Final #1

Mark the correct answer in each part of the following questions.

1. In this question we consider permutations σ of $\{1, 2, \ldots, n\}$, having the property $\sigma^{-1}(100) < \sigma^{-1}(50)$ (i.e., the number 100 appears in σ before the number 50).

   (a) If we go over all permutations of $\{1, 2, \ldots, n\}$ according to the lexicographic order, then the number of permutations we get, until we get for the first time a permutation of that type, is

   (i) $(n-49)! \cdot (1 + o(1))$.
   (ii) $(n-50)! \cdot (1 + o(1))$.
   (iii) $(n-99)! \cdot (1 + o(1))$.
   (iv) $(n-100)! \cdot (1 + o(1))$.
   (v) none of the above.

   (b) If we go over all permutations of $\{1, 2, \ldots, n\}$ with minimal changes (according to the order we have seen in class), then the number of permutations we get, until we get for the first time a permutation of that type, is

   (i) $49 \cdot (n-100)! + O(1)$.
   (ii) $50 \cdot (n-100)! + O(1)$.
   (iii) $49 \cdot \frac{n!}{100!} + O(1)$.
   (iv) $50 \cdot \frac{n!}{100!} + O(1)$.
   (v) none of the above.

   (c) We would like to select a uniformly random permutation having the property in question. Two methods have been proposed:

- Method A:
  a) Select a random permutation of $\{1, 2, \ldots, n-2\}$.
  b) Add 1 to all numbers from 50 on.
  c) Add 1 to all numbers from 100 on.
  d) Select a uniformly random integer k in the range $[1, n-1]$.
  e) If $k \leq n-2$, move the number at the k-th place to the $(n-1)$-st place.
  f) Place 50 at the k-th place.
  g) Select a uniformly random integer m in the range $[1, k]$.
  h) Move the number at the m-th place to the n-th place.
  i) Place 100 at the m-th place.
- Method B:
  Select a uniformly random permutation of $\{1, 2, \ldots, n\}$ until you obtain a permutation having the property in question.

(i) Both methods select a permutation having the property in question, but only the second selects it according to the required probability. Method A requires on the average linear time and method B requires $\Theta(n^2)$.

(ii) Both methods select a permutation having the property in question, but only the second selects it according to the required probability. Both require on the average linear time.

(iii) Both methods select a permutation as required, but only A requires on the average linear time, while B requires $\Theta(n^2)$.

(iv) Both methods select a permutation as required in average linear time.

(v) None of the above.

(d) Now we want to select a random permutation out of the set of all permutations, but not uniformly randomly; rather, the probability of permutations having the property in question should be twice as large as that of the other permutations. Two methods have been proposed:

- Method A: Select a uniformly random permutation. If it has the property in question, take it. Otherwise – interchange the numbers 50 and 100 with probability α.

- Method B: Select a uniformly random permutation. If it has the property in question, take it. Otherwise – with probability β take it, and with probability $1-\beta$ reject it and repeat the process.

(i) Method A selects a permutation according to the requirements if $\alpha = 1/3$. Method B does so if $\beta = 1/2$.

(ii) Method A selects a permutation according to the requirements if $\alpha = 1/3$. Method B does so if $\beta = 2/3$.

(iii) Method A selects a permutation according to the requirements if $\alpha = 1/2$. Method B does so if $\beta = 1/2$.

(iv) Method A selects a permutation according to the requirements if $\alpha = 1/2$. Method B does so if $\beta = 2/3$.

(v) None of the above.

2. Consider the algorithm presented in class for traversing the set of all partitions of $\{1, 2, \ldots, n\}$. Denote by a_n the number of partitions.

   When proceeding from partition to the next, the number of sets in the partition may increase, decrease or remain the same. The number of times (out of $a_n - 1$) it remains the same is

   (i) $a_n - a_{n-1} + a_{n-2} + O(1)$.

   (ii) $a_n - 2a_{n-1} + a_{n-2} + O(1)$.

   (iii) $a_n - a_{n-1} + 2a_{n-2} + O(1)$.

   (iv) $a_n - 2a_{n-1} + 2a_{n-2} + O(1)$.

   (v) none of the above.

3. In this question we consider trees over n labeled vertices, where n is even, and in which $n/2+1$ vertices are leaves, while the others are of degree 3.

   We would like to go over this family of trees with changes as small as possible. More formally, define:

- A *minimal change* in a tree consists of removing an edge of the form (i,j) and adding one of the four edges $(i+1,j),(i-1,j),(i,j+1),(i,j-1)$ in its stead (such that we still get a tree).
- A *tiny change* in a tree consists of removing an edge of the form (i,j) and adding an edge of the form (i,k) or (k,j) in its stead (such that we still get a tree).
- A *small change* in a tree consists of removing an edge (i,j) and adding any edge (k,l) in its stead (such that we still get a tree).

(i) It is possible to go over the family of trees above with minimal changes.

(ii) It is possible to go over the family of trees above with tiny changes, but not with minimal changes.

(iii) It is possible to go over the family of trees above with small changes, but not with tiny changes.

(iv) It is impossible to go over the family of trees above with small changes.

(v) none of the above.

4. Consider the set of rectangular Young tableaux, with an arbitrary fixed number of rows and two columns. Denote by $Y_{m,2}$ the number of such tableaux with m rows.

   (a) $Y_{m,2} =$

      (i) 2^n.

      (ii) $\binom{2n}{n}$.

      (iii) $\frac{\binom{2n}{n}}{n+1}$.

      (iv) 2^{2n-2}.

      (v) none of the above.

   (b) Consider the algorithm presented in class for traversing the set of all Young tableaux of a given shape, applied to our case with n rows. Denote by c_2 the number of times in which the location of the number 2 in the table changes, and by c_{2n-1} the number of times in which the location of the number $2n-1$ changes. Then:

(i) $c_2 = 2Y_{n-1,2} + O(1), \qquad c_{2n-1} = 1.$
(ii) $c_2 = 4Y_{n-1,2} + O(1), \qquad c_{2n-1} = 1.$
(iii) $c_2 = 2Y_{n-1,2} + O(1), \qquad c_{2n-1} = 2.$
(iv) $c_2 = 4Y_{n-1,2} + O(1), \qquad c_{2n-1} = 2.$
(v) None of the above.

Solutions

1. (a) First, we identify the first permutation satisfying the condition is question. Since a permutation appears earlier than another if and only if it has a lower number than the other at the first place they differ, we look for permutations with small numbers in the beginning. Obviously, the first 49 entries of the required permutation need to be $1, 2, \dots, 49$, in this order. Now the smallest number we can have is 51, and then $52, 53, \dots, 99$. At this point, the smallest number we can have is 100, then 50, and then $101, 102, \dots, n$. In conclusion, the permutation we are looking for is

$$(1, 2, \dots, 49, 51, 52, \dots, 99, 100, 50, 101, 102, \dots, n).$$

Now we count the steps until getting to this permutation when starting from the identity permutation. To replace the 50 at the 50-th place by 51, we need to go over all permutations of the last $n - 50$ elements $51, 52, \dots, n$, which takes $(n - 50)!$ steps. At this point we have the permutation

$$(1, 2, \dots, 49, 51, 50, 52, \dots, n).$$

Similarly, we need $(n-51)!$ steps to interchange 50 and 52, another $(n - 52)!$ steps to interchange 50 and 53, ..., $(n - 99)!$ steps to interchange 50 and 100. Altogether, the number of steps is

$$\begin{aligned}
&(n-50)! + (n-51)! + \ldots + (n-99)! \\
&\quad \le (n-50)! + (n-51)! \cdot (1 + 1/2 + 1/4 + \ldots) \\
&\quad = (n-50)! + 2 \cdot (n-51)! \\
&\quad = (n-50)! \cdot (1 + o(1)).
\end{aligned}$$

Thus, (ii) is true.

(b) For any $1 \le i \le n-1$ and any ordering of $1, 2, \ldots, i$, we see in succession all $n!/i!$ possibilities of completing this ordering to a permutation of $1, 2, \ldots, n$. Thus, it will take $n!/100!$ steps until the number 100 moves for the first time. (Recall our convention that, when two consecutive numbers interchange, we credit the move to the larger of the two.) It will take another $n!/100!$ steps until the number 100 moves for the second time, and so forth. Now the number 100 needs to move 50 times to the left to get before 50, so that the total number of steps required is $50 \cdot n!/100!$.

Thus, (iv) is true.

(c) Method B is a special case of a general method – selecting a random element until getting an item with the required properties. The number of times we need to repeat the process is distributed $G(p)$, where p is the probability of selecting an item as required. In our case, $p = 1/2$, so that we need to select two permutations on the average to get one satisfying $\sigma^{-1}(100) < \sigma^{-1}(50)$. Since we can select a uniformly random permutation in linear time, the method works in linear time as well.

Method A clearly works in linear time, and selects only permutations satisfying the property in question, but permutations are not selected with the required probabilities. Consider, for example, the permutation $(1, 2, \ldots, 49, 51, \ldots, 99, 101, \ldots, n, 100, 50)$. It will be selected by the algorithm if and only if we first select the identity permutation $(1, 2, \ldots, n-2)$ of the elements $1, 2, \ldots, n-2$, then choose $k = n-1$, and then $m = n-1$. The probability for these exact choices is

$$\frac{1}{(n-2)!} \cdot \frac{1}{n-1} \cdot \frac{1}{n-1} = \frac{1}{(n-1)(n-1)!} < \frac{2}{n!}.$$

Thus, (ii) is true.

(d) With Method A, we select any permutation possessing the property in question if we either select it in the first place, or select the required permutation, with 50 and 100 interchanged, and then at the second stage interchange 50 and 100. The total probability for this is

$$\frac{1}{n!} + \frac{1}{n!} \cdot \alpha = \frac{1+\alpha}{n!}.$$

Similarly, the probability for selecting any permutation without the required property is

$$\frac{1}{n!} \cdot (1 - \alpha).$$

To get the correct probabilities we need

$$1 + \alpha = 2(1 - \alpha),$$

which yields $\alpha = 1/3$.

Method B is a special case of a general method for selecting elements from a set randomly, but non-uniformly. As we have seen, if we select a uniformly random element, then we should accept it with a probability of $p/p_{\max}$, where p is the probability with which we want to select this element, and $p_{\max}$ is the probability of the element of maximal probability. In our case, this ratio is 1 for permutations possessing the property in question, and is 1/2 for the other permutations. Hence we should take $\beta = 1/2$.

Thus, (i) is true.

2. First, let us split the set of partitions into three pairwise disjoint classes:

   - The partitions in which n does not belong to the "last" (rightmost) set.
   - The partitions in which n belongs to the last set, but does not form a set by itself.
   - The partitions in which $\{n\}$ is one of the sets.

   The last two collections are obtained from the collection of all partitions of $\{1, 2, \ldots, n-1\}$ by adding the element n to the last set in each partition and by adding the singleton $\{n\}$ to each partition, respectively. Hence, each of these consists of a_{n-1} partitions, and therefore the first collection includes $a_n - 2a_{n-1}$ partitions. Now a change in a partition of the first type, when moving to its successor, is that we move n to the next set, which does not change the number of setsR. Hence all $a_n - 2a_{n-1}$ partitions of the first type possess the property that the number of sets does not change when we move to the next partition. In all partitions of the second type, the number of sets grows by 1, and they contribute nothing. A partition of the third type loses a set because n joins the first set in the partition. However, this loss is offset if the element $n-1$ belongs to the last set in the partition induced on $\{1, 2, \ldots, n-1\}$, but does not form a set by itself there. Namely, the induced partition is of type analogous to the second type above, but for $\{1, 2, \ldots, n-1\}$. By the same reasoning as before, the number of these partitions is a_{n-2}. Altogether, the number of partitions satisfying the required condition is

$$(a_n - 2a_{n-1}) + a_{n-2} = a_n - 2a_{n-1} + a_{n-2}.$$

Thus, (ii) is true.

3. All types of changes defined in the question change the degree of each vertex by at most 1. Hence, starting with an arbitrary tree in the family, it will be impossible by any small change to turn a leaf into a vertex of degree 3 and vice versa.

   Thus, (iv) is true.

4. (a) Square (i, j) has $n - i$ squares below it and $2 - j$ squares to its right for $1 \leq i \leq n, 1 \leq j \leq 2$. By the hook length formula:

$$\begin{aligned} Y_{n,2} &= \frac{(2n)!}{((n+1)n(n-1)\cdot 2)\cdot(n(n-1)\cdot\ldots\cdot 1)} \\ &= \frac{(2n)!}{(n+1)!n!} = \frac{\binom{2n}{n}}{n+1}. \end{aligned}$$

   Thus, (iii) is true.

   (b) The number 2 appears in any Young table either at the top of the second column or at the second entry of the second column. In our case, if 2 appears at the top of the second column, then the rest of the table may be any table of the given shape, but with

$n-1$ rows instead of n, filled by the integers between 3 and $2n$, instead of the integers between 1 and $2n-2$. Hence the number of these tables is $Y_{n-1,2}$. Now we claim that, whenever we have such a table, the 2 will move to the first column already in the next table (except for the last table in the sequence). Indeed, the least j which is not at the lowest corner, must be in the first column, and hence it is larger than 2. When we move to the next table, we will fill the numbers $1, 2, \ldots$ in the first column up to the place j occupies. In particular, 2 will move to the first column. Since 2 is located at the first column in the first table, and at the second column in the second table, this means that it will change its location exactly

$$Y_{n-1,2} + (Y_{n-1,2} - 1) = 2Y_{n-1,2} - 1$$

times.

The number $2n$ is at the bottom of the second column in all tables of the shape in question. Hence we may ignore it, and consider the shape consisting of one column of height n and a second column of height $n-1$. In all Young tables of this shape, the number $2n-1$ appears at the bottom of one of the columns. Clearly, all tables with $2n-1$ at the bottom of the second column precede all those with $2n-1$ at the bottom of the first column. Hence, $2n-1$ will move but once.

Thus, (i) is true.