Shape from Shading
Shape from Shading
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Shading is more than contours
Shape from Shading

Shading is more than edge contours
Shape from Shading

Inverting the image formation process

Image formation = “Shading from shape” (and light sources)
Image formation 1: Where is a world/object point projected in the image plane?

Image formation 2: What is the amount of light that is reflected in the direction of the camera?
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representation of directions

$L(\phi_r, \theta_r)$

$E(\phi_i, \theta_i)$

$\phi$ - Azimuth angle

$\theta$ - Zenith angle
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The Bidirectional Reflectance Distribution Function (BRDF)

\[ f_\lambda(\phi_i, \theta_i; \phi_r, \theta_r) = \frac{L_\lambda(\phi_r, \theta_r)}{E_\lambda(\phi_i, \theta_i)} \]

Helmholtz’s reciprocity

\[ f(\phi_i, \theta_i; \phi_r, \theta_r) = f(\phi_r, \theta_r; \phi_i, \theta_i) \]

Isotropic materials:

\[ f(\phi_i, \theta_i; \phi_r, \theta_r) = f(\phi_i - \phi_r, \theta_i, \theta_r) \]
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What is the intensity reflected in the direction of the camera?

\[ f_\lambda (\phi_i, \theta_i; \phi_r, \theta_r) = \frac{L_\lambda (\phi_r, \theta_r)}{E_\lambda (\phi_i, \theta_i)} \quad \rightarrow \quad L_\lambda (\phi_r, \theta_r) = f_\lambda (\phi_i, \theta_i; \phi_r, \theta_r)E_\lambda (\phi_i, \theta_i) \]
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Total surface reflection towards the camera

\[ L(\phi_r, \theta_r) = \int \omega f(\phi_i, \theta_i; \phi_r, \theta_r) \cdot E(\phi_i, \theta_i) \cdot \cos \theta \, d\omega \]
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Total surface reflection towards the camera

\[
L(\phi_r, \theta_r) = \int_{-\pi}^{\pi} \int_{0}^{\pi/2} f(\phi_i, \theta_i; \phi_r, \theta_r) \cdot E(\phi_i, \theta_i) \cdot \sin \theta_i \cdot \cos \theta_i \cdot \delta \theta_i \delta \phi_i
\]

\[L(\phi_r, \theta_r)\]
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Lambertian (perfectly diffused) surfaces

\[ f_L(\phi_i, \theta_i; \phi_r, \theta_r) = \text{const} = \overline{f} = \frac{1}{\pi} \]

\[ \int_{-\pi}^{\pi} \int_{0}^{\pi/2} \overline{f} \cdot \sin \theta_r \cdot \cos \theta_r \cdot \delta \theta_r \delta \phi_r = 1 \]

\[ \pi \overline{f} = 1 \]
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Mirrored (perfectly secular) surfaces

\[ f_S(\phi_i, \theta_i; \phi_r, \theta_r) = \frac{\delta(\theta_r - \theta_i)\delta(\phi_r - \phi_i - \pi)}{\sin \theta_i \cos \theta_i} \]
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Mixed surfaces

\[ f(\phi_i, \theta_i; \phi_r, \theta_r) = \alpha \cdot f_L(\phi_i, \theta_i; \phi_r, \theta_r) + (1 - \alpha) f_S(\phi_i, \theta_i; \phi_r, \theta_r) \]
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The fundamental radiometric relationship

\[ I = L \cdot \frac{\pi}{4} \left( \frac{d}{f} \right)^2 \cdot \cos^4 \alpha \]

\[ \Rightarrow \quad I \propto L \]
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Point light source from direction \((\phi_L, \theta_L)\)

\[
E(\phi_i, \theta_i) = E \cdot \frac{\delta(\theta_L - \theta_i) \cdot \delta(\phi_L - \phi_i)}{\sin \theta_L}
\]

\[
\int_{-\pi}^{\pi} \int_{0}^{\pi/2} E(\phi, \theta_i) \cdot \sin \theta_i \cdot \delta \theta_i \delta \phi_i = E
\]
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Surface brightness – appearance in the Lambertian case and point light source

\[ f_L(\phi_i, \theta_i; \phi_r, \theta_r) = \rho \frac{1}{\pi} \]

\[ E(\phi_i, \theta_i) = \frac{\delta(\theta_L - \theta_i) \delta(\phi_L - \phi_i)}{\sin \theta_L} \]

\[ I(x, y) \propto L(\phi_r, \theta_r) = \int_{-\pi}^{\pi} \int_{0}^{\pi/2} f(\phi_i, \theta_i; \phi_r, \theta_r) \cdot E(\phi_i, \theta_i) \cdot \sin \theta_i \cdot \cos \theta_i \cdot \delta \theta_i \delta \phi_i \]

\[ L = \rho \frac{1}{\pi} E \cos \theta_L \propto \rho (\hat{N} \cdot \hat{L}) \]
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Shape description – Tangent plane and normal vectors

\[ \vec{r}_x = \left( 1, 0, \frac{\partial H}{\partial x} \right) \quad \vec{r}_y = \left( 0, 1, \frac{\partial H}{\partial y} \right) \]

\[ \vec{N} = \vec{r}_x \times \vec{r}_y = (-p, -q, 1) \]

\[ \hat{N} = \frac{\vec{N}}{\|\vec{N}\|} = \frac{(-p, -q, 1)}{\sqrt{p^2 + q^2 + 1}} \]
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Shading on Lambertian surface – General point source

\[
I = \rho(\hat{N} \cdot \hat{L}) = \rho \frac{-p \cdot L_x - q \cdot L_y + L_z}{\sqrt{p^2 + q^2 + 1} \sqrt{L_x^2 + L_y^2 + L_z^2}} = \rho \frac{p \cdot p_L + q \cdot q_L + 1}{\sqrt{p^2 + q^2 + 1} \sqrt{p_L^2 + q_L^2 + 1}}
\]
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Shading on Lambertian surface – Overhead point source

\[
I(x, y) = \rho(\hat{N} \cdot [0, 0, 1]) = \rho \frac{1}{\sqrt{p^2 + q^2 + 1}} = R(p, q)
\]
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The Reflectance Map – Lambertian surface from overhead source position

\[ R(p, q) = \frac{1}{\sqrt{p^2 + q^2 + 1}} \]
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The Reflectance Map – Lambertian surface from general source position

\[ R(p, q) = \frac{p \cdot p_L + q \cdot q_L + 1}{\sqrt{p^2 + q^2} + 1 \sqrt{p_L^2 + q_L^2} + 1} \]

Gradient point of maximum brightness
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The Reflectance Map – typical real surfaces

\[ R(p, q) \]
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Surface orientation from shading

\[ I(x, y) = R(p, q) \]
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Photometric stereo
Shape from Shading

Photometric stereo

\[ I_1(x, y) = R_1(p, q) \]
\[ I_2(x, y) = R_2(p, q) \]
The SFS problem (special case)

Given $I(x,y)$ of an (orthographic) projection of $H(x,y)$, and the reflectance map $R(p,q)$, find $H(x,y)$ everywhere.

$$I(x, y) = R(p, q) = R\left(\frac{\partial}{\partial x} H(x, y), \frac{\partial}{\partial y} H(x, y)\right)$$
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Shape recovery via characteristic strips

\[ I(x, y) = R(p(x, y), q(x, y)) \]

\[ p(x, y) = \frac{\partial}{\partial x} H(x, y) \]

\[ q(x, y) = \frac{\partial}{\partial y} H(x, y) \]

\[ H(x + \delta x, y + \delta y) \approx H(x, y) + p \delta x + q \delta y \]

\[ p(x + \delta x, y + \delta y) \approx p(x, y) + \frac{\partial p}{\partial x} \delta x + \frac{\partial p}{\partial y} \delta y \]

\[ q(x + \delta x, y + \delta y) \approx q(x, y) + \frac{\partial q}{\partial x} \delta x + \frac{\partial q}{\partial y} \delta y \]
Shape from Shading

Shape recovery via characteristic strips

\[ I(x, y) = R(p(x, y), q(x, y)) \]

\[ \frac{\partial}{\partial x} I(x, y) = \frac{\partial R(p, q)}{\partial p} \frac{\partial p(x, y)}{\partial x} + \frac{\partial R(p, q)}{\partial q} \frac{\partial q(x, y)}{\partial x} \]

\[ \frac{\partial}{\partial y} I(x, y) = \frac{\partial R(p, q)}{\partial p} \frac{\partial p(x, y)}{\partial y} + \frac{\partial R(p, q)}{\partial q} \frac{\partial q(x, y)}{\partial y} \]
Shape from Shading

Shape recovery via characteristic strips

\[ I(x, y) = R(p(x, y), q(x, y)) \]

\[
\frac{\partial}{\partial x} I(x, y) = \frac{\partial R(p, q)}{\partial p} \frac{\partial p(x, y)}{\partial x} + \frac{\partial R(p, q)}{\partial q} \frac{\partial p(x, y)}{\partial y}
\]

\[
\frac{\partial}{\partial y} I(x, y) = \frac{\partial R(p, q)}{\partial p} \frac{\partial q(x, y)}{\partial x} + \frac{\partial R(p, q)}{\partial q} \frac{\partial q(x, y)}{\partial y}
\]
Shape from Shading

Shape recovery via characteristic strips

\[ I(x, y) = R(p(x, y), q(x, y)) \]

\[ \frac{\partial}{\partial x} I(x, y) = \nabla R \cdot \nabla p \]

\[ \frac{\partial}{\partial y} I(x, y) = \nabla R \cdot \nabla q \]

\[ \delta H \approx p \delta x + q \delta y \]

\[ \delta p \approx \frac{\partial p}{\partial x} \delta x + \frac{\partial p}{\partial y} \delta y = \nabla p \cdot (\delta x, \delta y) \]

\[ \delta q \approx \frac{\partial q}{\partial x} \delta x + \frac{\partial q}{\partial y} \delta y = \nabla q \cdot (\delta x, \delta y) \]

A smart choice

\[ \delta x = \frac{\partial R(p, q)}{\partial p} \delta s \]

\[ \delta y = \frac{\partial R(p, q)}{\partial q} \delta s \]

\[ \delta p \approx \frac{\partial}{\partial x} I(x, y) \cdot \delta s \]

\[ \delta q \approx \frac{\partial}{\partial y} I(x, y) \cdot \delta s \]
Shape from Shading

Shape recovery via characteristic strips

\[ \delta x = R_p \delta s \]
\[ \delta y = R_q \delta s \]
\[ \delta H = \left( pR_p + qR_q \right) \delta s \]
\[ \delta p = I_x \delta s \]
\[ \delta q = I_y \delta s \]
\[ \dot{x} = R_p \]
\[ \dot{y} = R_q \]
\[ \dot{H} = pR_p + qR_q \]
\[ \dot{p} = I_x \]
\[ \dot{q} = I_y \]
**Shape from Shading**

Shape recovery via characteristic strips

**Shape from Shading via Characteristic Curves**

Given

- $I(x,y)$ of an (orthographic) projection of unknown $H(x,y)$
- The reflectance map $R(p,q)$
- Initial data $x_0, y_0, H(x_0,y_0), p(x_0,y_0), q(x_0,y_0)$

Develop a curve solution on $H(x,y)$ by taking small steps of size $\delta s$ via the system

$$
\delta x = R_p \delta s \\
\delta y = R_q \delta s \\
\delta H = (pR_p + qR_q) \delta s \\
\delta p = I_x \delta s \\
\delta q = I_y \delta s
$$
Shape from Shading

Shape recovery via characteristic strips
Shape from Shading

Shape recovery via characteristic strips