Final #1

Mark the correct answer in each part of the following questions.

1. Consider the following 2-player game. We start with a given positive integer n. I and II in turn replace the current number m by a divisor m_1 thereof, such that the ratio m/m_1 is a prime or a prime power. (For example, if $n = 2^{100} \cdot 3^{10}$, then I may replace it by any of the 100 numbers $2^k \cdot 3^{10}$ with $0 \le k \le 99$, or by any of the 10 numbers $2^{100} \cdot 3^k$ with $0 \le k \le 9$. If I has replaced n by $2^{30} \cdot 3^{10}$, II moved to $2^{30} \cdot 3^2$, I moved to 3^2, and II moved to 1, then II has won.)

   (a) Denote by a the number of moves of I at his first turn, for which he will be able to ensure a win in the game, if $n = 2^{100} \cdot 3^{10}$, and by b the analogous number for $n = 2^{80} \cdot 3^{90} \cdot 5^{100}$.

   (i) $a = 1, b = 2$.
   (ii) $a = 1, b = 3$.
   (iii) $a = 2, b = 2$.
   (iv) $a = 2, b = 3$.
   (v) None of the above.

   (b) Let n_1 and n_2 be two positive integers, and consider the result of the game when we start with n_1 and when we start with n_2. (We assume that both I and II play optimally.)

   (i) If I wins both games, then he wins also when starting with $n_1 n_2$.
   (ii) The former claim is true if n_1 and n_2 are relatively prime, but not in general.
   (iii) If I loses both games, then he loses also when starting with $n_1 n_2$.

(iv) The former claim is true if n_1 and n_2 are relatively prime, but not in general.

(v) None of the above.

2. Let $A = (a_{ij})_{i,j=1}^{m,n}$ be the payoff matrix of a two-person zero-sum game. Let i_0 be the index of a certain row and j_0 the index of a certain column of A. For every real number x, denote by $V(x)$ the value of the game obtained by replacing $a_{i_0 j_0}$ with x.

   (i) $V(x) \underset{x\to\infty}{\longrightarrow} \infty$.

   (ii) The conclusion of (i) is true if and only if $m = 1$.

   (iii) The conclusion of (i) is true if and only if $n = 1$.

   (iv) The conclusion of (i) is true if and only if $m = n = 1$.

   (v) None of the above.

3. Let $A = ((-1)^{i+j}(i-j))_{i,j=1}^{n}$ be the payoff matrix of a two-person zero-sum game.

   (a) Suppose that n is odd. If II is a prophet (so that he knows what I is going to play), and I is aware of it, then I should:

      (i) Play just as he would have played against a regular player.

      (ii) Play strategy 1.

      (iii) Play strategy $(n+1)/2$.

      (iv) Play strategy n.

      (v) none of the above.

   (b) The value of the game is:

      (i) 0.

      (ii) 1.

      (iii) $1 + \frac{(-1)^{n-1}}{n}$.

(iv) $1 + \frac{(-1)^n}{n}$.
(v) none of the above.

4. Recall that we proved that, if K is a convex and closed subset of $\mathbf{R}^d$, that does not include the 0 vector, then there exist a vector $z \in \mathbf{R}^d$ and a number $c > 0$ for which

$$z^T x > c, \qquad x \in K. \tag{1}$$

(i) In fact, there exist an infinite set C of positive numbers and an infinite set Z of vectors in $\mathbf{R}^d$, such that (1) holds for every $c \in C$ and $z \in Z$.

(ii) In fact, there exists an infinite set C of positive numbers such that (1) holds for every $c \in C$. However, the claim in (i) is false.

(iii) In fact, there exists an infinite set Z of vectors in $\mathbf{R}^d$ such that (1) holds for every $z \in Z$. However, the claim in (i) is false.

(iv) None of the above.

5. Consider the following n-person game. Each player chooses an integer between 1 and $2n$. If a player has chosen i, then:

- If he is not the only one to have chosen i, he gets nothing.
- If he is the only one to have chosen i, and $1 \leq i \leq n$, he gets 1.
- If he is the only one to have chosen i, and $n + 1 \leq i \leq 2n$, he gets 2.

(a) The number of Nash equilibria in pure strategies is:

(i) $n!$.
(ii) $\binom{2n}{n}$.
(iii) $\frac{(2n!}{n!}$.
(iv) $(2n)!$.

(v) none of the above.

(b) Now consider Nash equilibria in fully mixed strategies, in which all players employ the same strategy. Moreover, each of them chooses all numbers between 1 and n with the same probability p_{n1}, and all numbers between $n+1$ and $2n$ with the same probability p_{n2}. (Thus, $n \cdot (p_{n1} + p_{n2}) = 1$.) Denote $\alpha = 2^{1/(n-1)}$.

(i) $p_{n1} = \frac{(\alpha-1)n}{(\alpha+1)n+\alpha}$.

(ii) $p_{n1} = \frac{(\alpha-1)n}{n+\alpha}$.

(iii) $p_{n1} = \frac{\alpha-(\alpha-1)n}{(\alpha+1)n}$.

(iv) $p_{n1} = \frac{\alpha-(\alpha-1)n}{(\alpha+1)n+\alpha}$.

(v) None of the above.

6. Consider the stable matchings problem. Suppose that the number n of boys and of girls is even, and that the preferences are given by a matrix $U = (u_{ij})_{i,j=1}^{n}$ measuring compatibility of possible pairs. Suppose that U has the following properties:

- u_{ij} increases as a function of j for each fixed i between 1 and n.
- u_{ij} increases as a function of i for each fixed j between 1 and $n/2$.
- u_{ij} decreases as a function of i for each fixed j between $n/2+1$ and n.

Suppose that, to find a stable matching, we run both the algorithm presented in class (boys propose, girls reject) and the algorithm with sexes reversed (girls propose, boys reject).

(a) Let T_1 be the number of stages it takes the first algorithm to produce a matching and T_2 the number of stages it takes the second algorithm.

(i) $T_1 = n/2, T_2 = n/2$.

(ii) $T_1 = n/2, T_2 = n$.

(iii) $T_1 = n, T_2 = n/2$.

(iv) $T_1 = n, T_2 = n$.

(v) none of the above.

(b) The number of boys matched with the same girl under both algorithms is:

(i) 1.

(ii) $n/2$.

(iii) $n/2 + 1$.

(iv) n.

(v) none of the above.

Solutions

1. (a) The simplest is to realize that the game is equivalent to Nim, as follows. Given as integer n, write its prime-power factorization

$$n = p_1^{e_2} p_2^{e_2} \dots p_k^{e_k}.$$

(The primes p_i are not necessarily ordered, but they are distinct.) Correspond to it a Nim instance with k heaps of sizes $e_1, e_2, \dots, e_k$. An allowed move in our game is to reduce one of the e_i-s, which corresponds to reducing the corresponding heap by the same amount. It follows that I wins for n if and only if the Nim-sum of the exponents $e_1, e_2, \dots, e_k$ is non-zero. Now:

$$\begin{aligned} 100_{10} &= 1100100_2, \\ 10_{10} &= 1010_2, \\ 80_{10} &= 1010000_2, \\ 90_{10} &= 1011010_2. \end{aligned}$$

Clearly, for both n-s in the question, the Nim-sum of the exponents is non-zero, so that I wins in both games. We know that, when I wins in Nim, he may play any heap, the base-2 expansion of whose size has a 1 at the highest digit where the Nim-sum is non-zero. Moreover, there is only one winning move in each such heap. I needs to reduce the heap so that its size will be the Nim-sum of all other heap sizes. With the numbers above, we have therefore $a = 1$ and $b = 3$.

Thus, (ii) is true.

(b) One can again use the equivalence with Nim, but we will solve this part independently of that.

I wins for $n_1 = 2$, as well as for $n_2 = 3$ (which are relatively prime), yet loses for $n = n_1 n_2 = 6$. Hence (i) and (ii) are false.

I loses for $n_1 = 6$, as well as for $n_2 = 10$, yet for $n = n_1 n_2 = 60$ (by passing to 15). Hence (iii) is false.

If I loses for both n_1 and n_2, and the numbers are relatively prime, then II can win for $n = n_1 n_2$ as follows. At each stage, the current number m can be written in a unique way in the form $m = m_1 m_2$, where m_1 is a divisor of n_1 and m_2 a divisor of n_2. II regards the game, in a sense, as two games. When I plays, he replaces either m_1 or m_2 by an appropriate divisor. II always replaces the number changed by I, as he would when playing from that n_i. Eventually, he will be the player to make both m_1 and m_2 equal to 1.

Thus, (iv) is true.

2. If $n > 1$, and II plays any strategy but the j_0-th, then I cannot win more the maximum in the corresponding column of A, so that V is bounded above. If $n = 1$, then II plays his only strategy, and I guarantees to get x by playing his i_0-th strategy. Hence, in this case, $V(x) \underset{x\to\infty}{\longrightarrow} \infty$.

   Thus, (iii) is true.

3. (a) Whichever row I chooses to play, II will choose the column that attains the minimum in that row. Hence I needs to play the row at which the minimum is maximal. Now it is easy to verify (using the fact that n is odd) that the minimum in the first row is $1-n$, in the second and third it goes down to $3-n$, and in the next two rows to $5-n$. This pattern continues until the middle row – row $(n+1)/2$. At the two rows just before it, the minimum is $(3-n)/2$, and at the middle row it goes down to $(1-n)/2$. At this point, the minimum starts decreasing. At the two following rows it is again $(3-n)/2$, and so forth. Hence I should play $(n+1)/2$, II should play either 1 or n, and I will lose $(n-1)/2$.

   Thus, (iii) is true.

   (b) Since A is anti-symmetric, the value of the game is 0.

   Thus, (i) is true.

4. Obviously, if (1) is satisfied for some $z \in \mathbf{R}^d$ and $c > 0$, then it satisfied with z replaced by any λz with $\lambda > 1$ and c replaced by any c' with $0 < c' \le c$. Hence (i) holds for the choices $Z = \{\lambda z : \lambda > 1\}$ and $C = \{c' : 0 < c' \le c\}$.

Thus, (i) is true.

5. (a) At any Nash equilibrium in pure strategies, it must be the case that all players play distinct numbers between $n+1$ and $2n$. Indeed, if any player does not, then at least one of the numbers in that range has not been chosen by anyone else, so that this player could have improved his gain by taking it. On the other hand, every point as above is clearly a Nash equilibrium. Since the number of possibilities of matching the numbers $n+1, n+2, \ldots, 2n$ with the players is $n!$, this is the number of Nash equilibria.

   Thus, (i) is true.

   (b) At a Nash equilibrium point, each pure strategy employed by a certain player with a positive probability has the property that, if employed with certainty, it yields for him the same gain. In our case, if a player deviates from the equilibrium strategy and plays 1 (or any other i up to n), he will get on the average $1 \cdot (1-p_{n1})^{n-1}$, while if he plays $n+1$ his expected gain is $2 \cdot (1-p_{n2})^{n-1}$. It follows that:

$$1 \cdot (1-p_{n1})^{n-1} = 2 \cdot (1-p_{n2})^{n-1}.$$

   Hence

$$\frac{1-p_{n1}}{1-p_{n2}} = \alpha,$$

   which yields

$$1 - p_{n1} = \alpha - \alpha \frac{1-np_{n1}}{n}.$$

   Routine calculations give:

$$p_{n1} = \frac{\alpha - (\alpha-1)n}{(\alpha+1)n}.$$

   Thus, (iii) is true.

6. (a) The first property of U means that all boys have exactly the same preferences. In such a case, the algorithm always takes n stages. First, all boys propose to the first girl on their common list, and all but one of them get rejected. Then all $n-1$ remaining boys propose to the second girl, and again all but one are rejected. After n stages, we get to the final matching.

    The two last properties mean that the first $n/2$ girls have the same preferences, and so do the last $n/2$ of them. However, whereas the girls in the first group have the first $n/2$ boys at the top half of their list, the last /2 girls have the first $n/2$ boys at the top. It follows that, in the first $n/2$ stages, the first girls will only propose to the last boys, and the last girls to the first boys. Consequently, by the end of stage $n/2$, all will be matched.

    Thus, (iii) is true.

   (b) We have proved that, when the preferences are determined by a matrix of compatibility measures, there is a unique stable matching. In particular, both algorithms match all n boys with the same n girls.

    Thus, (iv) is true.