Quiz Section 11 - Cauchy Sequences and Completeness

Definition:
Let V be a normed space, $\{u_n\}_{n=1}^{\infty} \subset V$ will be called a *Cauchy sequence*, if for every $\varepsilon > 0$ exists $n_0 \in \mathbb{N}$ such that for any $m, n > n_0$ $\|u_m - u_n\| < \varepsilon$.

Definition:
Let V be a normed space, V will be called *complete* if every Cauchy sequence in V converges. A complete normed space is called a *Banach* space. A complete inner-product space is called a *Hilbert* space.

Remark:
Do not confuse closedness and completeness. Closedness is a property a set may have in relation to a containing set. That is, a subspace may be closed even if not any Cauchy sequence converges, as long as any cauchy sequence that converges in the containing set converges to a point in the contained set. Completeness is a property that is independent of any containing set.

Prove/disprove the following:

1. Any convergent sequence is Cauchy.

Solution:
True. Let $\{u_n\}_{n=1}^{\infty} \subset V$ be a converging sequence and let $u \in V$ such that $u_n \xrightarrow[n\to\infty]{} u$. Let $\varepsilon > 0$, since $u_n \xrightarrow[n\to\infty]{} u$, exists $N \in \mathbb{N}$ such that for $n > N$ we have $\|u_n - u\| < \frac{\varepsilon}{2}$. Therefore for any $m, n > N$ we have:

$$\begin{aligned} \|u_n - u_m\| &= \|u_n - u + u - u_m\| \\ &\leq \|u_n - u\| + \|u - u_m\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon, \end{aligned}$$

therefore $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

2. Any Cauchy sequence is bounded.

Solution:

True. Let $\{u_n\}_{n=1}^{\infty} \subset V$ be a Cauchy sequence (V is a normed space). Then exists $n_0 \in \mathbb{N}$ such that for any $m, n \geq n_0$ we have $\|u_m - u_n\| < 1$ and in particular we have for any $n \geq n_0$ $\|u_n - u_{n_0}\| < 1$. Therefore for any $n \geq n_0$ we have:

$$\begin{aligned} \|u_n\| &= \|u_n - u_{n_0} + u_{n_0}\| \\ &\leq \|u_n - u_{n_0}\| + \|u_{n_0}\| \\ &< 1 + \|u_{n_0}\|. \end{aligned}$$

and we get that the sequence $\{u_n\}_{n=1}^{\infty}$ is bounded by: $\max\{\|u_1\|, \ldots, \|u_{n_0-1}\|, \|u_{n_0}\| + 1\}$.

3. If a Cauchy sequence $\{u_n\}_{n=1}^{\infty} \subset V$ has a converging subsequence, then $\{u_n\}_{n=1}^{\infty}$ converges as well.

Solution:

True. Assume $\{u_{n_k}\}_{k=1}^{\infty}$ is a converging subsequence of $\{u_n\}_{n=1}^{\infty}$. Let $\varepsilon > 0$. Since $\{u_n\}_{n=1}^{\infty}$ is a cauchy sequence, exists $N \in \mathbb{N}$ such that for any $m, n \geq N$ we have $\|u_n - u_m\| < \frac{\varepsilon}{2}$. Denote $u = \lim_{k\to\infty} u_{n_k}$. Since the subsequence converges to u, exists some k_0 such that $n_{k_0} > N$ and $\left\|u_{n_{k_0}} - u\right\| < \frac{\varepsilon}{2}$. Therefore, for every $n > n_{k_0}$ we have:

$$\begin{aligned} \|u_n - u\| &= \left\|u_n - u_{n_{k_0}} + u_{n_{k_0}} - u\right\| \\ &\leq \left\|u_n - u_{n_{k_0}}\right\| + \left\|u_{n_{k_0}} - u\right\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \epsilon, \end{aligned}$$

which implies that the entire sequence converges to u.

4. $C[-1,1]$ with $\|\cdot\|_1$ is complete.

Solution:

False. There are sequences of continuous functions which converge in $\|\cdot\|_1$ to functions which are not continuous, an example is:

$$f_n(x) = \begin{cases} 0, & -1 \leq x < -\frac{1}{n}, \\ n\left(x + \frac{1}{n}\right), & -\frac{1}{n} \leq x < 0, \\ 1, & 0 \leq x \leq 1. \end{cases}$$

It can be easily verified that this is a Cauchy sequence, but the sequence does not converge to a continuous function, hence the given space is not complete.

5. $C[-1,1]$ with $\|\cdot\|_2$ is complete.

Solution:
False. Can be disproved with the same example we used for $\|\cdot\|_1$.

6. $C[-1,1]$ with $\|\cdot\|_\infty$ is complete.

Solution:
True. It has been proved in class. The idea of the proof is if f_n is a Cauchy sequence, then for each x, $f_n(x)$ is a cauchy sequence of numbers and therefore converges, so f_n converges pointwise to some functions. Since the functions in the sequence are continuous and the convergence is uniform, the limit function must be continuous as well.

7. Let V be a Banach space, and let $\{u_n\}_{n=1}^{\infty} \subset V$ such that $\sum_{n=1}^{\infty} \|u_n\| < \infty$, then exists $u \in V$ such that $u = \sum_{n=1}^{\infty} u_n$.

Solution:
True. Construct a new sequence $v_n = \sum_{i=1}^{n} u_n$. Let $\varepsilon > 0$, since $\sum_{n=1}^{\infty} \|u_n\| < \infty$, exists $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} \|u_n\| < \varepsilon$. Let $m, n > N$ and assume without loss of generality that $m > n$ (if $m = n$ then $\|v_m - v_n\| = 0 < \varepsilon$). We have:

$$\begin{aligned} \|v_m - v_n\| &= \left\| \sum_{k=n+1}^{m} u_k \right\| \\ &\leq \sum_{k=n+1}^{m} \|u_k\| \\ &\leq \sum_{k=N}^{\infty} \|u_k\| \\ &< \varepsilon. \end{aligned}$$

Therefore $\{v_n\}_{n=1}^{\infty}$ is a Cauchy sequence. since V is a Banach space, the sequence converges, and the limit is $\sum_{n=1}^{\infty} u_n$.

8. Let V be a normed space, for every $r > 0$ and $u \in V$ denote $\overline{B_r(u)} = \{v \in V : \|u - v\| \leq r\}$. If V is not complete, then for every $r > 0$ and every $u \in V$, $\overline{B_r(u)}$ contains a non-converging Cauchy sequence.

Solution:
True. This means that if there are some holes in the space, then there are holes everywhere. The idea of the proof is that we can take a non-converging Cauchy sequence and move it inside a ball where every Cauchy sequence converges, find the limit there and then move it back. Here are the details: Let V be an incomplete normed space. Assume in negation that for some $x \in V$ and $r > 0$, $\overline{B_r(x)}$ does not contain any non-converging Cauchy sequences. V is not complete, therefore exists a Cauchy sequence $\{u_n\}_{n=1}^{\infty} \subset V$ which does

not converge. Since $\{u_n\}_{n=1}^{\infty}$ is Cauchy, exists n_0 such that for $n \geq n_0$ we have $\|u_n - u_{n_0}\| < r$. We define a new sequence: $v_n = u_n - u_{n_0} + x$. Note that for $n > n_0$ we have:

$$\begin{aligned} \|v_n - x\| &= \|u_n - u_{n_0} + x - x\| \\ &= \|u_n - u_{n_0}\| \\ &< r. \end{aligned}$$

Therefore, for $n > n_0$ we have $v_n \in \overline{B_r(x)}$. It is also easy to verify that v_n is a Cauchy sequence. The fact that perhaps $v_1, \ldots v_{n_0-1} \notin \overline{B_r(x)}$ will not bother us since we can omit these elements from v_n without changing the convergence property. v_n is a Cauchy sequence which is contained (aside from a finite number of elements) in $\overline{B_r(x)}$, therefore it converges, and we denote $v = \lim_{n\to\infty} v_n$. We now want to show that $\lim_{n\to\infty} u_n = v - x + u_{n_0} = u$. Let $\varepsilon > 0$, $v = \lim_{n\to\infty} v_n$, therefore exists N such that for $n > N$ $\|v_n - v\| < \varepsilon$. Therefore, for $n > N$:

$$\begin{aligned} \|u_n - u\| &= \|v_n + u_{n_0} - x - (v - x + u_{n_0})\| \\ &= \|v_n - v\| \\ &< \varepsilon. \end{aligned}$$

Therefore u_n converges, in contradiction to the assumption.

9. Let V be a normed space such that every series that is absolutely convergent is convergent (this means that for any sequence $\{u_n\}_{n=1}^{\infty} \subset V$, $\sum_{n=1}^{\infty} \|u_n\| < \infty$ implies that $\lim_{N\to\infty} \sum_{n=1}^{N} u_n$ exists). Then V is a Banach space.

Solution:

True. We need to prove that every Cauchy sequence in V converges. Let $\{v_n\}_{n=1}^{\infty} \subset V$ be a Cauchy sequence. It is sufficient to show that $\{v_n\}_{n=1}^{\infty}$ has a converging subsequence $\{v_{n_k}\}_{k=1}^{\infty}$. We define a new sequence recursively. Since $\{v_n\}_{n=1}^{\infty}$ is Cauchy, we can define n_1 such that for any $n > n_1$, $\|v_n - v_{n_1}\| < \frac{1}{2}$. We define the first element of the subsequence to be v_{n_1}. Assume we already defined $n_1, \ldots n_{k-1}$. We choose $n_k \in \mathbb{N}$ to be some number $n_k > n_{k-1}$ such that for any $n > n_k$ we have $\|v_n - v_{n_k}\| < \frac{1}{2^{-k}}$. We want to prove that this subsequence converges. We define a new sequence, $u_1 = v_{n_1}$, and for $k > 1$ $u_k = v_{n_k} - v_{n_{k-1}}$.

$$\begin{aligned}
\sum_{k=1}^{\infty}\|u_k\| &= \|v_{n_1}\| + \sum_{k=2}^{\infty}\left\|v_{n_k} - v_{n_{k-1}}\right\| \\
&\underset{*}{\leq} \|v_{n_1}\| + \sum_{k=2}^{\infty}\frac{1}{2^{-k-1}} \\
&= \|v_{n_1}\| + \sum_{k=1}^{\infty}\frac{1}{2^{-k}} \\
&= \|v_{n_1}\| + 1 \\
&< \infty.
\end{aligned}$$

Therefore $\sum_{n=1}^{\infty} u_n$ converges. Since $\sum_{j=1}^{k} u_j = v_{n_k}$, we have that $\{v_{n_k}\}_{k=1}^{\infty}$ also converges. So $\{v_n\}_{n=1}^{\infty}$ is a Cauchy sequence with a converging subsequence, and therfore converges, hence V is Banach.

10. Let V be the space of sequences with a finite number of non-zero coordinates with the norm $\|x\| = \sqrt{\sum_{n=1}^{\infty}|x_i|^2}$. Then V is complete.

Solution:

False. For example consider the sequence $u^{(n)}$ defined by $\left(u^{(n)}\right)_i = \begin{cases} \frac{1}{k}, & k \leq n, \\ 0, & k > n. \end{cases}$. $u^{(1)} = (1, 0, 0\ldots)$, $u^{(2)} = \left(1.\frac{1}{2}, 0, 0\ldots\right)$, $u^{(3)} = \left(1, \frac{1}{2}, \frac{1}{3}, 0, 0, \ldots\right)$ and so on. We first verify that this is indeed a Cauchy sequence. Let $N \in \mathbb{N}$ and $m, n > N$. Assume without loss of generality $m > n$.

$$\begin{aligned}
\left\|u^{(m)} - u^{(n)}\right\| &= \sqrt{\sum_{k=n+1}^{m}\frac{1}{k^2}} \\
&\leq \sqrt{\sum_{k=N}^{\infty}\frac{1}{k^2}} \\
&\xrightarrow[N\to\infty]{} 0.
\end{aligned}$$

But it is clear that $u^{(n)}$ cannot converge to any vector with finite number of non-zero coordinates, therefore V is not complete.

11. $L^1_C(\mathbb{R})$ is complete.

Solution:

False. We can construct an example very similar to part (3):

$$f_n(x) = \begin{cases} 0, & x < -\frac{1}{n}, \\ n\left(x + \frac{1}{n}\right), & -\frac{1}{n} \le x < 0, \\ e^{-x}, & 0 \le x. \end{cases}$$

It is easy to very that this is a Cauchy sequence which converges to a function which is not continuous at $x = 0$.

12. $L^2_C(\mathbb{R})$ is complete.

Solution:
False. We can utilize our example from the previous part.

13. $L^\infty_C(\mathbb{R})$ is complete.

Solution:
True. The proof is similar to the proof of (5).

14. l^1 is complete.

Solution:
True. It is enough to prove that every absolutely convergent series converges. Let $\left\{u^{(n)}\right\}_{n=1}^{\infty} \subset l^1$ be a sequence such that $\sum_{n=1}^{\infty} \left\|u^{(n)}\right\| < \infty$. We denote by $u_i^{(n)}$ the i'th element of the n'th. For each i we have $\left|u_i^{(n)}\right| \le \left\|u^{(n)}\right\|$, therefore $\sum_{n=1}^{\infty} \left|u_i^{(n)}\right| < \infty$ and we define a new sequence of complex numbers v by $v_i = \sum_{n=1}^{\infty} u_i^{(n)}$. $\mathbb{C}$ is complete, the series defining each element of v is absolutely convergent, therefore convergent, and the sequence is well defined. We need to prove that $v \in l^1$ and that $v = \sum_{n=1}^{\infty} u^{(n)}$. First we show that $v \in l^1$, recall that l^1 is the space of absolutely summable sequences of complex numbers.

$$
\begin{aligned}
\sum_{i=1}^{\infty} |v_i| &= \lim_{K\to\infty} \sum_{i=1}^{K} |v_i| \\
&= \lim_{K\to\infty} \sum_{i=1}^{K} \left| \sum_{n=1}^{\infty} u_i^{(n)} \right| \\
&= \lim_{N,K\to\infty} \sum_{i=1}^{K} \left| \sum_{n=1}^{N} u_i^{(n)} \right| \\
&\leq \lim_{N,K\to\infty} \sum_{i=1}^{K} \sum_{n=1}^{N} \left| u_i^{(n)} \right| \\
&= \lim_{N,K\to\infty} \sum_{n=1}^{N} \sum_{i=1}^{K} \left| u_i^{(n)} \right| \\
&\leq \lim_{N,K\to\infty} \sum_{n=1}^{N} \sum_{i=1}^{\infty} \left| u_i^{(n)} \right| \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \sum_{i=1}^{\infty} \left| u_i^{(n)} \right| \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \left\| u^{(n)} \right\| \\
&= \sum_{n=1}^{\infty} \left\| u^{(n)} \right\| \\
&< \infty
\end{aligned}
$$

Therefore $v \in l^1$. We need to show that $v = \sum_{n=1}^{\infty} u^{(n)}$.

$$\begin{aligned}
\left\| v - \sum_{n=1}^{N} u^{(n)} \right\| &= \sum_{i=1}^{\infty} \left| v_i - \sum_{n=1}^{N} u_i^{(n)} \right| \\
&= \sum_{i=1}^{\infty} \left| \sum_{n=N+1}^{\infty} u_i^{(n)} \right| \\
&\leq \sum_{i=1}^{\infty} \sum_{n=N+1}^{\infty} \left| u_i^{(n)} \right| \\
&= \sum_{n=N+1}^{\infty} \sum_{i=1}^{\infty} \left| u_i^{(n)} \right| \\
&= \sum_{n=N+1}^{\infty} \left\| u^{(n)} \right\| \\
&\xrightarrow[N\to\infty]{} 0.
\end{aligned}$$

In summary, we wanted to show that l^1 is complete, from a previous section, we know that it suffices to show that any absolutely convergent series is convergent, so we took an arbitrary absolutely convergent series and showed that it converges to a vector in l^1, hence l^1 is complete.

15. l^2 is complete.

Solution:
True. See the proof for l^1.

16. l^∞ is complete.

Solution:
True. See the proof for l^2.

17. Let V be a Banach space, $W \subset V$ a closed subspace, then W is complete.

Solution:
True. Let $\{u_n\}_{n=1}^\infty \subset W$ be a Cauchy sequence. $\{u_n\}_{n=1}^\infty$ is also a Cauchy sequence in V. Since V is complete, exists $u \in V$ such that $u_n \xrightarrow[n\to\infty]{} u$. Now, since W is closed, $u \in W$. This proves that W is complete.

18. Let V be a Banach space, $W \subset V$ a subspace, if W is complete then W is closed.

Solution:
True. Let $\{u_n\}_{n=1}^\infty \subset W$ be a sequence that converges to some vector $u \in V$. The sequence converges in V therefore it is Cauchy. Since W is complete and $\{u_n\}_{n=1}^\infty$ is Cauchy, it converges to a limit in W and therefore W is closed

19. Let V be the space of piecewise constant functions from $[0,1]$ to $\mathbb{C}$. Then V is complete.

Solution:

False. We have already seen that any function in $L^2_{PC}[0,1]$ can be approximated arbitrarily closely by piecewise constant functions. Therefore the space of piecewise constant functions is not closed in $L^2_{PC}[0,1]$ (because its closure is $L^2_{PC}[0,1]$). This means we can construct a sequence of functions in V which converges to a function in $L^2_{PC}[0,1] \setminus V$, the sequence will be Cauchy (since it converges in $L^2_{PC}[0,1]$), but it will not converge in V. Therefore V is not complete.