Graph Theory

Exercises

1 Graphic Sequences

1. We have shown in class that, in a finite graph, the number of vertices with an odd number of neighbors is even. Show that the result does not hold in an infinite graph even if we assume that each vertex has a finite degree and that the number of vertices with an odd number of neighbors is finite.

2. In a class with n students, each student is required to send Good-Year cards to k of his classmates. For which pairs (n, k) can it be the case that each student will receive cards exactly from the students he has sent cards to?

3. Employing the Havel-Hakimi Theorem, decide whether each of the following sequences is graphic. If it is – construct a suitable graph, otherwise – prove it is not.

(a) $(4, 3, 2, 2, 1, 1, 1)$.

(b) $(4, 4, 3, 1, 1, 1, 0)$.

4. For which pairs of non-negative integers (a, b) does there exist a connected graph on $a + b$ vertices, of which a vertices are of even degrees and b of odd degrees?

5.

(a) Show that, in the sequence of vertex degrees of a graph, there must be at least two equal numbers.

(b) Show that the claim is false for multi-graphs. Is it true if the multi-graph is not allowed to contain loops?

6.

(a) Show that, for each n, there exist sequences $(d_1, d_2, \ldots, d_n)$ for which there exist unique (up to isomorphism) graphs with these sequences of degrees.

(b) Show that, for sufficiently large n, there exist non-isomorphic n-vertex graphs with the same sequence of degrees.

(c) Show that you cannot always determine by the sequence of degrees of a graph whether it is connected or not.

(d) Show that you cannot always determine by the sequence of degrees of a graph whether it is a tree or not.

7. Prove that, for each k, the sequence

$$(k, k, k-1, k-1, k-2, k-2, \ldots, 2, 2, 1, 1)$$

is graphic.

8. Show that a non-increasing sequence $d_1, d_2, \ldots, d_n$ of non-negative integers is the sequence of degrees of a multi-graph without loops if and only if $\sum_{i=1}^{n} d_i$ is even and $d_1 \leq d_2 + d_3 + \ldots + d_n$.

9. Let $d_1, d_2, \ldots, d_n$ be a non-increasing sequence of non-negative integers, and let $1 \leq k \leq n$. Show that the given sequence is graphic if and only if, deleting d_k from the sequence and reducing the d_k largest remaining elements by 1 each, we obtain a graphic sequence.

10. Let $d_1, d_2, \ldots, d_n$ be a non-increasing sequence of non-negative integers with even sum, satisfying $d_1 \leq n-1$ and $d_1 \leq d_n + 1$. Prove that the sequence is graphic.

11. Redo Problem 3, employing the Erdős-Gallai Theorem instead of the Havel-Hakimi Theorem.

12. The sequences $d_1, d_2, \ldots, d_m$ and $d'_1, d'_2, \ldots, d'_n$ are both graphic. For each of the following sequences, determine whether it is necessarily graphic or not. (Hint: When suspecting that the answer is affirmative, try to realize the sequence directly rather than using the Havel-Hakimi Theorem or the Erdős-Gallai Theorem.)

(a) $d_1 + 1, d_2 + 1, \ldots, d_m + 1$.

(b) $d_1 - 1, d_2 - 1, \ldots, d_m - 1$, where it is given that m is even and all d_i's are strictly positive.

(c) $2d_1, 2d_2, \ldots, 2d_m$.

(d) $d_1, d_2, \ldots, d_m, d'_1, d'_2, \ldots, d'_n$.

(e) $d_1 + n, d_2 + n, \ldots, d_m + n, d'_1 + m, d'_2 + m, \ldots, d'_n + m$.

(f) $d_1+1, d_2+1, \ldots, d_m+1, d'_1+1, d'_2+1, \ldots, d'_m+1$, where in this part it is assumed that $n = m$.

(g) $(d_i + d'_j)_{i,j=1,1}^{m,n}$.

(h) $(d_i d'_j)_{i,j=1,1}^{m,n}$.

13. Let $d_1, d_2, \ldots, d_n$ and $d'_1, d'_2, \ldots, d'_n$ be two sequences of non-negative integers. The latter is obtained from the former by a *balancing step* if, for some indices k and l satisfying $d_k \geq d_l + 2$, we have $d'_k = d_k - 1$ and $d'_l = d_l + 1$, while for all other indices i we have $d'_i = d_i$. The sequence $d'_1, d'_2, \ldots, d'_n$ is *more balanced* than $d_1, d_2, \ldots, d_n$ if it is obtained from it by several balancing steps.

(a) Prove that, if $d_1, d_2, \ldots, d_n$ is a graphic sequence, and $d'_1, d'_2, \ldots, d'_n$ is more balanced than it, then $d'_1, d'_2, \ldots, d'_n$ is a graphic sequence as well.

(b) Show that it is possible for $d'_1, d'_2, \ldots, d'_n$ to be a graphic sequence, even though $d_1, d_2, \ldots, d_n$ is not.

14. Let the *mean deviation* of a finite sequence of numbers $(a_i)_{i=1}^n$ be defined by $\frac{1}{n}\sum_{i=1}^n |a_i - \bar{a}|$, where $\bar{a} = \frac{1}{n}\sum_{i=1}^n a_i$. Given two finite sequences $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ of the same length and total sum, the second is *at least as concentrated as* the first if its mean deviation does not exceed that of the first.

(a) Prove that, if a sequence is more balanced (see Problem 13) than another, then it is at least as concentrated.

(b) Is a sequence, which is at least as concentrated as some given graphic sequence, necessarily graphic as well?

2 Graph Coloring

15.

(a) Prove that, for every positive integers n and k, with $1 \leq k \leq n$, there exists a graph G on n vertices such that $\chi(G) = k$.

(b) Prove the following claim (which implies the one in the preceding part): Let $G = (V, E)$, where $|V| = n$ and $\chi(G) = l$. Then for every $1 \leq k < l$ there exists a set $E' \subset E$ such that, denoting $G' = (V, E')$, we have $\chi(G') = k$. Also, for every $l < k \leq n$ there exists a set $E'' \supset E$ such that, denoting $G'' = (V, E'')$, we have $\chi(G') = k$.

16. Find a graph G and a vertex v such that, removing the vertex v (and the incident edges) from G, the chromatic number of G decreases, and the same holds for $\overline{G}$.

17. Find a 3-coloring of $C_5 \,\square\, C_5$, such that the numbers of vertices colored in the 3 colors are $9, 8$ and 8.

18. Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two graphs on the same set of vertices, and let $G = (V, E_1 \cup E_2)$. Show that it is not necessarily the case that $\chi(G) \leq \chi(G_1) + \chi(G_2)$.

19. The plane is divided into regions by means of finitely many straight lines. Show that the resulting map may be colored using 2 colors.

20. Let G be the graph on the vertex set $\{0, 1, \ldots, n-1\}$, where i and j are adjacent if $j - i = \pm 1, \pm 2, \ldots, \pm k$ modulo n, for some specific positive integer $k < \frac{n-1}{2}$. Prove that, if n is divisible by $k+1$, then $\chi(G) = k+1$, while if it is not then $\chi(G) \geq k+2$.

21. Find the clique number of the following graphs:

(a) K_n.

(b) $K_{n_1, n_2, \ldots, n_k}$.

(c) C_n.

(d) P_n.

(e) $K_m \,\square\, K_n$.

22. Find the independence number of the following graphs:

(a) K_n.

(b) $K_{n_1, n_2, \ldots, n_k}$.

(c) C_n.

(d) P_n.

(e) $K_m \,\square\, K_n$.

(f) Perfect binary tree of height n (namely, a binary tree of height n, having 2^k nodes at each level $k \leq n$).

23. Identify the following graphs:

(a) $K_m \vee K_n$.

(b) $\overline{K}_m \vee \overline{K}_n$.

24. Prove that it is impossible to bound from above the chromatic number of a graph in terms of the average of all vertex degrees.

25. Prove that every graph admits an ordering of the vertices, for which greedy coloring yields an optimal coloring.

26.

(a) Suppose we have finitely many straight lines in the plane, no three of which meet at a single point. Form a graph, whose vertices are all intersection points of two of the lines, where two vertices are adjacent if they are consecutive intersection points on the same line. Show that the chromatic number of the graph is at most 3. (Hint: Explain why we may assume that no two vertices have the same x-coordinate. Use greedy coloring, where the order of the vertices is related to their x-coordinates.)

(b) Is the condition, whereby no three of the lines meet at a single point, required?

27. Let $A_1, A_2, \ldots, A_k$ be finite sets and $V = A_1 \times A_2 \times \ldots \times A_k$. Let $G = (V, E)$, where E consists of all pairs $(a_1, a_2, \ldots, a_k)$ and $(a'_1, a'_2, \ldots, a'_k)$ satisfying $a_i \neq a'_i$ for every $1 \leq i \leq k$. Determine $\chi(G)$ (in terms of the sizes $|A_i|$).

28. Show that the following polynomials are not chromatic polynomials of any graph:

(a) $k^{10} - 4k^8$.

(b) $k^5 - 5k^4 + 4k^3$.

(c) $k^7 + 2k^6 - 14k^5 + 26k^4$.

(d) $k^9 - 12k^8 - 14k^7 + 5k^6$.

29. Show that there is a unique (up to isomorphism) graph whose chromatic polynomial is:

(a) k^n.

(b) $k(k-1)(k-2)\ldots(k-n+1)$.

(c) $k^{n-1}(k-1)$.

(d) $k^{n-2}(k-1)(k-2)$.

30. Show that there are exactly two non-isomorphic graphs whose chromatic polynomial is $k^{n-2}(k-1)^2$.

31. Find all chromatic polynomials of the form $k^{10} - 3k^9 + \sum_{i=0}^{8}(-1)^i a_i k^{8-i}$.

32. Show that for each N there exists a polynomial such that there exist at least N mutually non-isomorphic graphs whose chromatic polynomial is the given polynomial.

33. Prove that the zeros of the chromatic polynomial of any n-vertex graph do not exceed $n-1$.

34.

(a) Write the chromatic polynomial of the graph $G \vee K_n$ in terms of the chromatic polynomial of G.

(b) The *wheel graph* on n vertices is denoted by W_n, and defined as the graph obtained from C_{n-1} by adding to it one vertex and connecting this vertex with all others. Calculate $\chi(W_n, k)$.

35. Find the chromatic polynomials of the following graphs:

(a) $P_n \square K_2$.

(b) $S_n \square K_2$.

36. Let G be a connected graph and $\chi(G,k) = \sum_{i=0}^{n}(-1)^i a_i k^{n-i}$ its chromatic polynomial. Prove that $a_i \geq \binom{n-1}{i}$ for each i.

37. Express the third coefficient in the chromatic polynomial of G (namely, the coefficient of $k^{|V|-2}$) in terms of $|V|$, $|E|$ and the number of independent sets of size 3 in G.

38. Consider the lower and upper bounds we obtained on the number of possibilities of filling in any row $k+1$ of a Latin square $n \times n$ we construct, given the preceding k rows.

(a) Explain why, for $k = 1$, namely when filling in the second row, the number of possibilities does not depend on our choice of the first row.

(b) Find the number of possibilities for filling in the second row.

(c) Show that the ratio between this exact number and both the upper and the lower bounds we obtained is multiplicatively negligible as $n \to \infty$ (namely, it is $1 + o(1)$). Conclude that we would have gained very little by replacing our bounds by the correct value.

39. Show that, contrary to the situation in the preceding exercise regarding the number of possibilities for filling in the second row of a Latin square, the number of possibilities for filling in the third row depends in a non-trivial way on the choices of the first two rows.

40.

(a) Show that, for an arbitrary fixed k, the ratio between the upper and the lower bounds we obtained on the number of possibilities for filling in row $k+1$ of a Latin square $n \times n$ we construct, given the preceding k rows, tends to 1 as $n \to \infty$.

(b) Conclude that the number of possibilities for filling in row $k+1$ is $\frac{n!}{e^k} \cdot (1 + o(1))$.

(c) Show that the result of part (a) does not hold if k is allowed to vary over the whole range $[1, n-1]$. Specifically, show that the ratio in question behaves as $C\sqrt{n}$ for some constant $C > 0$ for $k = n/2 - 1$ (and even n).

41.

(a) Show that the ratio between the upper and the lower bounds we obtained on the number of possibilities for filling in row $n-1$ of a Latin square $n \times n$ we construct, given the preceding $n-2$ rows, grows exponentially as a function of n.

(b) Show that, in fact, already the ratio between the true value and the lower bound grows exponentially as a function of n.

(c) Show that the ratio between the upper bound and the true value is exponentially large for some fillings of the first $n-2$ rows.

(d) Show that, on the other hand, for some fillings of the first $n-2$ rows (and even n), the upper bound actually yields exactly the true value.

(e) Show that, for the last row, the lower bound misses the true value by an exponentially growing factor, while the upper bound yields the correct value.

42. Consider the set $M_{2,n}$ of all $2 \times n$ rectangles $(a_{ij})_{i,j=1,1}^{2,n}$, each of whose entries is marked by one of the numbers $1, 2, \ldots, n$, satisfying the following constraints:

(i) All labels in each row are distinct.

(ii) Each label in each row is different from the label above/below it, as well as the labels on the sides of that entry. Namely, $a_{ij} \neq a_{i+1,j-1}, a_{i+1,j}, a_{i+1,j+1}$ for $i = 1, 2$ and $1 \leq j \leq n$. Here, we consider the first index modulo 2 and the second modulo n.

Thus, the label at each entry should be different than the other $n-1$ labels on that row and from 3 of the labels in the other row.

(a) Employ the method, used in class to bound from below the number of Latin squares, to bound $|M_{2,n}|$ from below.

(b) Same for an upper bound.

(c) How close are the two bounds you obtained?

43. Let M denote the matrix of order n, all of whose diagonal elements are a and all off-diagonal elements are b, where a and b are any constants. Express the permanent of M in terms of the derangement numbers d_k (counting the number of permutations on a k-element set, having no fixed points).

44. For each of the following posets, find $\mu(m, x)$ for every x in the poset, where m is the minimal element of the poset.

(a) P – the set of all finite subsets of even size of $\{1, 2, \ldots, 10\}$, with $A \preceq B$ if $A \subseteq B$.

(b) P – the set of all subsets of the form $A' \times B'$ of the set $A \times B$, where $|A| = 2$ and $|B| = 3$, with $X \preceq Y$ if $X \subseteq Y$.

(c) P – the set of all points with integer coordinates between 0 and 3 in the plane, with $(x_1, y_1) \preceq (x_2, y_2)$ if both $x_1 \leq x_2$ and $y_1 \leq y_2$.

(d) Same for points in 3-dimensional space.

(e) P – the set of all divisors of of an arbitrary fixed positive integer n, where $a \preceq b$ if a divides b.

(f) P – the set of all (non-degenerate) triangles with integer sides between 1 and 3, with $T_1 \preceq T_2$ if T_1 can be obtained from T_2 by shortening some of the sides.

(g) P – the set of all linear subspaces of $(\mathbf{Z}/p\mathbf{Z})^2$, where p is a prime, with $V_1 \preceq V_2$ if $V_1 \subseteq V_2$.

(h) P – the set of all substrings of an arbitrary fixed string over some finite alphabet, where $s_1 \preceq s_2$ if s_1 is a substring of s_2. (The minimal element is the empty string.)

45. Same as the preceding question for the following graph-related posets:

(a) P – the set of all spanning subgraphs of an arbitrary fixed graph G, with $G_1 \preceq G_2$ if G_1 is a subgraph of G_2.

(b) P – the set of all subgraphs of P_3, with $G_1 \preceq G_2$ if G_1 is a subgraph of G_2.

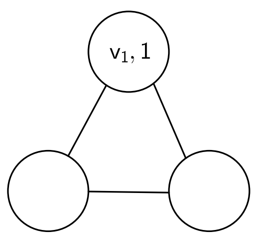
46. Let T be a rooted tree, and $\preceq$ the partial order defined on $V(T)$ by: $v_2 \preceq v_1$ if v_2 is a descendant of v_1. Let $f : V(T) \longrightarrow \mathbf{Z}$ be the constant function 1.

(a) Let g be defined in terms of f as in the Möbius inversion formula. What does g signify?

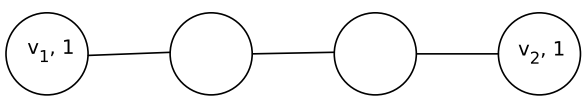
(b) Prove directly that the Möbius inversion formula is satisfied in this case.

47. Calculate $\chi(G, V_0, k)$ for the following graphs using the machinery developed in class (although it is completely unnecessary in these simple cases). The set V_0 is the set of vertices whose label is shown, and the color of the vertices is according to the specification next to their label.

(a)



(b)



3 Spanning Trees

48.

(a) Find the number of spanning paths of K_n.

(b) Same for cycles instead of paths.

(c) Same as the preceding two parts for K_{mn}.

49.

(a) How many spanning trees does an n-vertex graph have on the average? (More precisely, we take all graphs on n labeled vertices, and take the average of the number of spanning trees over this set.)

(b) Same for spanning paths instead of spanning trees.

(c) Same for spanning stars.

50. List all non-isomorphic trees on:

(a) 5 vertices.

(b) 6 vertices.

51. Find $\tau(G)$ for the following graphs G:

(a) K_n, with one edge removed.

(b) $K_{m,n}$, with one edge removed.

52. Find $\tau(G)$ for the graphs $G = (V, E)$, where:

(a)

$$
\begin{aligned}
V &= \{v_{ij} : 1 \leq i \leq n, 1 \leq j \leq 4\}, \\
E &= \{(v_{ij}, v_{i,j+1}) : 1 \leq i \leq n, 1 \leq j \leq 4\} \\
&\quad \cup \{(v_{i3}, v_{i+1,1}) : 1 \leq i \leq n\}.
\end{aligned}
$$

(Here i is to be understood modulo n and j modulo 4, so that E includes, for example, (v_{n3}, v_{11}) and (v_{14}, v_{11}).)

(b)

$$
\begin{aligned}
V &= \{v_{ij} : 1 \leq i \leq n, 1 \leq j \leq 4\}, \\
E &= \{(v_{ij}, v_{i,j+1}) : 1 \leq i \leq n, 1 \leq j \leq 4\} \\
&\quad \cup \{(v_{i2}, v_{i4}) : 1 \leq i \leq n\} \\
&\quad \cup \{(v_{i3}, v_{i+1,1}) : 1 \leq i \leq n\}.
\end{aligned}
$$

53. More generally than the preceding question, find $\tau(G)$ for a graph $G = (V, E)$, constructed out of n given graphs $G_i = (V_i, E_i), 1 \leq i \leq n$ (with pairwise disjoint V_i's), as follows: V is made of the union of all V_i's, and E is made of the union of all E_i's, with one additional edge between V_i and V_{i+1} for each i. (Your answer should be in terms of the numbers $\tau(G_i)$.)

54. Find $\tau(G)$ for the following multi-graphs G:

(a) K_n, with two specific vertices having any number d of edges between them.

(b) $K_{m,n}$, with two specific vertices having any number d of edges between them.

55. Given a (multi-)graph G and an integer $k > 1$, let G' be the multi-graph obtained from G by replacing each edge by k edges between the same vertices. Express $\tau(G')$ in terms of $\tau(G)$, the parameters of G and k.

56. Given a graph G and an integer $k > 1$, let G' be the graph obtained from G by replacing each edge by a path of length k (where, for each edge of G, we thus add $k-1$ new vertices). Express $\tau(G')$ in terms of $\tau(G)$, the parameters of G and k.

57. Denote by τ_n the number of spanning trees of $P_n \,\square\, P_2$ for $n \geq 1$. Prove that the sequence $(\tau_n)_{n=1}^{\infty}$ satisfies the recurrence

$$\tau_n = 4\tau_{n-1} - \tau_{n-2}, \qquad n \geq 3.$$

58. Denote by τ_n the number of spanning trees of $P_n \vee P_1$ for $n \geq 1$. Prove that the sequence $(\tau_n)_{n=1}^{\infty}$ satisfies the recurrence

$$\tau_n = 3\tau_{n-1} - \tau_{n-2}, \qquad n \geq 3.$$

59. Provide an alternative proof of the formula for $\tau(K_{mn})$, using the eigenvalues of the matrix Q. (Hint: Start by finding a simple lower bound for the dimension of the nullspace of $Q - mI$, and similarly $Q - nI$, where you may want to distinguish between the cases $m = n$ and $m \neq n$. Use the fact that 0 is an eigenvalue of Q for every graph, and finally use $\mathrm{tr}(Q)$ to find the last eignevalue of Q.)

60. Find the number of spanning trees of the graph obtained from C_n by adding a single edge. (Note that the answer depends on the distance in C_n between the endpoints of the new edge.)

61.

(a) Show that there exists a constant $\tau > 1$ such that, for every sufficiently large n, there exists a graph G_n with n vertices and $2n$ edges such that $\tau(G_n) > \tau^n$.

(b) Show that, for every $\epsilon > 0$, there exists a graph G_n with n vertices and $2n$ edges such that $\tau(G_n) < (1 + \epsilon)^n$.