Automata and Formal Languages

Solutions to Selected Exercises

1 Review of some Basic Notions

5. By induction on n. For $n = 24$:

$$24 = 5 \cdot 2 + 7 \cdot 2.$$

Assume that for some $n \geq 24$ we have $n = 5a + 7b$ for appropriately chosen non-negative integers a and b. Clearly, we must have either $b \geq 2$ or $a \geq 4$. In the first of these cases we have $n+1 = 7(b-2) + 5(a+3)$ and in the second $n+1 = 7(b+3) + 5(a-4)$.

10. Each of the people knows either 0 other people, or 1, or 2, ..., or $n-1$. It is impossible for one of the people to know 0 others and for another to know $n-1$. Suppose, say, nobody knows $n-1$ people. There are $n-1$ possibilities regarding the number of other people each of the people knows. Using the pigeonhole principle, we conclude that there are two people who know the same number of people.
The case where each person knows at least one other person is dealt with in the same way.

13.

(a) Reflexive, symmetric, transitive.

(b) Symmetric.

(c) Reflexive, symmetric.

(d) Reflexive, symmetric, transitive.

(e) Transitive.

(f) Reflexive, transitive.

(g) Symmetric.

(h) Reflexive, symmetric, transitive.

(i) Reflexive, symmetric.

(j) Reflexive, symmetric, transitive.

(k) Reflexive, symmetric, transitive.

(l) Reflexive, transitive.

(m) Reflexive, symmetric.

(n) Reflexive, symmetric.

17. Let $G = (V, E)$ be a graph with the property described in the question. We use induction on the size of the graph. For $|V| = 2$, denote by v_1, v_2 the two vertices of G. Then either $(v_1, v_2) \in E$ or $(v_2, v_1) \in E$, and therefore there is a Hamiltonian path in G. Now assume that that the claim holds for graphs with $|V| = n$, and let $G = (V, E)$ be a graph with $|V| = n + 1$ satisfying the property in the question. Consider the subgraph G' obtained by omitting the some vertex $v \in V$ from G. By the induction hypothesis there exists a Hamiltonian path in G'. Let $v_1, v_2, \ldots, v_n$ be such a Hamiltonian path. If the edge (v, v_1) is in E, then $v, v_1, v_2, \ldots, v_n$ is a Hamiltonian path in G. Otherwise, $(v_1, v) \in E$. Consider the first $1 \le i \le n - 1$ (if there exists any) such that $(v_i, v) \in E$ and $(v, v_{i+1}) \in E$. If such an i exists, then $v_1, \ldots, v_i, v, v_{i+1}, \ldots, v_n$ is a Hamiltonian path in G. Otherwise, $(v_n, v) \in E$, and therefore $v_1, v_2, \ldots, v_n, v$ is a Hamiltonian path in G.

2 Decidable and Undecidable Problems

19. Let $z = a + bi \ne 0, \pm 1, \pm i$. The only potential divisors $y = c + di$ of z are those for which $2 \le c^2 + d^2 < a^2 + b^2$. Since there are only finitely many such numbers, we can check for each of these whether or not it divides z.

24. Let $a \in [1, n]$ be a number for which the equation is solvable. Note that it suffices to consider non-negative solutions (x, y, z). If (x, y, z) is such a solution, then $x, y, z \le \sqrt[4]{n}$. Since the number of triples (x, y, z) with integral x, y, z in the range $[0, \sqrt[4]{n}]$ is at most $(n^{1/4} + 1)^3 = o(n)$, for most integers a there is no solution.

3 Alphabets and Languages

25.

(a) Let us show that two words $x, y \in \Sigma^*$ commute if and only if $x = w^k$ and $y = w^l$ for some $w \in \Sigma^*$ and $k, l \geq 0$.
In fact, the "if" part is straightforward. For the "only if" part, we have to show that, if $xy = yx$, then $x = w^k$ and $y = w^l$ for some $w \in \Sigma^*$ and $k, l \geq 0$. We use induction on $|x| \cdot |y|$. If $|x| \cdot |y| = 0$ the claim is trivial. Suppose the claim is correct for all pairs of words $x, y \in \Sigma^*$ such that $|x| \cdot |y| < n$ for some $n \geq 1$. Let $x, y \in \Sigma^*$ be two words with $|x| \cdot |y| = n$. Assume, say, $1 \leq |x| = p \leq q = |y|$. The equality $xy = yx$ shows, in particular, that the prefix of length p of the words xy and yx coincide. Thus, x is a prefix of y. Write $y = xu$, where $|u| = q - p$. Since $xy = yx$, we have $xxu = xux$, so that $xu = ux$. Since $|x| \cdot |u| = p(q-p) < n$, there exist some $w \in \Sigma^*$ and $k, l' \geq 0$ such that $x = w^k$ and $u = w^{l'}$. Hence, $y = xu = w^k w^{l'} = w^{k+l'} = w^l$ for $l = k + l'$, and thereby the "only if" part is proven.

28.

(a) To prove the inclusion $L_1L_3 \cup L_2L_3 \subseteq (L_1 \cup L_2)3$, we note that, in general, if $L' \subseteq L''$, then $L'L \subseteq L''L$ for every L. In our case, since both $L_1 \subseteq L1 \cup L_2$ and $L_2 \subseteq L1 \cup L_2$, we have $L_1L_3 \subseteq (L_1 \cup L_2)\, L_3$ and $L_2L_3 \subseteq (L_1 \cup L_2)\, L_3$, thus proving the required inclusion.
To prove the converse inclusion, let $w \in (L_1 \cup L_2)\, L_3$. Write $w = uv$, where $u \in L_1 \cup L_2$ and $v \in L_3$. If $u \in L_1$, then $uv \in L_1L_3$, and so $w = uv \in L_1L_3 \cup L_2L_3$. Otherwise, $u \in L_2$, $uv \in L_1L_2$, and so $w = uv \in L_1L_3 \cup L_2L_3$.

(b) Let $L_1 = \{1\}, L_2 = \{10\}, L_3 = \{1, 01\}$, and $w = 101$. Then $w \in L_1L_3 \cap L_2L_3$, but $w \notin (L_1 \cap L_2)L_3$.

4 Regular Expressions

34.

a. 1^*0.

b. $(0 \cup 1)^*$.

c. $(0 \cup 1)^*1 \cup \phi^* \cup (01^*100)^*$.

d. $(a \cup b)^*$.

e. b^*a.

f. $b(a \cup b)^* \cup (a \cup b)^*b \cup \phi^*$.

g. $b^*a(a \cup b)^*$

5 Deterministic Finite Automata

6 Non-Deterministic Finite Automata

7 Pumping Lemma

8 Context-Free Grammars

9 Regular Grammars

10 Pushdown Automata

11 Closure Properties; Non-Context-Freeness Proofs

12 Turing Machines