Final #1

Mark the correct answer in each of the following questions.

1. Let V be the vector space of all real-valued continuous functions over the interval $[-1, 1]$, let v be the constant function 1 and $W = \text{span}\{w\}$, where $w = x$. Denote by w_1^*, w_2^*, and w_∞^* a vector in W that is closest to v within W, according to the norms $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$, respectively.

   (a) w_1^* and w_2^* are uniquely determined and are equal. However, there are infinitely many equally good choices for w_∞^*.

   (b) w_1^* and w_2^* are uniquely determined and distinct. However, there are infinitely many equally good choices for w_∞^*.

   (c) w_2^* and w_∞^* are uniquely determined and are equal. However, there are infinitely many equally good choices for w_1^*.

   (d) w_2^* and w_∞^* are uniquely determined and distinct. However, there are infinitely many equally good choices for w_1^*.

   (e) None of the above.

2. Let V be a vector space and $\|\cdot\|$ a norm on V. Define (for the sake of this question only) a set of vectors $\{v_n\}_{n=1}^{\infty}$ as *generating* if for every vector $v \in V$ and $\varepsilon > 0$ there exist a positive integer n and complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\|v - (\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n)\| < \varepsilon$.

   Now consider the spaces ℓ_1, ℓ_2, and ℓ_∞ of all sequences $\mathbf{x} = (x_1, x_2, \ldots)$ of complex numbers, satisfying the condition $\sum_{n=1}^{\infty} |x_n| < \infty$, the condition $\sum_{n=1}^{\infty} |x_n|^2 < \infty$, and the condition $\sup_{1 \le n < \infty} |x_n| < \infty$, respectively.

Define vectors $e_n,\ n = 1, 2, \ldots,$ as follows. For each n, all entries of e_n are 0, except for the n-th entry, which is 1. Note that each e_n resides in all three spaces $\ell_1,\ \ell_2$, and ℓ_∞.

(a) The set $\{e_n\}_{n=1}^{\infty}$ is generating in all three spaces $\ell_1,\ \ell_2$, and ℓ_∞.

(b) The set $\{e_n\}_{n=1}^{\infty}$ is generating in ℓ_1 and ℓ_2, but not in ℓ_∞.

(c) The set $\{e_n\}_{n=1}^{\infty}$ is generating in ℓ_2 and ℓ_∞, but not in ℓ_1.

(d) The set $\{e_n\}_{n=1}^{\infty}$ is generating in ℓ_2, but neither in ℓ_1 nor in ℓ_∞.

(e) None of the above.

3. Let f be a complex-valued function, defined on the interval $[0, \pi]$ only. Suppose $f \in L^2_{\mathrm{PC}}[0, \pi]$.

   (a) There does not necessarily exist a Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converging to f in $\|\cdot\|_2$-norm (namely, such that the partial sums S_n satisfy $\int_0^\pi |f(x) - S_n(x)|^2\, dx \underset{n\to\infty}{\longrightarrow} 0$).

   (b) There exists exactly one Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converging to f in $\|\cdot\|_2$-norm.

   (c) There exist infinitely many Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converging to f in $\|\cdot\|_2$-norm. However, there does not necessarily exist a series of cosines $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ with this property, nor does there necessarily exist a series of sines $\sum_{n=1}^{\infty} b_n \sin nx$ with this property.

   (d) There exist infinitely many Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converging to f in $\|\cdot\|_2$-norm. Moreover, there exist a unique series of cosines $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ with this property and a unique series of sines $\sum_{n=1}^{\infty} b_n \sin nx$ with this property.

   (e) None of the above.

4. Given a function $f \in L^2_{\mathrm{PC}}[-\pi, \pi]$, we have considered in class the sequence of functions $(S_n)_{n=1}^{\infty}$ of partial sums of its Fourier series. We have also considered the sequence $(\sigma_n)_{n=1}^{\infty}$, defined by:

$$\sigma_n(x) = \frac{S_0(x) + S_1(x) + \ldots + S_n(x)}{n+1}, \qquad -\pi \le x \le \pi.$$

Here we consider an additional sequence, $(\tau_n)_{n=1}^{\infty}$, defined by:

$$\tau_n(x) = \frac{\sigma_0(x) + \sigma_1(x) + \ldots + \sigma_n(x)}{n+1}, \qquad -\pi \le x \le \pi.$$

(a) The sequence (τ_n) has strictly better convergence properties than (σ_n) in the following sense: If $\sigma_n(x) \underset{n\to\infty}{\longrightarrow} f(x)$ uniformly on $[-\pi, \pi]$, then $\tau_n(x) \underset{n\to\infty}{\longrightarrow} f(x)$ uniformly as well. However, the converse is not true in general.

(b) $\sigma_n(x) \underset{n\to\infty}{\longrightarrow} f(x)$ uniformly on $[-\pi, \pi]$ if and only if $\tau_n(x) \underset{n\to\infty}{\longrightarrow} f(x)$ uniformly on $[-\pi, \pi]$.

(c) If $S_n(x) \underset{n\to\infty}{\longrightarrow} f(x)$ uniformly on $[-\pi, \pi]$, then $\sigma_n(x) \underset{n\to\infty}{\longrightarrow} f(x)$ uniformly on $[-\pi, \pi]$. However, if $S_n(x) \underset{n\to\infty}{\longrightarrow} f(x)$ only pointwise, it is not necessarily the case that $\sigma_n(x) \underset{n\to\infty}{\longrightarrow} f(x)$ pointwise.

(d) It is possible that $S_n(x) \underset{n\to\infty}{\longrightarrow} f(x)$ pointwise, but the convergence $\tau_n(x) \underset{n\to\infty}{\longrightarrow} f(x)$ does not hold pointwise.

(e) None of the above.

5. The function $f \in L^2_{\mathrm{PC}}[-\pi, \pi]$ is defined by:

$$f(x) = e^{\frac{\sin x}{|x|}}.$$

(Note that the right-hand side is undefined for $x = 0$, but, due to the definition of L^2_{PC}, this is irrelevant.) The Fourier series of f converges at the point 0 to:

(a) $\sinh 1$.

(b) $\cosh 1$.

(c) $\tanh 1$.

(d) $\coth 1$.

(e) None of the above.

Reminder: The hyperbolic functions are defined by:

$$\sinh x = \tfrac{e^x - e^{-x}}{2}, \quad \cosh x = \tfrac{e^x + e^{-x}}{2},$$

$$\tanh = \tfrac{\sinh x}{\cosh x}, \qquad \coth x = \tfrac{\cosh x}{\sinh x}.$$

6. Consider the family of functions $\{f_a : a \in \mathbf{C}\}$, defined by:

$$f_a(x) = e^{ax}, \qquad -\pi \leq x \leq \pi.$$

(a) For every complex number a, the Fourier series of f_a converges uniformly on $[-\pi, \pi]$ to f_a.

(b) There exist infinitely many complex numbers a for which the Fourier series of f_a converges uniformly on $[-\pi, \pi]$ to f_a. However, there also exist infinitely many complex numbers a for which the Fourier series of f_a does not converge uniformly on $[-\pi, \pi]$ to f_a.

(c) There exists a unique complex number a for which the Fourier series of f_a converges uniformly on $[-\pi, \pi]$ to f_a.

(d) For no complex number a does the Fourier series of f_a converge uniformly on $[-\pi, \pi]$ to f_a.

(e) None of the above.

7. For each $n = 1, 2, 3, \ldots$, define the function $f_n : [0, 1] \to \mathbf{C}$ by:

$$f_n(x) = \begin{cases} 1, & \tfrac{1}{n+1} < x \leq \tfrac{1}{n}, \\ 0, & \text{otherwise.} \end{cases}$$

Next define functions $g_n : [0, 1] \to \mathbf{C}$ by:

$$g_n(x) = f_1(x) + \frac{1}{2}f_2(x) + \frac{1}{3}f_3(x) + \ldots + \frac{1}{n}f_n(x), \qquad n = 1, 2, 3, \ldots.$$

(a) The sequence $(g_n)_{n=1}^{\infty}$ converges uniformly to some function $g \in L^2_{\mathrm{PC}}[0, 1]$.

(b) The sequence $(g_n)_{n=1}^{\infty}$ converges uniformly to some function g, that does not necessarily belong to $L^2_{\text{PC}}[0,1]$. The sequence is a Cauchy sequence with respect to $\|\cdot\|_2$, but does not converge in $L^2_{\text{PC}}[0,1]$.

(c) The sequence $(g_n)_{n=1}^{\infty}$ converges pointwise on $[0,1]$, but not uniformly. It is a Cauchy sequence with respect to $\|\cdot\|_2$, but does not converge in $L^2_{\text{PC}}[0,1]$.

(d) The sequence $(g_n)_{n=1}^{\infty}$ does not converge pointwise on $[0,1]$, neither is it a Cauchy sequence with respect to $\|\cdot\|_2$.

(e) None of the above.

8. We have claimed in class that the transform of the function f, defined by

$$f(x) = \begin{cases} 1, & a \le x \le b, \\ 0, & \text{otherwise}, \end{cases}$$

is given by

$$\hat{f}(\omega) = \frac{e^{-i\omega a} - e^{-i\omega b}}{2\pi i\omega}.$$

Now, the right-hand side of this formula is clearly undefined for $\omega = 0$. The value of $\hat{f}(0)$:

(a) exists, and is a real number, depending in a non-trivial way on a and b. Moreover $\hat{f}$ is continuous at the point 0.

(b) exists, and is 0 for every choice of a and b. Moreover $\hat{f}$ is continuous at the point 0.

(c) exists, but $\hat{f}$ is not continuous at the point 0.

(d) is undefined.

(e) None of the above.

9. Let f be the function given by:

$$f(x) = e^{-x^4},$$

and let $y = \hat{f}$. Then the function y satisfies the differential equation:

(a)

$$y' = \frac{\omega}{2} y.$$

(b)

$$y'' = \frac{\omega}{3} y.$$

(c)

$$y''' = \frac{\omega}{4} y.$$

(d)

$$y^{(4)} = \frac{\omega}{5} y.$$

(e) None of the above.

Solutions

1. The function v is even, while w is odd, and hence they are orthogonal, which yields $w_2^* = 0$.

   For every α we have

$$\|v - \alpha w\|_\infty \geq (v - \alpha w)(0) = 1.$$

   Now, if $\alpha = 0$, then clearly $\|v - \alpha w\|_\infty = 1$, but for $\alpha \neq 0$ we have:

$$\|v - \alpha w\|_\infty \geq \max\{(v - \alpha w)(1), (v - \alpha w)(-1)\} = 1 + |\alpha| > 1.$$

   It follows that $w_\infty^* = 0$.

   We turn to find w_1^*. For every α:

$$\|v - \alpha w\|_1 = \int_{-1}^{1} |1 - \alpha x| dx \geq \left| \int_{-1}^{1} (1 - \alpha x) dx \right| = 2.$$

   We have equality here if and only if the expression $1 - \alpha x$ has the same sign throughout the interval $[-1, 1]$, which is the case if and only if

$-1 \leq \alpha \leq 1$. Hence, the vector αx is a possible choice for w_1^* for every $-1 \leq \alpha \leq 1$.

Thus, (c) is true.

2. First we show that the set is generating in ℓ_1. Indeed, let $\mathbf{x} = (x_1, x_2, \ldots) \in \ell_1$. Then:
$$\left\| \mathbf{x} - \sum_{i=1}^{n} x_k e_k \right\|_1 = \sum_{j=n+1}^{\infty} |x_k| \underset{n\to\infty}{\longrightarrow} 0.$$
The analogous claim for ℓ_2 is similarly proved.

   Next we show that the set is not generating in ℓ_∞. Let $v = (1, 1, \ldots) \in \ell_\infty$. Obviously, for any $\alpha_1, \alpha_2, \ldots, \alpha_n$, all entries of the vector $v - \sum_{k=1}^{n} \alpha_k e_k$ from the $(n+1)$-st place on are still 1, and therefore :
$$\left\| v - \sum_{i=1}^{n} \alpha_k e_k \right\|_\infty \geq 1.$$

Thus, (b) is true.

3. The main observation is that, for every extension of f to a piecewise continuous function $\tilde{f}$ on $[-\pi, \pi]$, there exists a unique Fourier series converging to $\tilde{f}$ in $\|\cdot\|_2$ in $L^2_{\mathrm{PC}}[-\pi, \pi]$. All Fourier series obtained this way certainly converge to f in $\|\cdot\|_2$ in $L^2_{\mathrm{PC}}[0, \pi]$.

   A series of cosines converging to f on $[0, \pi]$ converges to an even function on $[-\pi, \pi]$, and therefore is the Fourier series of the function obtained from f by requiring its extension to be even, namely $\tilde{f}(x) = f(|x|)$ for $x \in [-\pi, \pi]$. Similarly, a series of sines converging to f on $[0, \pi]$ converges to an odd function on $[-\pi, \pi]$, and therefore is the Fourier series of the function obtained from f by requiring it to be odd, namely $\tilde{f}(x) = \mathrm{sgn}(x) \cdot f(|x|)$ for $x \in [-\pi, \pi]$. Hence there are a unique series of cosines and a unique series of sines with the required property.

Thus, (d) is true.

4. As we have seen, averaging improves convergence. Thus, if $S_n \underset{n\to\infty}{\longrightarrow} f$ in some sense (pointwise or uniformly), then $\sigma_n \underset{n\to\infty}{\longrightarrow} f$ in the same sense. Similarly, if $\sigma_n \underset{n\to\infty}{\longrightarrow} f$, then $\tau_n \underset{n\to\infty}{\longrightarrow} f$. If $\tau_n \underset{n\to\infty}{\longrightarrow} f$ uniformly, then f is the uniform limit of a sequence of continuous functions, and is therefore continuous. By Féjer's Theorem, this implies that $\sigma_n \underset{n\to\infty}{\longrightarrow} f$ uniformly.

   Thus, (b) is true.

5. Since $\lim_{x\to 0} \sin x/x = 1$, we have $\lim_{x\to 0+} \sin x/|x| = 1$ and $\lim_{x\to 0-} \sin x/|x| = -1$. Hence the Fourier series of f converges at 0 to

$$\frac{f(0+) + f(0-)}{2} = \frac{e^1 + e^{-1}}{2} = \cosh 1.$$

   Thus, (b) is true.

6. For every $a \in \mathbf{C}$, the function f_a (after extending it to a periodic function on $\mathbf{R}$) is continuous and piecewise continuously differentiable on $[-\pi, \pi]$, with the possible exception of the point π. At the point π it is continuous if and only if $f_a(-\pi) = f_a(\pi)$, which is the case if and only if $e^{2a\pi} = 1$, namely a is an integer multiple of i.

   Thus, (b) is true.

7. We start by observing that the intervals where the functions f_n, $n = 1, 2, \ldots$, do not vanish are pairwise disjoint. Hence, for each $x \in [0, 1]$, the sequence $(g_n(x))_{n=1}^{\infty}$ is eventually constant, and in particular convergent. In fact, this shows that the sequence (g_n) converges pointwise as $n \to \infty$ to the function g, given by

$$g(x) = \begin{cases} \frac{1}{n}, & \frac{1}{n+1} < x \leq \frac{1}{n},\ n = 1, 2, \ldots, \\ 0, & x = 0. \end{cases}$$

Alternatively:

$$g(x) = \begin{cases} 0, & x = 0, \\ \left(\left\lfloor \frac{1}{x}, \right\rfloor\right)^{-1} & x > 0. \end{cases}$$

As all points $1/n$ are discontinuity points of g, the function does not belong to $L^2_{\mathrm{PC}}[0, 1]$.

We claim that the convergence is uniform. Since, for each n, the function $|g - g_n|$ is bounded above by $\frac{1}{n+1}$ throughout $[0, 1]$, the convergence $g_n \underset{n\to\infty}{\longrightarrow} g$ is uniform. In particular, (g_n) is a Cauchy sequence in $\|\cdot\|_\infty$, and hence in $\|\cdot\|_2$ as well.

Thus, (b) is true.

8. Since the function f belongs to $L^1_{\mathrm{PC}}(-\infty, \infty)$, the transform is well-defined on the whole of $\mathbf{R}$ and is continuous. Its value at 0 is calculated directly from the definition:

$$\begin{aligned} \hat{f}(0) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{i0x} dx \\ &= \frac{1}{2\pi} \int_a^b dx \\ &= \frac{b-a}{2\pi}. \end{aligned}$$

Alternatively, since $\hat{f}$ is known to be continuous, one could also calculate $\hat{f}(0)$ by finding the limit of the expression $\frac{e^{-i\omega a} - e^{-i\omega b}}{2\pi i \omega}$ as $\omega \to 0$.

Thus, (a) is true.

9. We have $f'(x) = -4x^3 e^{-x^4}$. Now:

$$\begin{aligned} y^{(3)}(\omega) &= \frac{1}{i^3}\widehat{x^3 f(x)}(\omega) \\ &= -\frac{1}{4} i \cdot (-\widehat{4x^3 f(x)})(\omega) \\ &= -\frac{1}{4} i \widehat{f'}(\omega) \\ &= -\frac{1}{4} i \cdot i\omega \hat{f}(\omega) \\ &= \frac{\omega}{4} y(\omega), \end{aligned}$$

Thus, (c) is true.