Fourier Analysis

Solutions to Selected Exercises

1 Review Questions in Linear Algebra

1. Recall that a non-empty subset of a vector space forms a subspace if and only if it is closed under addition and under multiplication by scalar. Equivalently, it is a linear subspace if and only if, for every two vectors v_1, v_2 in the subset and every two scalars a, b, the combination $av_1 + bv_2$ belongs to the subset as well.

(a) Yes. In fact,

$$a(x_1, x_1) + b(x_2, x_2) = (ax_1 + bx_2, ax_1 + bx_2).$$

(b) No. For example, the vector $(1, 2)$ belongs to the subset in question, yet the vector $-1 \cdot (1, 2) = (-1, -2)$ does not.

(g) Yes. Given any vectors $v_k = (x_k, y_k)$ satisfying $4.7x_k + 6.8y_k = 0$ for $k = 1, 2$, and reals a, b, we have

$$av_1 + bv_2 = (ax_1 + bx_2, ay_1 + by_2)$$

and

$$4.7(ax_1 + bx_2) + 6.8(ay_1 + by_2) = a(4.7x_1 + 6.8y_1) + b(4.7x_2 + 6.8y_2) = 0.$$

2.

(c) No. For example, both points $(0, 1, 1)$ and $(0, 1, -1)$ belong to the set in question, but the sum $(0, 1, 1) + (0, 1, -1) = (0, 2, 0)$ does not.

3.

(a) Yes, directly from the definitions.

(b) No. For example, both polynomials x^2 and $-x^2$ belong to the set in question, but the sum $x^2 + (-x^2) = 0$ does not.

4.

(a) No. In fact, one verifies easily that the sum of no two functions in the indicated subset belongs to the subset.

(d) No. For example, the constant function $f(x) = 10$ belongs to the set, yet $2 \cdot f$ does not.

2 Inner Product Spaces

5.

(b) Yes. It is a special case of the inner product defined by $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{k=1}^{n} w_k x_k y_k$ with arbitrary weights $w_1, \ldots, w_n$, mentioned in class.

(d) No. All properties hold, except that $\langle \mathbf{x}, \mathbf{x} \rangle$ may well be negative. For example, for $n = 2$, $\mathbf{x} = (1, -1)$ we have $\langle \mathbf{x}, \mathbf{x} \rangle = 1 \cdot (-1) + (-1) \cdot 1 = -2$.

6.

(d) No. All properties hold, except that we may have $\langle f, f \rangle = 0$ for $f \neq 0$. For example, this is the case for $f(x) = x$.

(e) No. All properties hold, except that we may have $\langle f, f \rangle = 0$ for $f \neq 0$. For example, this is the case for $f(x) = |x| - 1/2$.

3 Normed Spaces

12. It is easy to see that the function g must also be bounded in absolute value by M. Therefore $|g_n(x) - g(x)| \leq 2M$ for every $n \in \mathbf{N}$ and $x \in \mathbf{R}$. Hence:

$$\int_{-\infty}^{\infty} |g_n(x) - g(x)|^2 dx \leq \int_{-\infty}^{\infty} 2M |g_n(x) - g(x)| dx = 2M \|g_n - g\|_1 \underset{n \to \infty}{\longrightarrow} 0.$$

4 Orthogonal Systems

15. We have to show that $\langle v_k, v_l \rangle = 0$ for $k \neq l$. Indeed, since $\zeta^k = e^{2\pi ki/n} \neq e^{2\pi li/n} = \zeta^l$, we have

$$\begin{aligned}
\langle v_k, v_l \rangle &= \sum_{j=0}^{n-1} e^{2\pi kji/n}\overline{e^{2\pi lji/n}} \\
&= \sum_{j=0}^{n-1} e^{2\pi(k-l)ji/n} \\
&= \frac{1-\left(e^{2\pi(k-l)i/n}\right)^n}{1-e^{2\pi(k-l)i/n}} = 0.
\end{aligned}$$

16.

(c) We use induction on n. For $n = 0$:

$$\int_0^\infty x^n e^{-x} dx \int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = 1 = 0!.$$

Assume the required equality holds for $n = k$ and take $n = k+1$. Then:

$$\begin{aligned}
\textstyle\int_0^\infty x^n e^{-x} dx &= -x^{k+1}e^{-x}\Big|_0^\infty - \textstyle\int_0^\infty -(k+1)x^k e^{-x} dx \\
&= 0 + (k+1)k! = (k+1)!.
\end{aligned}$$

(b) We need to show that the inner product of each of the three vectors with itself is 1, while each of the inner products of two distinct vectors vanishes. Let us show two of the required equalities. The inner product of the last two vectors is

$$\begin{aligned}
\langle x-1, x^2/2-2x+1 \rangle &= \textstyle\int_0^\infty (x-1)(x^2/2-2x+1)e^{-x} dx \\
&= \textstyle\int_0^\infty (x^3/2-5x^2/2+3x-1)e^{-x} dx \\
&= 3!/2 - 5\cdot 2!/2 + 3\cdot 1! - 0! = 0,
\end{aligned}$$

and that of the last vector with itself is

$$\begin{aligned}
\langle x^2/2-2x+1, x^2/2-2x+1 \rangle &= \textstyle\int_0^\infty (x^4/4-2x^3+5x^2-4x+1)e^{-x} dx \\
&= 4!/4 - 2\cdot 3! + 5\cdot 2! - 4\cdot 1! + 0! = 1.
\end{aligned}$$

19. Let $\{e_1, e_2, \ldots, e_n\}$ and $\{e'_1, e'_2, \ldots, e'_n\}$ be the orthonormal bases we obtain when running the Gram-Schmidt process starting from $\{v_1, v_2, \ldots, v_n\}$ and $\{u_1, u_2, \ldots, u_n\}$, respectively. The relation between the two bases, and the fact that in the Gram-Schmidt process the first k vectors of the basis we construct span the same subspace as the first k vectors of the original basis for each k, imply that $\text{span}\{e_1, e_2, \ldots, e_k\} = \text{span}\{e'_1, e'_2, \ldots, e'_k\}$ for each k. Hence, when we proceed to find the $(k+1)$-st vector in each of the bases we construct, we subtract from the next vector in the original basis its projection in the same subspace. Let w be the projection of v_{k+1} in $\text{span}\{e_1, e_2, \ldots, e_k\}$. Since $\alpha_{k+1,1}v_1 + \alpha_{k+1,2}v_2 + \ldots + \alpha_{k+1,k}v_k$ belongs to this subspace, the projection of u_{k+1} in $\text{span}\{e_1, e_2, \ldots, e_k\}$ is the vector $\alpha_{k+1,1}v_1 + \alpha_{k+1,2}v_2 + \ldots + \alpha_{k+1,k}v_k + \alpha_{k+1,k+1}w$. Hence the vector we obtain from u_{k+1} before normalization is $\alpha_{k+1,k+1}(v_{k+1} - w)$. Hence after the normalization we will obtain a multiple of e_{k+1} by some scalar. Since $\|e_{k+1}\| = \|e'_{k+1}\| = 1$, the scalar must be of modulus 1.

In summary, we must have $e'_k = \beta_k e_k,\ 1 \le k \le n$, for some scalars $\beta_1, \ldots, \beta_n$ of modulus 1 each.

5 Best Approximations

20. Let d be the distance between v and the closest points in W. Then, for any two closest points $w_1^*, w_2^* \in W$ and $\alpha \in [0, 1]$:

$$\begin{aligned}
\|v - (\alpha w_1^* + (1-\alpha)w_2^*)\| &= \|\alpha(v - w_1^*) + (1-\alpha)(v - w_2^*)\| \\
&\le \|\alpha(v - w_1^*)\| + \|(1-\alpha)(v - w_2^*)\| \\
&= \alpha\|v - w_1^*\| + (1-\alpha)\|v - w_2^*\| \\
&= \alpha d + (1-\alpha)d = d.
\end{aligned}$$

Since d is the minimal distance, we actually have:

$$\|v - (\alpha w_1^* + (1-\alpha)w_2^*)\| = d.$$

Thus, the point $\alpha w_1^* + (1-\alpha)w_2^*$ is closest to v as well.

6 Convergence in Normed Spaces

30.

(a) Since the sequence converges uniformly, for some n_0 we have $|f_{n_0}(x) - f(x)| < 1$. Therefore:

$$\int_0^\infty |f_{n_0}(x) - f(x)|^2 e^{-x} dx \le \int_0^\infty e^{-x} dx = 1 < \infty.$$

Therefore $f_{n_0} - f \in V$. We also know that $f_{n_0} \in V$. Therefore, since V is a vector space, $f_{n_0} - (f_{n_0} - f) = f \in V$.

We still need to show convergence in norm. Let $\varepsilon > 0$. Then there exists an n_0 such that for every $n > n_0$ we have $|f_n(x) - f(x)| < \varepsilon$. Therefore for every $n > n_0$:

$$\int_0^\infty |f_n(x) - f(x)|^2 e^{-x} dx \le \varepsilon^2 \int_0^\infty e^{-x} dx = \varepsilon^2.$$

This means that $\|f_n - f\| \underset{n\to\infty}{\longrightarrow} 0$ as n tends to infinity, and therefore we have convergence in norm, as required.

(b) Define:

$$f_n(x) = \begin{cases} 0, & 0 \le x < n, \\ x - n, & n \le x < n+1, \\ 1, & n+1 \le x. \end{cases}$$

It is not hard to verify that this sequence meets the requirements.

(c) Define:

$$f_n(x) = \begin{cases} n\sqrt{x} e^{x/2}, & 0 \le x < \frac{1}{n}, \\ n\sqrt{\left(\frac{2}{n} - x\right)} e^{x/2}, & \frac{1}{n} \le x < \frac{2}{n}, \\ 0, & \frac{2}{n} \le x. \end{cases}$$

Pointwise convergence to the zero function is easy to verify. Also $\|f_n\|^2 = 1$, and therefore the sequence does not converge in norm.
Remark: the reason behind choosing this specific sequence is that, in the integral defining the norm, the graph of the integrand is now a triangle with base of length $\frac{2}{n}$ and height n, and therefore the area under it remains constant.

32.

(a) True. By Bessel's inequality $\sum_{n=1}^{\infty} |\langle u, e_n\rangle|^2 \le \|u\|^2 < \infty$, and therefore $\lim_{n\to\infty} |\langle u, e_n\rangle|^2 = 0$. Hence also $\lim_{n\to\infty} \langle u, e_n\rangle = 0$.

(b) True; see the proof of the previous part.

(c) False. Let $V = \mathbf{R}^2$, $n = 1$, $e_1 = (2, 0)$, and $u = (1, 1)$. Then

$$\tilde{u} = \frac{\langle u, e_1\rangle}{\|e_1\|} e_1 = \frac{2}{2}(2, 0) = (2, 0),$$

but the best approximation of u in W is $(1, 0)$. If the division was by $\|e_n\|^2$ instead of $\|e_n\|$, it would have worked.

(d) True. Since the norm of each basis vector is 1, this definition coincides with the formula for best approximation we have shown in class.

(e) True, it follows from uniqueness of the best approximation in inner-product spaces.

(f) True, it does not matter that the spanning set is not orthogonal. In inner-product spaces, the best approximation and the orthogonal projection are the same.

(g) True, follows from the previous part.

33. Put:

$$f_n(x) = \begin{cases} \frac{1}{n}, & 0 \leq x < n^2, \\ 0, & n^2 < x. \end{cases}$$

Clearly, $f_n \underset{n\to\infty}{\longrightarrow} 0$ uniformly on $\mathbf{R}$. However:

$$\|f_n\|_2 = \sqrt{\int_0^\infty |f_n(x)|^2 \, dx} = \sqrt{\int_0^{n^2} \frac{1}{n^2} dx} = 1.$$

Therefore, f_n does not converge in norm.

35. Let

$$f_n(x) = \begin{cases} 1, & x \in I_n, \\ 0, & x \notin I_n, \end{cases}$$

where $I_1 = [0,1]$, $I_2 = \left[0, \frac{1}{2}\right]$, $I_3 = \left[\frac{1}{2}, \frac{2}{2}\right]$, $I_4 = \left[0, \frac{1}{4}\right]$, $I_5 = \left[\frac{1}{4}, \frac{2}{4}\right]$, $I_6 = \left[\frac{2}{4}, \frac{3}{4}\right]$, $I_7 = \left[\frac{3}{4}, \frac{4}{4}\right]$, $I_8 = \left[0, \frac{1}{8}\right]$, and so on.
(Formally,

$$I_n = \left[\frac{n - 2^{\lfloor \log n \rfloor}}{2^{\lfloor \log n \rfloor}}, \frac{n - 2^{\lfloor \log n \rfloor} + 1}{2^{\lfloor \log n \rfloor}}\right]$$

which means that we cover the interval $[0,1]$ with closed intervals of length $\frac{1}{2^k}$ with minimal overlapping.)

7 Fourier Series

36.

(a) This function is already given in the form of its real Fourier series.

(b) We have:

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \\ &= \frac{1}{\pi} \left(\int_{-\pi}^{0} \cos x \cos nx dx + \int_0^{\pi} \sin x \cos nx dx \right) \end{aligned}$$

For convenience, we deal with each of these integrals separately, using trigonometric identities:
For $n > 1$:

$$\int_{-\pi}^{0} \cos x \cos nx dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos x \cos nx dx = 0$$

since $\cos x$, $\cos nx$ are orthogonal. Throughout this exercise we use the fact that some integrands are even to expand the integral to a symmetrical interval and use the fact that the integral can be interpreted as an inner-product of functions from an orthonormal set. For $n = 1$:

$$\int_{-\pi}^{0} \cos x \cos x dx = \frac{1}{2} \int_{-\pi}^{0} (1 + \cos 2x)\, dx = \frac{\pi}{2}.$$

For $n = 0$:

$$\int_{-\pi}^{0} \cos x \cos nx dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos x dx = 0.$$

Now the other integral: For $n > 1$:

$$\begin{aligned}
\int_{0}^{\pi} \sin x \cos nx dx &= \frac{1}{2} \left(\int_{0}^{\pi} \sin (n+1)\, x dx - \int_{0}^{\pi} \sin (n-1)\, x dx \right) \\
&= \frac{1}{2} \left(-\frac{1}{n+1} \cos (n+1)\, x|_{0}^{\pi} + \frac{1}{n-1} \cos (n-1)\, x|_{0}^{\pi} \right) \\
&= \frac{1}{2} \left(\frac{1}{n-1} (\cos (n-1)\, \pi - 1) - \frac{1}{n+1} (\cos (n+1)\, \pi - 1) \right) \\
&= \frac{1}{2} \left((-1)^{n+1} - 1 \right) \frac{2}{n^2 - 1} \\
&= -\frac{(-1)^n + 1}{n^2 - 1}.
\end{aligned}$$

For $n = 1$:

$$\int_{0}^{\pi} \sin x \cos x dx = \frac{1}{2} \int_{0}^{\pi} \sin 2x dx = 0.$$

For $n = 0$:

$$\int_{0}^{\pi} \sin x dx = -\cos \pi + \cos 0 = 2.$$

Collecting everything together we have:

$$a_n = \begin{cases} \frac{2}{\pi}, & n = 0, \\ \frac{1}{2}, & n = 1, \\ -\frac{(-1)^n + 1}{\pi (n^2 - 1)}, & n > 1. \end{cases}$$

Now we calculate b_n:

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \\ &= \frac{1}{\pi}\left(\int_{-\pi}^{0}\cos x\sin nx dx + \int_{0}^{\pi}\sin x\sin nx dx\right) \end{aligned}$$

Again we solve the integrals separately:
For $n > 1$:

$$\begin{aligned} \int_{-\pi}^{0}\cos x\sin nx dx &= \frac{1}{2}\left(\int_{-\pi}^{0}\sin(n+1)x dx + \int_{-\pi}^{0}\sin(n-1)x dx\right) \\ &= \frac{1}{2}\left(-\frac{1}{n+1}\cos(n+1)x|_{-\pi}^{0} - \frac{1}{n-1}\cos(n-1)x|_{-\pi}^{0}\right) \\ &= \frac{1}{2}\left(-\frac{1}{n+1}(1-\cos(n+1)\pi) - \frac{1}{n-1}(1-\cos(n-1)\pi)\right) \\ &= \frac{1}{2}\left(-\frac{1}{n+1}(1-\cos(n+1)\pi) - \frac{1}{n-1}(1-\cos(n+1)\pi)\right) \\ &= -\frac{1+(-1)^n}{2}\frac{2n}{n^2-1} \\ &= -\frac{(1+(-1)^n)n}{n^2-1}. \end{aligned}$$

For $n = 1$:

$$\int_{-\pi}^{0}\cos x\sin x dx = 0.$$

Now to the other integral:
For $n > 1$:

$$\int_{0}^{\pi}\sin x\sin nx dx = \frac{1}{2}\int_{-\pi}^{\pi}\sin x\sin nx dx = 0,$$

since $\sin x, \sin nx$ are orthogonal for $n > 1$.
For $n = 1$:

$$\int_{0}^{\pi}\sin x\sin x dx = \frac{1}{2}\int_{-\pi}^{\pi}\sin x\sin x dx = \frac{\pi}{2}.$$

We combine the results:

$$b_n = \begin{cases} \frac{1}{2}, & n = 1, \\ -\frac{(1+(-1)^n)n}{\pi(n^2-1)}, & n > 1. \end{cases}$$

The Fourier series is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx).$$

38.

(a) We have

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx} dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin\frac{x}{2} e^{-inx} dx \\
&= \frac{1}{4i\pi}\int_{-\pi}^{\pi} \left(e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}\right) e^{-inx} dx \\
&= \frac{1}{4i\pi}\int_{-\pi}^{\pi} \left(e^{i\frac{x}{2}-inx} - e^{-i\frac{x}{2}-inx}\right) dx \\
&= \frac{1}{4i\pi}\int_{-\pi}^{\pi} \left(e^{i\left(\frac{1}{2}-n\right)x} - e^{-i\left(\frac{1}{2}+n\right)x}\right) dx \\
&= \frac{1}{4i\pi}\left(\frac{1}{i\left(\frac{1}{2}-n\right)} e^{i\left(\frac{1}{2}-n\right)x}|_{-\pi}^{\pi} + \frac{1}{i\left(\frac{1}{2}+n\right)} e^{-i\left(\frac{1}{2}+n\right)x}|_{-\pi}^{\pi}\right) \\
&= \frac{1}{4\pi}\left(\frac{2}{2n-1}\left(e^{i\left(\frac{1}{2}-n\right)\pi} - e^{-i\left(\frac{1}{2}-n\right)\pi}\right) - \frac{2}{2n+1}\left(e^{-i\left(\frac{1}{2}+n\right)\pi} - e^{i\left(\frac{1}{2}+n\right)\pi}\right)\right) \\
&= \frac{i}{\pi}\left(\frac{1}{2n-1}\sin\left(\frac{1}{2}-n\right)\pi + \frac{1}{2n+1}\sin\left(\frac{1}{2}+n\right)\pi\right) \\
&= \frac{i}{\pi}\left(\frac{1}{2n-1}\cos n\pi + \frac{1}{2n+1}\cos n\pi\right) \\
&= \frac{i}{\pi}\frac{4n}{4n^2-1}(-1)^n,
\end{aligned}$$

and the Fourier series is $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$.

(b) We first calculate the following integral for $n \neq 0$:

$$
\begin{aligned}
\int_{-\pi}^{\pi} x^2 e^{-inx} dx &= -\frac{1}{in} \int_{-\pi}^{\pi} x^2 d\left(e^{-inx}\right) \\
&= -\frac{1}{in} \left(x^2 e^{-inx} |_{-\pi}^{\pi} - 2 \int_{-\pi}^{\pi} x e^{-inx} dx \right) \\
&= -\frac{1}{in} \left(\pi^2 e^{-in\pi} - \pi^2 e^{in\pi} - 2 \int_{-\pi}^{\pi} x e^{-inx} dx \right) \\
&= -\frac{1}{in} \left(-\pi^2 + \pi^2 - 2 \int_{-\pi}^{\pi} x e^{-inx} dx \right) \\
&= -\frac{2i}{n} \int_{-\pi}^{\pi} x e^{-inx} dx \\
&= \frac{2}{n^2} \int_{-\pi}^{\pi} x d\left(e^{-inx}\right) \\
&= \frac{2}{n^2} \left(x e^{-inx} |_{-\pi}^{\pi} - \int_{-\pi}^{\pi} e^{-inx} \right) \\
&= \frac{2}{n^2} \left(\pi e^{-in\pi} + \pi e^{in\pi} - 0 \right) \\
&= \frac{4\pi}{n^2} (-1)^n .
\end{aligned}
$$

Therefore for $n \neq 0$

$$
c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\pi - x^2\right) e^{-inx} dx = \frac{2}{n^2} (-1)^{n+1} ,
$$

and for $n = 0$:

$$
\begin{aligned}
c_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\pi - x^2\right) dx \\
&= \frac{1}{2\pi} \left(2\pi^2 - \frac{x^3}{3} |_{-\pi}^{\pi} \right) \\
&= \pi - \frac{1}{2\pi} \left(2 \frac{\pi^3}{3} \right) \\
&= \pi - \frac{1}{3}\pi^2 .
\end{aligned}
$$

47.

(a) First we calculate the Fourier coefficients. Since $f(x)$ is an even function, $b_n = 0$ for every $n \in \mathbf{N}$.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos ax dx \\ &= \frac{1}{a\pi}\sin ax|_{-\pi}^{\pi} \\ &= \frac{2\sin a\pi}{a\pi}. \end{aligned}$$

For $n > 0$:

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos ax \cos nx dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\cos(n+a)x + \cos(n-a)x)\,dx \\ &= \frac{1}{2\pi}\left(\frac{1}{n+a}2\sin(n+a)\pi + \frac{1}{n-a}2\sin(n-a)\pi\right) \\ &= \frac{1}{\pi}\left(\frac{1}{n+a}(-1)^n\sin a\pi - \frac{1}{n-a}(-1)^n\sin a\pi\right) \\ &= \sin(a\pi)(-1)^n\left(\frac{1}{a\pi+n\pi} + \frac{1}{a\pi-n\pi}\right). \end{aligned}$$

Since the function meets the conditions of Dirichlet's theorem, the Fourier series converges at $x = 0$ to $\cos(a\cdot 0) = 1$:

$$\begin{aligned} 1 &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\cdot 0 \\ &= \frac{\sin a\pi}{a\pi} + \sin a\pi \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{a\pi+n\pi} + \frac{1}{a\pi-n\pi}\right). \end{aligned}$$

Dividing by $\sin a\pi$ we get

$$\frac{1}{\sin a\pi} = \frac{1}{a\pi} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{a\pi+n\pi} + \frac{1}{a\pi-n\pi}\right),$$

as required.

(b) Since $f(\pi) = f(-\pi)$, and all other conditions of Dirichlet's theorem are met, the Fourier series converges to f on $x = \pi$ as well. Therefore:

$$\begin{aligned}
\cos a\pi &= \frac{\sin a\pi}{a\pi} + \sin a\pi \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{a\pi + n\pi} + \frac{1}{a\pi - n\pi} \right) \cos \pi n \\
&= \frac{\sin a\pi}{a\pi} + \sin a\pi \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{a\pi + n\pi} + \frac{1}{a\pi - n\pi} \right) (-1)^n \\
&= \frac{\sin a\pi}{a\pi} + \sin a\pi \sum_{n=1}^{\infty} \left(\frac{1}{a\pi + n\pi} + \frac{1}{a\pi - n\pi} \right).
\end{aligned}$$

Once again, dividing by $\sin a\pi$ we get

$$\frac{\cos a\pi}{\sin a\pi} = \cot a\pi = \frac{1}{a\pi} + \sum_{n=1}^{\infty} \left(\frac{1}{a\pi + n\pi} + \frac{1}{a\pi - n\pi} \right),$$

as required.