Game Theory

Exercises

1 Combinatorial Games

1. Consider the game of Tic-Tac-Toe.

(a) Provide a tree representation of the game, starting at the position where the points $(1, 1)$ and $(3, 3)$ belong to player I, while $(2, 2)$ and $(2, 3)$ belong to II.

(b) Same, where before each turn a fair coin is tossed; if it shows H, then I is to play, whereas if shows T then II does.

2. Consider the following 2-player game. A finite set S of positive integers is given. We define a graph $G = (S, E)$ as follows: distinct elements s_1, s_2 of S are connected by an edge if they are relatively prime (namely, the greatest common divisor of s_1 and s_2 is 1). Each player in turn chooses an element of S, not chosen earlier, until all elements belong to one of the players. Let S_1 and S_2 be the sets of elements chosen by players I and II, respectively. I wins the game if the subgraph of G induced by S_1 contains more edges than the subgraph induced by S_2, and loses if it contains less edges. In case of equality, the game is drawn.

(a) Show that I has a strategy guaranteeing him at least a draw.

(b) Present (non-trivial examples of) sets S for which I has a winning strategy and sets for which II can force at least a draw.

3. Consider the following 2-player game. A non-empty graph is given. Each player in turn removes an edge of the graph. The player who breaks a connected component of the graph into two components loses.

(a) Show that either I or II has a winning strategy.

(b) Why is the game (almost completely) trivial?

4. Consider the following 2-player game. A disconnected graph is given. Each player in turn adds an edge to the graph. The player who connects the graph wins.

(a) Explain why the game is not interesting if the graph comprises two connected components.

(b) Suppose the graph comprises three connected components. Which parameters of the graph determine whether I has a winning strategy or II does?

5. I and II play Chomp. Both are superb players; each of them will certainly win in a winning position. To make the game interesting, it has been agreed that the first move of I will be chosen uniformly randomly out of all legal moves, and from that point on the game will continue as usual.

(a) Suppose the board is $n \times n$ for some $n > 1$. What is I's probability of winning?

(b) Now suppose that the board is $m \times n$, where $(m, n) \neq (1, 1)$. Explain why both I and II have a strictly positive probability of winning. Try to find lower bounds for these probabilities (as large as you can).

6. Design a winning strategy for player I in Chomp for a $2 \times n$ board.

7. I and II play Chomp on an infinite board. More precisely, the squares are in correspondence with the set $\mathbf{N}^2$ of all pairs of positive integers. As in the finite version, when a player chooses (i, j), all pairs (i', j') with $i' \geq i$ and $j' \geq j$ are excluded.

(a) Can you conclude from the theorem, proved in class, regarding games of perfect information, that either I has a winning strategy, or II has one, or each of them has a strategy guaranteeing at least a draw?

(b) Show that a draw (namely, an infinite game) is impossible.

(c) Show directly that I has a winning strategy.

8. Consider the following 2-player game. A positive integer n is given. I and II in turn select a divisor n' of n (including n itself) such that no divisor of n' has already been selected earlier. The player who selects 1 loses. (For example, suppose $n = 1000$. Let I choose 250. None of the players is allowed to select the numbers 1000, 500 and 250 later. Suppose II now selects 100, which excludes the possibility of selecting 200 and 100 subsequently. If I selects now 50, next II selects 8, and finally I selects 1, then I has lost.)

(a) Suppose $n = 10^{100}$. Design a winning strategy for I.

(b) Suppose $n = 20^{100}$. Show that I has a winning strategy. Can you find such a strategy?

9. Consider the space $\{1, 2, \dots, n\}^d$ for some integers $n \geq 2$ and $d \geq 2$.

(a) How many combinatorial lines are there in the space?

(b) Define (for the purposes of this exercise only) a *generalized combinatorial line*, as follows. Start from a d-tuple $(a_1, a_2, \dots, a_d)$, in which each a_i is either an element of $\{1, 2, \dots, n\}$ or x or $\bar{x}$, at least one entry being of one of the two latter types. A generalized combinatorial line is obtained from this d-tuple by letting x and $\bar{x}$ vary over $\{1, 2, \dots, n\}$ in such a way that, when x assumes any value k, we let $\bar{x}$ assume the value $n+1-k$. How many generalized combinatorial lines are there in the space?

10. Several players – $P_1, P_2, \dots, P_c$ – play Tic-Tac-Toe on the space $\{1, 2, \dots, n\}^d$. The first to capture all points on some combinatorial line (or generalized combinatorial line, as defined in the preceding question) wins.

(a) Show that none of $P_2, \dots, P_c$ has a winning strategy.

(b) Does it imply that P_1 does have a winning strategy (for large enough d)?

(c) Design a winning strategy for P_1 if $n = 2$ (for large enough d).

11. A Nim game is played with d heaps, the number of matches in each being uniformly distributed between 0 and $2^k - 1$ for some k. What is the probability that I has a winning strategy?

12. A student tried proving that I has a winning strategy in Nim if and only if the Nim-sum of the heap sizes, when these sizes are written in base 3 and added accordingly, is non-zero. Where did his proof fail?

13. Consider the following 2-player game. A positive integer n is given. I and II in turn replace the current number n' by a proper divisor n'' thereof, such that the ratio n'/n'' is a prime power. (For example, 100 may be replaced by one of the numbers 50, 25, 20, and 4, but not by any other.) The player who gets to 1 wins. For which initial numbers n does I have a winning strategy?

14.

(a) Bi-Nim is a special case of Nim, where there are n heaps, of sizes $1, 2, 2^2, \dots, 2^{n-1}$. Show that, in Bi-Nim with any $n \geq 1$, player I has exactly one winning move at the initial position.

(b) Fibo-Nim is a special case of Nim, where there are n heaps, whose sizes are the Fibonacci numbers $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, \dots, F_n$. Show that, in Fibo-Nim with any $n \geq 1$, player I has at most two winning moves at the initial position.

15. Given a positive integer n, for which numbers k can you generate an instance of Nim with n heaps, such that I has exactly k distinct winning moves in his first turn?

16. Infi-Nim is an extension of Nim, where some of the heaps may be infinite. When a player plays an infinite heap, all matches must be removed, except for at most finitely many. Find out which positions in this game are winning for I and which for II, and design corresponding winning strategies.

2 Two-Person Zero-Sum Games

17. Which of the following payoff matrices admit a Nash equilibrium in pure strategies? Find optimal mixed strategies for the others.

(a)

$$\begin{pmatrix} 5 & 4 \\ 8 & 1 \end{pmatrix}.$$

(b)

$$\begin{pmatrix} 2 & 7 \\ 4 & 3 \end{pmatrix}.$$

(c)

$$\begin{pmatrix} 1 & 3 \\ 9 & 4 \end{pmatrix}.$$

(d)

$$\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}.$$

(e)

$$\begin{pmatrix} 0 & 2 & -6 \\ -2 & 0 & -3 \\ 6 & -3 & 0 \end{pmatrix}.$$

(f)

$$\begin{pmatrix} 0 & -1 & 2 & -1 \\ 1 & 0 & 1 & -1 \\ -2 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}.$$

18.

(a) Let A be the payoff matrix of a two-person zero-sum game, and let V be the value of this game. Show that, for every $\lambda \geq 0$, the value of the game with payoff matrix λA is λV.

(b) Show that the above result is false in general for $\lambda < 0$.

19. Let A be the payoff matrix of a two-person zero-sum game, and let V be the value of this game. Show that, if we add a constant c to all entries of A, then the value of the game represented by the new matrix is $V + c$.

20. Show that there is no necessary relation between the value of a game represented by some payoff matrix A and the value of the game represented by A^T. Moreover, for every two real numbers V, V', one can find a matrix A such that the value of the first game is V and that of the other is V'.

21. I and II are going to play two two-person zero-sum games, one after the other. The payoff matrices of the games are A_1 and A_2.

(a) Explain how you can present the combined game as a single game. How can you construct the payoff matrix of this game from A_1 and A_2?

(b) Let x_1^* and y_1^* be optimal strategies of I and II, respectively, in the first game, and x_2^*, y_2^* optimal strategies in the second game. Find optimal strategies for the combined game.

(c) What is the relation between the values of the given games and that of the combined game?

22.

(a) Find optimal strategies x^* and y^* for both players in the game defined by the $(2n+1) \times (2n+1)$ payoff matrix $A = (a_{ij})_{i,j=-n}^{n}$, where $a_{ij} = |i - j|$ for $-n \leq i, j \leq n$. (Hint: Guess what x^* and y^* are. Then show that x^* guarantees I an expected win of at least some amount V, while y^* guarantees II an expected loss of at most the same amount V.)

(b) Same as (a) if $a_{ij} = (i - j)^2$ for $-n \leq i, j \leq n$.

(c) Same as (a) if $a_{ij} = \sqrt{|i - j|}$ for $-n \leq i, j \leq n$.

23.

(a) Prove that the algebraic sum $F + K$ of a closed set $F \subseteq \mathbf{R}^d$ and a closed bounded set $K \subset \mathbf{R}^d$ is closed.

(b) Show that the conclusion fails in general if both sets are only assumed to be closed.

24. Prove that the algebraic sum of two convex subsets of $\mathbf{R}^d$ is convex.

25. Let $(K_n)_{n=1}^{\infty}$ be a sequence of closed subsets of $\mathbf{R}^d$. Assume that there exists a sequence $(\varepsilon_n)_{n=1}^{\infty}$ of positive numbers with $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, such that $\|v\| \leq \varepsilon_n$ for every $n \geq 1$ and $v \in K_n$, where $\| \cdot \|$ is some norm on $\mathbf{R}^d$. Prove that the set of all points $v \in \mathbf{R}^d$, which may be written in the form $v = \sum_{n=1}^{\infty} v_n$ for some sequence $(v_n)_{n=1}^{\infty}$ with $v_n \in K_n$ for each n, is closed and bounded.

26. Consider the following family of two-person zero-sum games. Let B be some set and $\mathcal{F}$ a family of functions from B to $\mathbf{R}$. I selects some $f \in \mathcal{F}$ and II selects some $b \in B$. The payoff is $f(b)$. (Note that, depending on the cardinalities of B and $\mathcal{F}$, each of the players may have infinitely many pure strategies.)

(a) Suppose $B = [0, \pi/2]$ and $\mathcal{F} = \{\sin, \cos\}$. Show that I has a strategy x^* guaranteeing him an expected win of at least V, and II has a strategy y^* guaranteeing him an expected loss of at most V, for some number V. (For simplicity, consider for II only mixed strategies assigning positive probabilities to finitely many points.)

(b) Same as (a) for $B = [0, \pi/2]$ and $\mathcal{F} = \{1 - \sin, 1 - \cos\}$.

3 General-Sum Games

27. Players I and II choose one of the numbers 0 or 1 each. If they both choose the same number, none of them gets a prize. If they have chosen differently, the one to have chosen 0 gets IS3 and the one who has chosen 1 gets IS2.

(a) Find all Nash equilibria in pure strategies.

(b) Show that there is a unique Nash equilibrium in mixed strategies.

28. Consider the following dilemma – volunteer or not. A group of k people is given the following choice each. A person may volunteer or not. If none of them volunteers, they all get nothing. If at least one person volunteers, then all volunteers get IS2, while all others get IS3. (The people have no communication between them.)

(a) Show that there exists a unique mixed Nash equilibrium in which all people volunteer with the same probability p_k.

(b) Calculate p_k and show that $p_k = \frac{\log 3 + o(1)}{k}$ as $k \to \infty$.

(c) Compare the outcome with what the players might have received if they could cooperate and make a binding agreement prior to the beginning of the game.

29. Consider the following k-person game, where $k \geq 3$. Each player chooses one of the numbers 0 or 1. A player gets a prize only if none of the others has chosen the same number as him. If he is the only one to have chosen this number, then he gets IS3 if the number is 0 and IS2 if the number is 1.

(a) Find all Nash equilibria in pure strategies.

(b) Show that there exists a unique mixed Nash equilibrium in which all players choose 0 with the same probability p_k.

(c) Calculate p_k, show that $p_k \underset{k\to\infty}{\longrightarrow} 1/2$, and estimate the error as best you can.

30. Consider the following symmetric k-person game. Each player has m pure strategies $s_1, s_2, \ldots, s_m$. A player gets a prize only if he is the only one to have used some strategy s_i; in this case he gets u_i. (All possible prizes u_i are strictly positive.)

(a) Show that there are numerous Nash equilibria in pure strategies.

(b) Show that there exists a unique fully mixed Nash equilibrium in which all players use each s_i with the same probability p_{ki}.

(c) Estimate the p_{ki}-s as best you can.