial of an n-vertex graph $G = (V, E)$ is $-|E|$. In the graph X_3, there are $3^4 = 81$ vertices, each of which has 20 neighbors (8 on the same row, another 8 on the same column, and another 4 in the same 3×3-square but not the same row or column). Hence the total number of edges in X_3 is $81 \cdot 20/2 = 810$. It follows that the coefficient of k^{80} in $\chi(X_3, k)$ is -810.

   Thus, (i) is true.

5. Consider the chromatic polynomials $\chi(G_1, k)$ and $\chi(G_2, k)$, and put $P(k) = \chi(G_1, k) - \chi(G_2, k)$. Since the chromatic polynomial provides the number of proper colorings of a graph, $P(k)$ is the excess (positive, negative or 0) of the number of proper colorings of G_1 by k colors over the analogous number for G_2. Now if Q is any non-constant polynomial, we have $Q(x) \underset{x \to \infty}{\longrightarrow} \infty$ if the leading coefficient of Q is positive and $Q(x) \underset{x \to \infty}{\longrightarrow} -\infty$ if it is negative. It follows that either for every sufficiently large k the number of proper colorings of G_1 by k colors exceeds that of G_2, or for every sufficiently large k the two are equal, or for every sufficiently large k the second exceeds the first.

   Claim I certainly holds if G_1 and G_2 are isomorphic. However, the converse is false in general. For example, we have seen that all trees on the same number of vertices have the same chromatic polynomial.

   Thus, (ii) is true.

6. The greedy algorithm, applied to any graph G, yields a coloring by at most $\Delta(G) + 1$ colors. Hence, applied to C_n, it will always yield a coloring by at most 3 colors. As $\chi(C_n) = 3$ for odd n, the greedy algorithm will always yield a coloring of C_n by $\chi(C_n)$ colors for such n. Now consider the case of even n, where we let the vertices of C_n be $0, 1, \ldots, n-1$, each vertex i neighboring the vertices $i \pm 1$ modulo n. Depending on the ordering of the vertices, we may get a coloring by $\chi(C_n)$ colors (for example, if we first color the vertices $0, 2, 4, \ldots, n-2$) or by $\chi(C_n) + 1$ colors (for example, if the first two vertices to be colored are 0 and 3).

   Thus, (ii) is true.